# *Polynomial extensions in* $H(\mathbf{curl})$

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# Traces

Traces of Sobolev spaces are well studied.

Scalar trace: $\operatorname{trc} \phi = \phi\big|_{\partial K}$

Normal trace: $\operatorname{trc}_n \boldsymbol{\phi} = (\boldsymbol{\phi} \cdot \boldsymbol{n})\big|_{\partial K}$,

Tangential trace: $\operatorname{trc}_\tau \boldsymbol{\phi} = \big(\boldsymbol{\phi} - (\boldsymbol{\phi} \cdot \boldsymbol{n})\boldsymbol{n}\big)\big|_{\partial K}$,

*Ranges:*

$$H^{1/2}(\partial K) = \operatorname{trc} H^1(K)$$

$$H^{-1/2}(\partial K) = \operatorname{trc}_n \boldsymbol{H}(\mathrm{div})$$

$$\boldsymbol{X}^{-1/2}(\partial K) = \operatorname{trc}_\tau \boldsymbol{H}(\mathbf{curl})$$

*Trace maps are continuous:*

$$H^1(K) \xrightarrow{\operatorname{trc}} H^{1/2}(\partial K)$$

$$\boldsymbol{H}(\mathrm{div}) \xrightarrow{\operatorname{trc}_n} H^{-1/2}(\partial K)$$

$$\boldsymbol{H}(\mathbf{curl}) \xrightarrow{\operatorname{trc}_\tau} H^{-1/2}(\partial K)$$

# Extensions

- Extension operators are right inverses of trace maps.
- Traditionally they appear in Sobolev space theory in proving the surjectivity of trace maps. [Lions, 1972]
- A *polynomial extension* operator is an extension with the additional property that whenever the function on $\partial K$ to be extended is the trace of a polynomial on $K$, the extended function is also a polynomial. (Many standard extensions – e.g., [Lions]'s – are not polynomial extensions.)
- Polynomial extensions are important in high order finite elements ($hp$ FEM).

# Background

- $1^{st}$polynomial extension [Babuška & Suri,1987] for $H^1$(triangle).
- Used later by [Maday, 1989] (for interpolation), and [Babuška, Craig, Mandel & Pitkäranta, 1991] (preconditioning).
- Polynomial extension for $H^1(\text{cube})$: [Ben Belgacem, 1994].
- For $H^1(\text{tetrahedron})$: [Muñoz-Sola, 1997].
- Two-dimensional $H(\mathrm{curl})$: [Demkowicz & Babuška, 2003], [Ainsworth & Demkowicz, 2007] (Hardy integral operators).
- Tetrahedral $\boldsymbol{H}(\mathbf{curl})$ case? $\boldsymbol{H}(\mathrm{div})$ case?

  We develop a new technique of constructing *commuting polynomial extensions* for all first order Sobolev spaces $H^1(K)$, $\boldsymbol{H}(\mathbf{curl})$, and $\boldsymbol{H}(\mathrm{div})$, on a tetrahedron.

# $H^1(K)$ *polynomial extension*

*Problem in $H^1(K)$:* For any tetrahedron $K$, construct a map

$$\mathcal{E}_K^{\text{grad}} : H^{1/2}(\partial K) \mapsto H^1(K)$$

with the following properties:

- *Extension property:* $\operatorname{trc} \mathcal{E}_K^{\text{grad}} u = u.$
- *Continuity:* $\mathcal{E}_K^{\text{grad}}$ is a continuous operator.
- *Polynomial preservation:* ($P_p =$ polynomials of degree $\le p$.)

$$u = \operatorname{trc} \phi_p \text{ for some } \phi_p \in P_p \implies \mathcal{E}_K^{\text{grad}} u \in P_p.$$

# $\boldsymbol{H}(\mathbf{curl})$ *polynomial extension*

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*Problem in $\boldsymbol{H}(\mathbf{curl})$:* Construct an operator

$$\mathcal{E}_K^{\mathrm{curl}} : \boldsymbol{X}^{-1/2}(\partial K) \mapsto \boldsymbol{H}(\mathbf{curl})$$

with the following properties:

- *Extension property:* $\operatorname{trc}_\tau \mathcal{E}_K^{\mathrm{curl}} \boldsymbol{u} = \boldsymbol{u}.$
- *Continuity:* $\mathcal{E}_K^{\mathrm{curl}}$ is a continuous operator.
- *Polynomial preservation:* ($\boldsymbol{N}_p =$ Nédélec space.)

$$\boldsymbol{u} = \operatorname{trc}_\tau \boldsymbol{\phi}_p \text{ for some } \boldsymbol{\phi}_p \in \boldsymbol{N}_p \implies \mathcal{E}_K^{\mathrm{curl}} \boldsymbol{u} \in \boldsymbol{N}_p.$$

$$\boldsymbol{u} = \operatorname{trc}_\tau \boldsymbol{\phi}_p \text{ for some } \boldsymbol{\phi}_p \in \boldsymbol{P}_p \implies \mathcal{E}_K^{\mathrm{curl}} \boldsymbol{u} \in \boldsymbol{P}_p.$$

# Commutative diagram

*Goal:* Construct polynomial extension operators satisfying

$$
\begin{array}{ccccc}
H^{1/2}(\partial K) & \xrightarrow{\mathbf{grad}_\tau} & \boldsymbol{X}^{-1/2}(\partial K) & \xrightarrow{\mathrm{curl}_\tau} & H^{-1/2}(\partial K) \\
\Big\downarrow \mathcal{E}_K^{\mathrm{grad}} & & \Big\downarrow \boldsymbol{\mathcal{E}}_K^{\mathrm{curl}} & & \Big\downarrow \boldsymbol{\mathcal{E}}_K^{\mathrm{div}} \\
H^1(K) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl}) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div})
\end{array}
$$

and establish the continuity estimates.

# Overview of techniques

- *Primary extensions:* Extensions from a plane.
- *Correction operators:* to fix traces on multiple faces.
- *Commutativity:* to move from left to right in the sequence

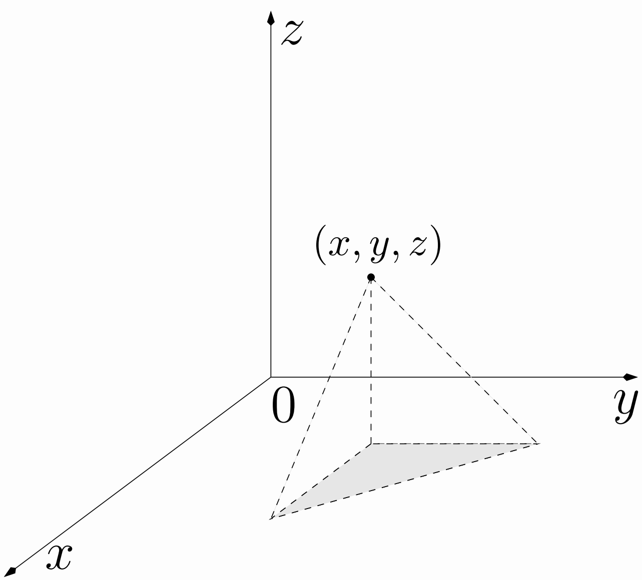
$$H^1(K) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl}) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2(K).$$

- *Regular decomposition of traces:* to obtain negative norm continuity from positive norm continuity.
- *Weighted norm estimates:* for integral operators defining the extensions.

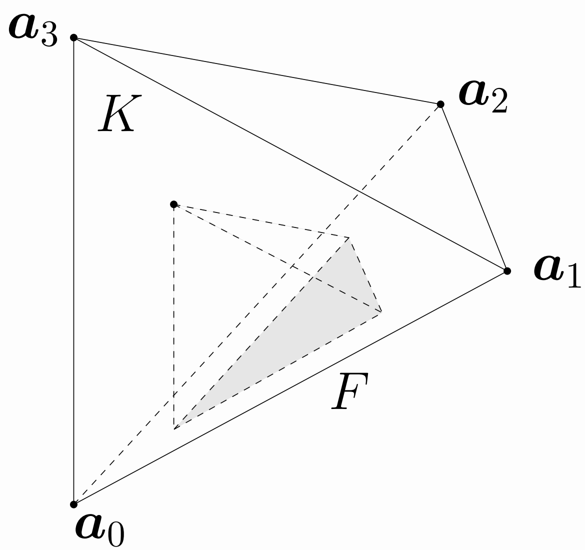
# Extensions from a plane

$$\mathcal{E}^{\text{grad}} u\,(x,y,z) = \frac{2}{z^2}\int_x^{x+z}\int_y^{x+y+z-x'} u(x',y')\,dy'\,dx'$$

$$= 2\int_0^1\int_0^{1-s} u(x+sz,\,y+tz)\,dt\,ds.$$




*Region of integration*




*Extension mapped to a general tetrahedron* $K$

# $H(\mathrm{curl})$ *primary extension*

How to define $\mathcal{E}^{\mathrm{curl}}$?

*Motivation:* We'd like to have commutativity

$$\mathcal{E}^{\mathrm{curl}}(\mathbf{grad}_\tau u) = \mathbf{grad}(\mathcal{E}^{\mathrm{grad}} u) \;\ldots$$

$$\left[ 2\,\mathbf{grad} \int_0^1 \int_0^{1-t} u(x+sz, y+tz)\, ds\, dt = 2 \int_0^1 \int_0^{1-t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ s & t \end{pmatrix} \mathbf{grad}_\tau\, u(x+sz, y+tz)\, ds\, dt \right.$$

Hence, define

$$\mathcal{E}^{\mathrm{curl}} \boldsymbol{v}(x,y,z) = 2 \int_0^1 \int_0^{1-t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ s & t \end{pmatrix} \boldsymbol{v}(x+sz, y+tz)\, ds\, dt.$$

# Properties

The operator $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}$ has the following properties:

- $\mathbf{grad}(\mathcal{E}^{\mathrm{grad}} u) = \boldsymbol{\mathcal{E}}^{\mathrm{curl}}(\mathbf{grad}_\tau\, u)$ for all smooth $u$.

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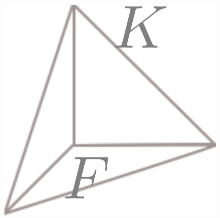
$$\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\boldsymbol{v}\,(x,y,z) = 2\int_0^1\int_0^{1-t}\begin{pmatrix}1 & 0\\ 0 & 1\\ s & t\end{pmatrix}\boldsymbol{v}(x+sz, y+tz)\,ds\,dt$$

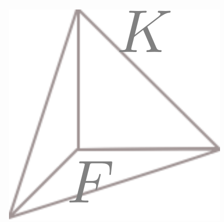


# Properties

The operator $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}$ has the following properties:

- $\mathbf{grad}(\mathcal{E}^{\mathrm{grad}}u) = \boldsymbol{\mathcal{E}}^{\mathrm{curl}}(\mathbf{grad}_\tau\, u)$ for all smooth $u$.
- If $\boldsymbol{v}$ is in $\boldsymbol{P}_p(F)$, then its extension $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\boldsymbol{v}$ is in $\boldsymbol{P}_p(K)$.
  If $\boldsymbol{v}$ is in $\boldsymbol{N}_p(F)$, then its extension $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\boldsymbol{v}$ is in $\boldsymbol{N}_p(K)$.

$$\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\boldsymbol{v}\ (x,y,z) = 2\int_0^1\int_0^{1-t}\begin{pmatrix}1 & 0\\ 0 & 1\\ s & t\end{pmatrix}\boldsymbol{v}(x+sz, y+tz)\, ds\, dt$$



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- $(\mathrm{trc}_\tau\, \boldsymbol{\mathcal{E}}^{\mathrm{curl}}\boldsymbol{v})\big|_F = \boldsymbol{v}$, for all smooth $\boldsymbol{v}$.

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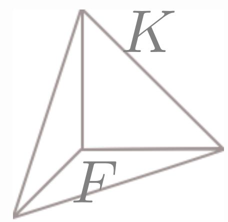

# Properties

The operator $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}$ has the following properties:

- $\mathbf{grad}(\mathcal{E}^{\mathrm{grad}} u) = \boldsymbol{\mathcal{E}}^{\mathrm{curl}}(\mathbf{grad}_\tau\, u)$ for all smooth $u$.
- If $\boldsymbol{v}$ is in $\boldsymbol{P}_p(F)$, then its extension $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\boldsymbol{v}$ is in $\boldsymbol{P}_p(K)$.
  If $\boldsymbol{v}$ is in $\boldsymbol{N}_p(F)$, then its extension $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\boldsymbol{v}$ is in $\boldsymbol{N}_p(K)$.
- $(\mathrm{trc}_\tau\, \boldsymbol{\mathcal{E}}^{\mathrm{curl}}\boldsymbol{v})\big|_F = \boldsymbol{v}$, for all smooth $\boldsymbol{v}$.
- $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}$ is a continuous map from $\boldsymbol{H}^{1/2}(F)$ into $\boldsymbol{H}^1(K)$.

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# Properties

The operator $\mathcal{E}^{\rm curl}$ has the following properties:

- $\mathbf{grad}(\mathcal{E}^{\rm grad} u) = \mathcal{E}^{\rm curl}(\mathbf{grad}_\tau\, u)$ for all smooth $u$.
- If $\boldsymbol{v}$ is in $\boldsymbol{P}_p(F)$, then its extension $\mathcal{E}^{\rm curl}\boldsymbol{v}$ is in $\boldsymbol{P}_p(K)$.
  If $\boldsymbol{v}$ is in $\boldsymbol{N}_p(F)$, then its extension $\mathcal{E}^{\rm curl}\boldsymbol{v}$ is in $\boldsymbol{N}_p(K)$.
- $(\mathrm{trc}_\tau\, \mathcal{E}^{\rm curl}\boldsymbol{v})\big|_F = \boldsymbol{v}$, for all smooth $\boldsymbol{v}$.
- $\mathcal{E}^{\rm curl}$ is a continuous map from $\boldsymbol{H}^{1/2}(F)$ into $\boldsymbol{H}^1(K)$.

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This can be proved using positive norm estimates and Peetre's $K$-functional.

But, we need continuity from the negative norm trace space . . .

# Trace space decompositions

PROPOSITION. $\boldsymbol{X}^{-1/2}(\partial K)$ *admits the stable decomposition*

$$\boldsymbol{X}^{-1/2}(\partial K) = \mathbf{grad}_\tau H^{1/2}(\partial K) + \operatorname{trc}_\tau \boldsymbol{H}^1(K).$$

# Trace space decompositions

PROPOSITION. $\boldsymbol{X}^{-1/2}(\partial K)$ *admits the stable decomposition*

$$\boldsymbol{X}^{-1/2}(\partial K) = \mathbf{grad}_\tau H^{1/2}(\partial K) + \underbrace{\mathrm{trc}_\tau \boldsymbol{H}^1(K)}_{\subseteq \boldsymbol{H}^{1/2} \text{ on faces}}.$$

- Thus, even though $\boldsymbol{X}^{-1/2} \subseteq \boldsymbol{H}^{-1/2}$ (negative norm), analysis is possible using $H^{1/2}$-norm (positive norm).
- Restrictions of traces to faces are well defined:

$$\boldsymbol{X}^{-1/2}(F) = \mathbf{grad}_\tau H^{1/2}(F) + \boldsymbol{H}^{1/2}(F).$$

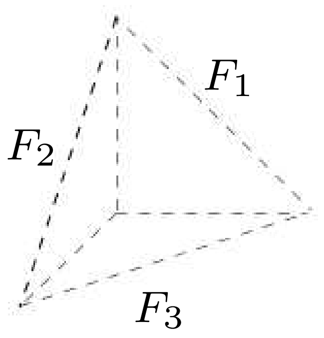
- It is also possible to similarly characterize traces of $\boldsymbol{H}(\mathbf{curl})$ functions that weakly vanish on some faces.

# *Primary extension theorem*

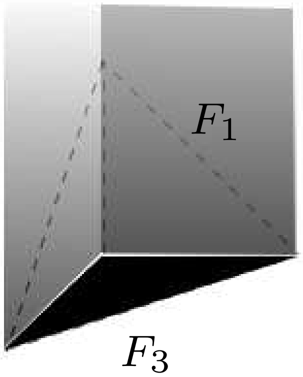
THEOREM. *The primary extension* $\mathcal{E}^{\text{curl}}$ *satisfies the following:*

1. Continuity: $\mathcal{E}^{\text{curl}}$ *is a continuous map from* $\boldsymbol{X}^{-1/2}(F)$ *into* $\boldsymbol{H}(\mathbf{curl}\,)$.
2. Commutativity: $\mathbf{grad}(\mathcal{E}^{\text{grad}}u) = \mathcal{E}^{\text{curl}}(\mathbf{grad}_\tau\, u)$ *for all* $u$ *in* $H^{1/2}(F)$.
3. Extension property: *The tangential trace of* $\mathcal{E}^{\text{curl}}\boldsymbol{v}$ *on* $F$ *equals* $\boldsymbol{v}$ *for all* $\boldsymbol{v}$ *in* $\boldsymbol{X}^{-1/2}(F)$.
4. Polynomial preservation:
   *If* $\boldsymbol{v}$ *is in* $\boldsymbol{P}_p(F)$, *then its extension* $\mathcal{E}^{\text{curl}}\boldsymbol{v}$ *is in* $\boldsymbol{P}_p(K)$.
   *If* $\boldsymbol{v}$ *is in* $\mathbf{N}_p(F)$, *then its extension* $\mathcal{E}^{\text{curl}}\boldsymbol{v}$ *is in* $\mathbf{N}_p(K)$.

# Two face problem

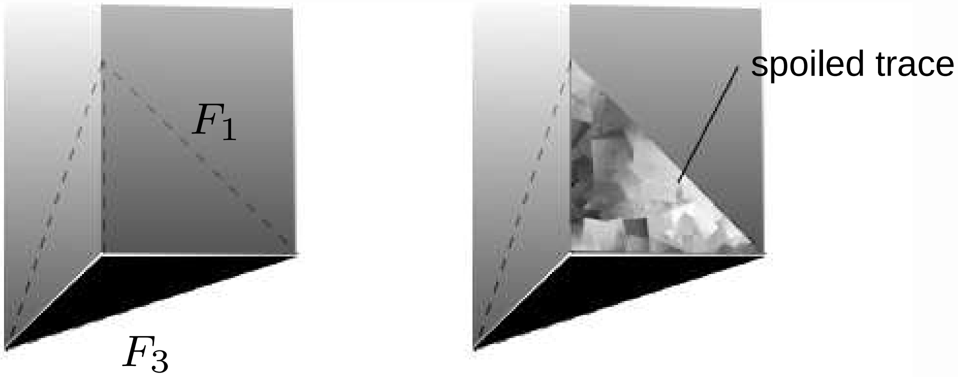


# Two face problem




$$U_1 = \mathcal{E}^{\text{grad}} u$$

# Two face problem

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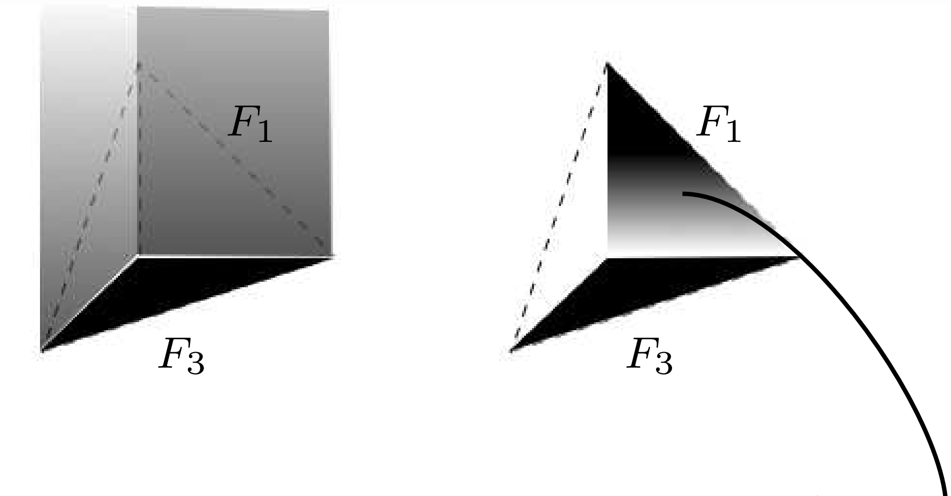
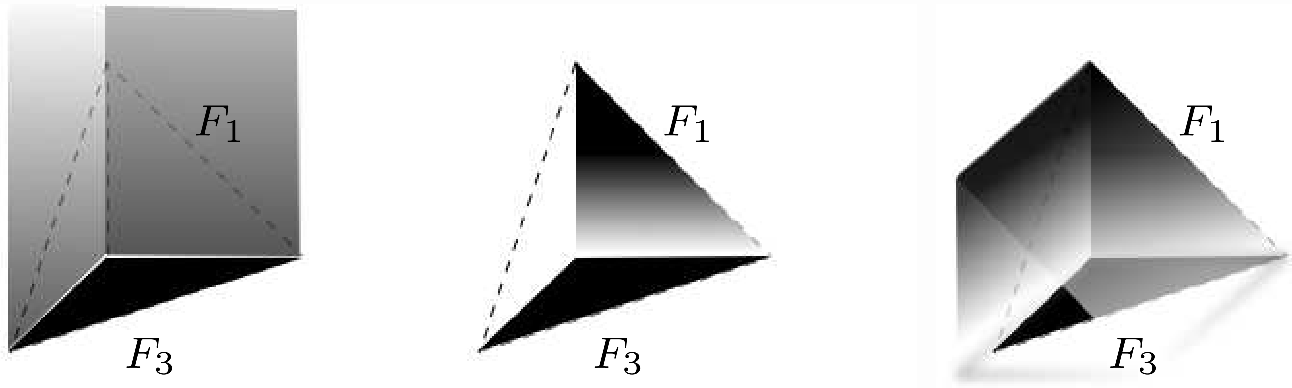
# Two face problem




$$U_1 = \mathcal{E}^{\text{grad}} u \qquad (u - U_1)\big|_{F_1}$$

Note: This difference is zero on the edge that $F_1$ shares with $F_3$.

# Two face problem




$U_1 = \mathcal{E}^{\mathrm{grad}} u$ $\qquad$ $U_2 = \mathring{\mathcal{E}}_1^{\mathrm{grad}} (u - U_1)\big|_{F_1}$

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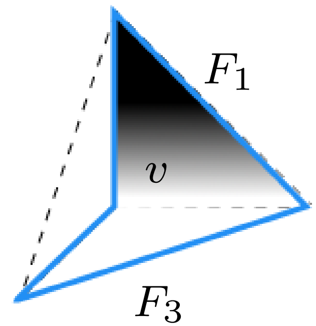
Zero preserving 2-face extension: We need an extension $\mathring{\mathcal{E}}_1^{\mathrm{grad}} v$ of $v$ from $F_1$ that has zero trace on $F_3$.



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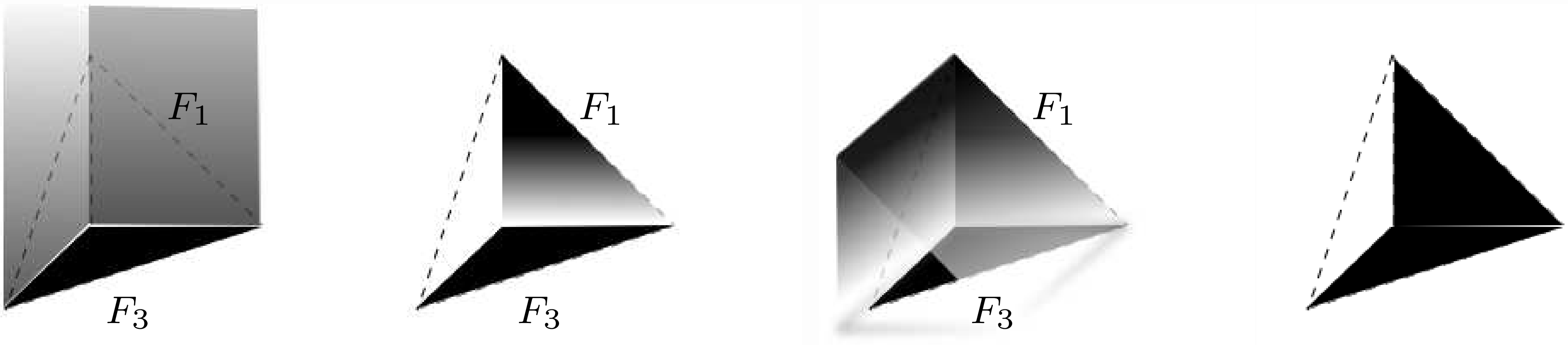



$$U_1 = \mathcal{E}^{\mathrm{grad}} u \qquad\qquad U_2 = \mathring{\mathcal{E}}_1^{\mathrm{grad}} (u - U_1)\big|_{F_1}$$

---

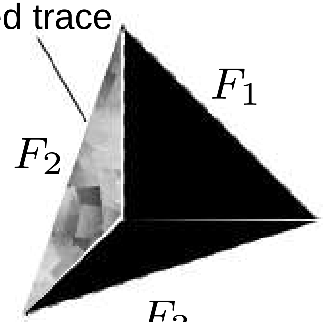
If we have such a $\mathring{\mathcal{E}}_1^{\mathrm{grad}}$, then the combined extension process

$$U_1 + U_2$$

achieves correct traces on **two faces** $F_1 \cup F_3$.

# Three face problem

The previous extension process that obtains correct traces on $F_1$ and $F_3$ does not produce the right trace on $F_2$.




$$U_1 + U_2$$

# Three face problem

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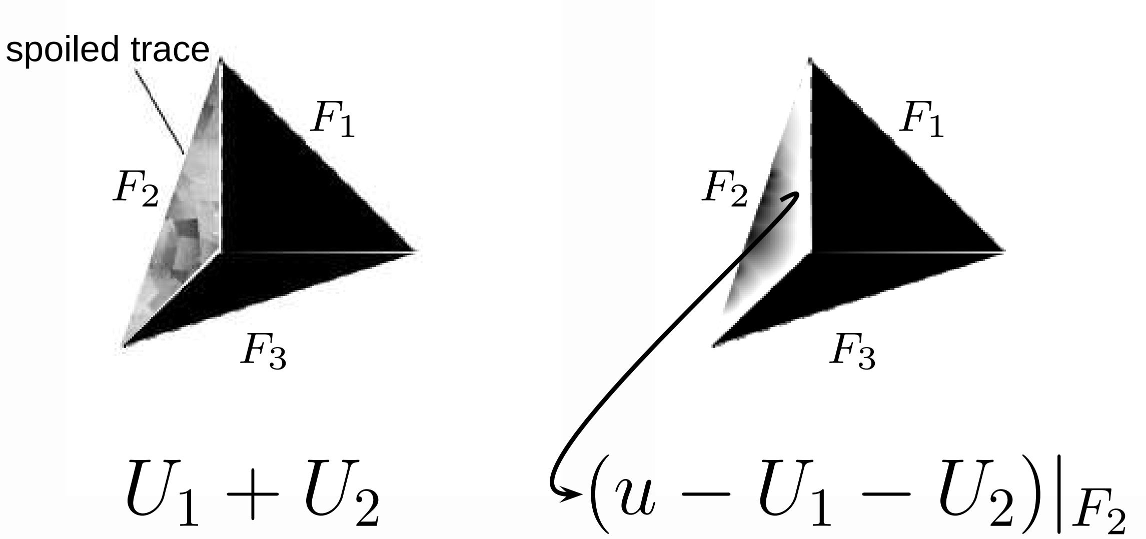


Note: This difference is zero on the two edges that $F_2$ shares with $F_3$ and $F_1$.

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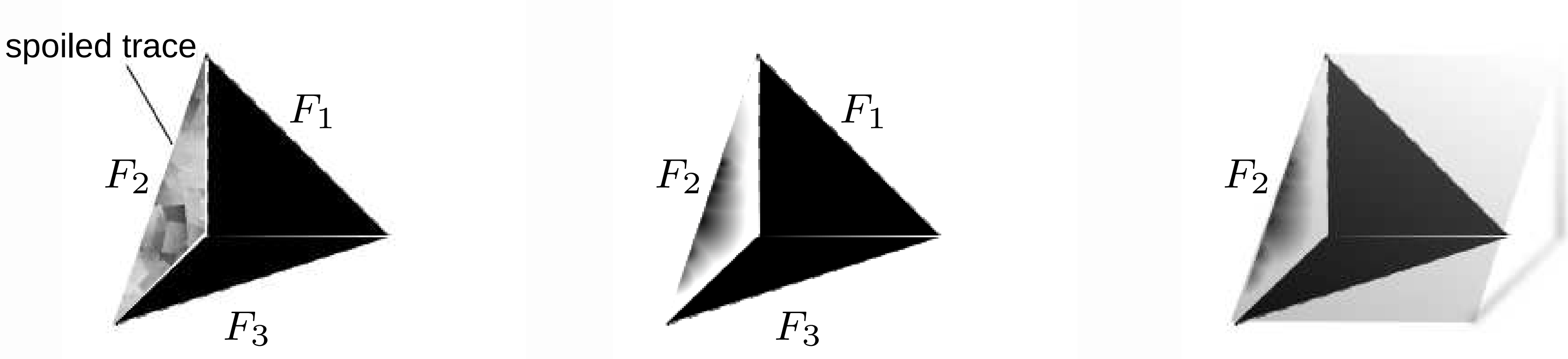



$$U_1 + U_2 \qquad U_3 = \mathring{\mathcal{E}}_2^{\text{grad}}(u - U_1 - U_2)|_{F_2}$$

The extension

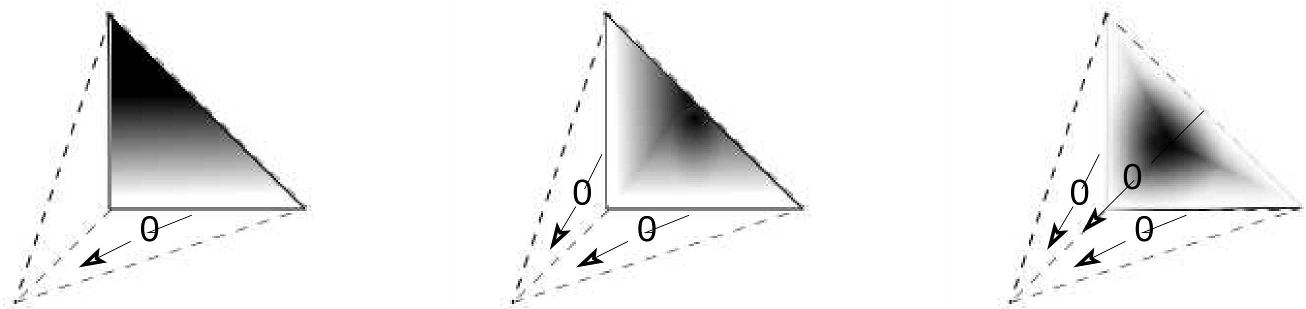
$$U_1 + U_2 + U_3$$

obtains correct traces on 3 faces ($F_1 \cup F_2 \cup F_3$), *provided* we have the zero preserving 3-face extension $\mathring{\mathcal{E}}_2^{\text{grad}}$.

# *Zero preserving extensions*

The two-face, three-face, and finally a four-face problem, can be solved, provided we have the required zero preserving extensions.

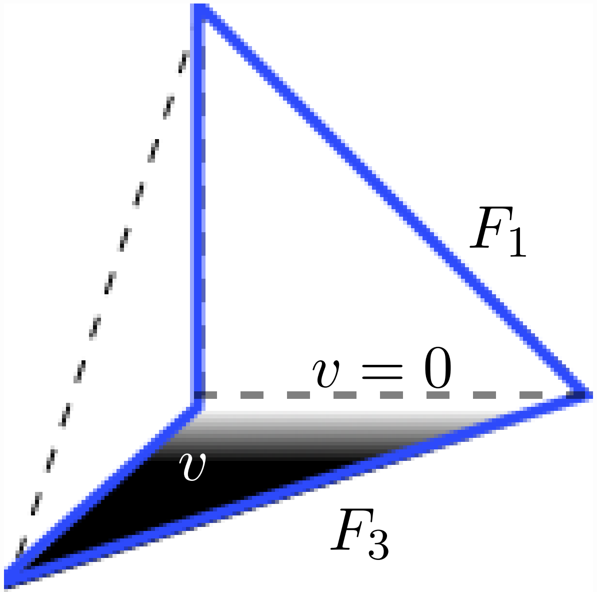
Thus, the whole extension from $\partial K$ can be constructed provided we have the following zero preserving extensions.




$\mathring{\mathcal{E}}_1^{\text{grad}}$ $\mathring{\mathcal{E}}_2^{\text{grad}}$ $\mathring{\mathcal{E}}_3^{\text{grad}}$

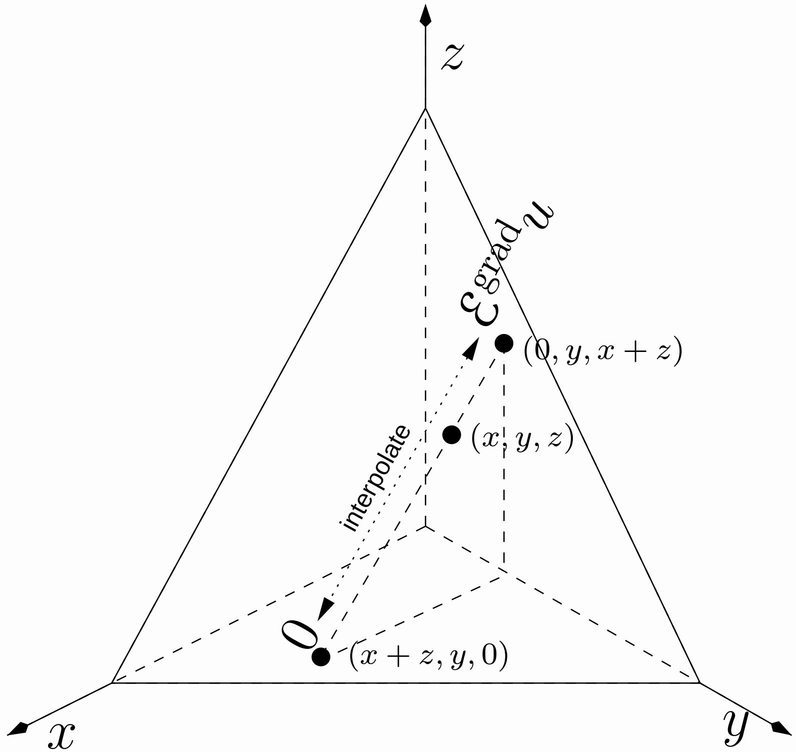
We will only discuss the construction of the first operator $\mathring{\mathcal{E}}_1^{\text{grad}}$.

$$\mathring{\mathcal{E}}_1^{\text{grad}} v = \mathcal{E}^{\text{grad}} v - \mathcal{E}_{\text{corr}}^{\text{grad}} v$$




First apply $\mathcal{E}^{\text{grad}}$ from $F_3$.

Then, we must fix face $F_1$, as $\mathcal{E}^{\text{grad}} v \neq 0$ on $F_1$.



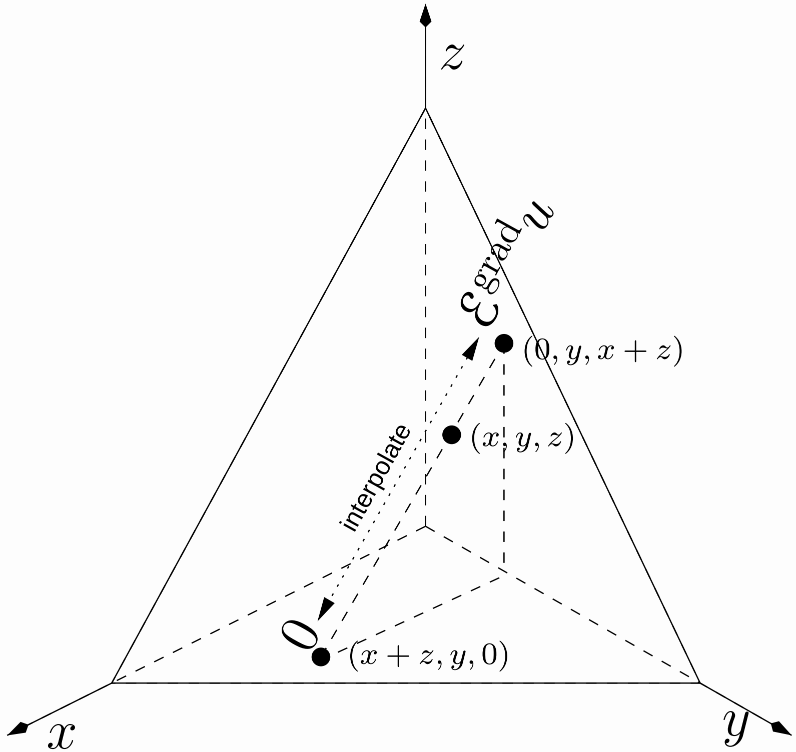

$$\mathcal{E}_{\text{corr}}^{\text{grad}} v(x, y, z) = \frac{z}{x+z} \mathcal{E}^{\text{grad}} v(0, y, x+z)$$

# The extension $\mathring{\mathcal{E}}_1^{\mathrm{grad}}$

$$\mathring{\mathcal{E}}_1^{\mathrm{grad}} v = \mathcal{E}^{\mathrm{grad}} v - \mathcal{E}_{\mathrm{corr}}^{\mathrm{grad}} v$$

Whenever $v$ is a polynomial, the correction $\mathcal{E}_{\mathrm{corr}}^{\mathrm{grad}} v$ is also a polynomial.

The denominator $x+z$ cancels out because $x$ divides $v(x,y)$.




$$\mathcal{E}_{\mathrm{corr}}^{\mathrm{grad}} v(x,y,z) = \frac{z}{x+z} \mathcal{E}^{\mathrm{grad}} v(0, y, x+z)$$

# The $H(\text{curl})$ correction

Construction motivated by commutativity.

Recall: $$\mathcal{E}^{\text{grad}}_{\text{corr}} v(x,y,z) = \frac{z}{x+z}\mathcal{E}^{\text{grad}} v(0,y,x+z)$$

Want $\mathcal{E}^{\text{curl}}_{\text{corr}}$ satisfying: $$\mathbf{grad}(\mathcal{E}^{\text{grad}}_{\text{corr}} u) = \mathcal{E}^{\text{curl}}_{\text{corr}}(\mathbf{grad}_\tau u).$$

$$= \frac{z}{x+z}\,\mathbf{grad}\,\mathcal{E}^{\text{grad}} u(0,y,x+z) + \mathcal{E}^{\text{grad}} u(0,y,x+z)\,\mathbf{grad}\left(\frac{z}{x+z}\right)$$

$= \ldots$ (technical) $\ldots =$ expression depending only on $\mathbf{grad}_\tau\, u$.

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$$\mathcal{E}^{\text{curl}}_{\text{corr}} \boldsymbol{v} \stackrel{\text{def}}{=} \frac{2z}{x+z}\int_0^1\int_0^{1-t} \begin{pmatrix} s & t \\ 0 & 1 \\ s & t \end{pmatrix} \boldsymbol{v}(s(x+z), y+t(x+z))\, ds\, dt$$

$$+ \frac{1}{x+z}\begin{pmatrix} -z \\ 0 \\ x \end{pmatrix}\int_0^1\int_0^{1-t} \begin{pmatrix} 1-s \\ -t \end{pmatrix} \cdot \boldsymbol{v}(s(x+z), y+t(x+z))\, ds\, dt.$$

# Main result

THEOREM. *By the above described techniques we can construct* $\boldsymbol{\mathcal{E}}_K^{\mathrm{curl}} : \boldsymbol{X}^{-1/2}(\partial K) \mapsto \boldsymbol{H}(\mathbf{curl})$ *satisfying:*

1. Continuity: $\boldsymbol{\mathcal{E}}_K^{\mathrm{curl}}$ *is continuous.*
2. Commutativity: $\mathbf{grad}(\mathcal{E}_K^{\mathrm{grad}} u) = \boldsymbol{\mathcal{E}}_K^{\mathrm{curl}}(\mathbf{grad}_\tau\, u)$ *for all* $u$ *in* $H^{1/2}(\partial K)$.
3. Extension property: *The tangential trace* $\mathrm{trc}_\tau(\boldsymbol{\mathcal{E}}_K^{\mathrm{curl}} \boldsymbol{v})$ *coincides with* $\boldsymbol{v}$ *for all* $\boldsymbol{v}$ *in* $\boldsymbol{X}^{-1/2}$.
4. Full polynomial preservation: *If* $\boldsymbol{v}$ *is the tangential trace of a function in* $\boldsymbol{P}_p(K)$, *then* $\boldsymbol{\mathcal{E}}_K^{\mathrm{curl}} \boldsymbol{v}$ *is in* $\boldsymbol{P}_p(K)$.
5. Nédélec polynomial preservation: *If* $\boldsymbol{v}$ *is the tangential trace of a function in* $\boldsymbol{N}_p(K)$, *then* $\boldsymbol{\mathcal{E}}_K^{\mathrm{curl}} \boldsymbol{v}$ *is in* $\boldsymbol{N}_p(K)$.