New DPG techniques for designing numerical schemes

Jay Gopalakrishnan

University of Florida

Collaborator: Leszek Demkowicz

October 2009

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Thanks: NSF

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The philosophy

The new:

- Discontinuous Petrov-Galerkin (DPG) methods
- Remarkable stability (through natural test space design)

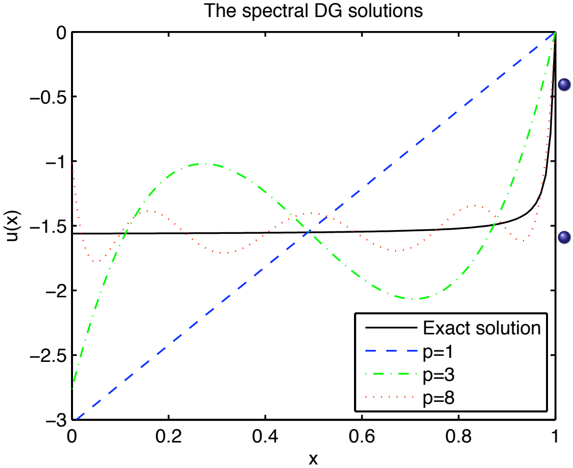
The old:

- DG methods (upwind stabilization, or stability by penalty parameters)
- SUPG methods (stability through artificial streamline diffusion)

Outline

1. How does the new compare to the old?
   - Sample comparisons between DPG and DG results.
2. Elements of design of schemes.
   - The example of simple 1D transport equation.
3. DPG method for the transport equation.
   - Extension of the 1D idea to 2D.
   - The spectral DPG method.
   - The composite DPG method on a mesh.
4. Extensions.
   - The DPG-X method.
   - Optimal test functions.
   - *hp*-results.
   - A method for all seasons?

Comparison: 1D, 1 element case



- Experiment: Solve 1D transport equation using DG and DPG on one element.
- Exact solution has a sharp layer at $x = 1$.

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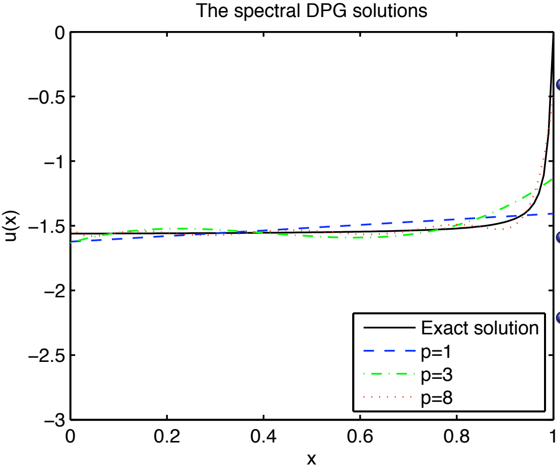
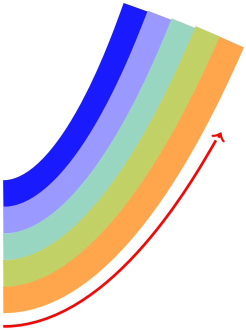
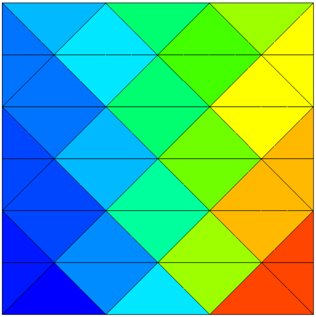


- Experiment: Solve 1D transport equation using DG and DPG on one element.
- Exact solution has a sharp layer at $x = 1$.
- *DPG solutions oscillate an order of magnitude less.*

Comparison: Crosswind diffusion



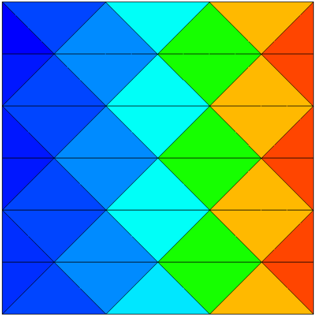
Pure transport should not diffuse materials crosswind.



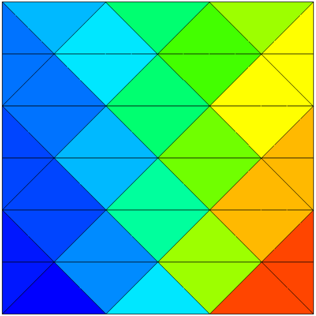
But most numerical methods do.

Experiment: Use DG and DPG for simulating vertically upward transport of linearly varying density from the bottom of the unit square.

Comparison: Crosswind diffusion



DPG doesn't.



DG has crosswind diffusion.

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Comparison: Convergence rates

Brief history of finite element methods for stationary transport:

- [Reed & Hill 1973]: First DG method proposed

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- DPG: 1^{st} method with provably optimal h and p rates.

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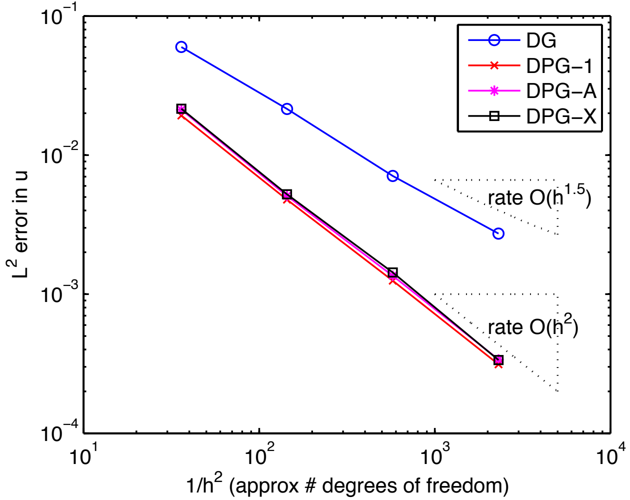

Experiment: Apply DG and three different DPG methods (with $p = 1$) to Peterson's transport example.

Next

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"Petrov-Galerkin" schemes

Petrov-Galerkin schemes are distinguished by different trial and test spaces.

The problem: [P.D.E.+ boundary conditions.

↓

Variational form: [Find u in a trial space satisfying

$$b(u, v) = l(v)$$

for all v in a test space.

↓

Discretization: [Find u_n in a discrete trial space X_n satisfying

$$b(u_n, v_n) = l(v_n)$$

for all v_n in a discrete test space V_n.

Petrov-Galerkin schemes have $X_n \neq V_n$.

Designing a simple PG scheme

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Example: A simple continuous Petrov-Galerkin (CPG) scheme

1D transport eq. $\left[\begin{array}{ll} u' = f & \text{in } (0,1), \\ u(0) = u_0 & \text{(inflow b.c.)} \end{array}\right.$

Variational form: $\left[\begin{array}{l} \text{Find } u \text{ in } H^1, \text{ satisfying } u(0) = u_0, \& \\ \underbrace{\int_0^1 u' v}_{b(u,v)} = \underbrace{\int_0^1 f v}_{l(v)}, \qquad \text{for all } v \text{ in } L^2. \end{array}\right.$

Spectral method: $\left[\begin{array}{l} \text{Find } u_p \in P_p, \text{ satisfying } u_p(0) = u_0, \& \\ b(u_p, v) = l(v), \qquad \forall v \in P_{p-1}. \end{array}\right.$

u_p : trial fn.
v : test fn.

(Notation: P_p = set of polynomials of degree at most p.)

Babuška's theorem

Let $u \in X$ and $u_n \in X_n \equiv$ trial space be exact and approximate solutions,

$$b(u - u_n, v_n) = 0 \qquad \forall v_n \in V_n \equiv \text{test space},$$

and $b(\cdot, \cdot)$ be bounded in $X \times V_n$.

Theorem (A simple version of Babuška's theorem)

If

$$C_1 \|w_n\|_X \le \sup_{v_n \in V_n} \frac{b(w_n, v_n)}{\|v_n\|_{V_n}} \qquad \forall w_n \in X_n,$$

then

$$\|u - u_n\|_X \le C_2 \inf_{w_n \in X_n} \|u - w_n\|_X.$$

Guiding principle: *While we must choose trial spaces with good approximation properties, we may design test spaces solely to obtain good stability properties.*

Choice of spaces

Example: The 1D spectral CPG scheme (contd.)

Spectral method: $\left[\begin{array}{c} \text{Find } u_p \in P_p, \text{ satisfying } u_p(0) = u_0, \& \\ b(u_p, v) = l(v), \qquad \forall v \in P_{p-1}. \end{array}\right.$ $\left| \begin{array}{l} u_p : \text{trial fn.} \\ v : \text{test fn.} \end{array}\right.$

Q: Why the choice of spaces P_p and P_{p-1}?
A:

- Since $b(u,v) = \int_0^1 u'\, v$, the fraction $\dfrac{b(u,v)}{\|v\|_{L^2}}$ is maximized by $v = u'$, which we call the the optimal test function for the given u.
- If u is in P_p, then $v = u'$ is in P_{p-1}.
- Babuška's theorem $\implies$ stability, for these choice of spaces.

What is DPG?

- DPG schemes (Discontinuous Petrov-Galerkin schemes) uses nonequal DG spaces (no interlement continuity) for trial and test spaces.

- The name "DPG" was used previously for methods with DG test spaces augmented with bubbles etc:
  - [Bottasso, Micheletti & Sacco 2002]: DPG for elliptic problems
  - [Bottasso, Micheletti & Sacco 2005]: Multiscale DPG
  - [Causin, Sacco & Bottasso, 2005]: DPG for advection diffusion.
  - [Causin & Sacco 2005]: Hybridized DPG for Laplace's equation

- The DPG methods of this talk *differs* from the above works in our approach to the test space design.

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Example: DPG for 1D transport equation

1D transport eq. $\begin{cases} u' = f & \text{in } (0,1), \\ u(0) = u_0 & \text{(inflow b.c.)} \end{cases}$

L^2 variational form: $\begin{cases} \text{Find } u \in L^2, \text{ and a number } \hat{u}_1, \text{ satisfying} \\ \underbrace{-\int_0^1 uv' + \hat{u}_1 v(1)}_{b(\,(u,\hat{u}_1),\,v)} = \underbrace{\int_0^1 fv + u_0 v(0)}_{l(v)}, \qquad \forall v \in H^1. \end{cases}$

Spectral method: $\begin{cases} \text{Find } u_p \in P_p, \text{ and a number } \hat{u}_1 \text{ satisfying} \\ b(\,(u_p, \hat{u}_1), v) = l(v), \qquad \forall v \in P_{p+1}. \end{cases}$

This leads to a stable discontinuous Petrov-Galerkin (DPG) scheme.

Q: Why did we set the trial space to P_{p+1}?

A simple DPG method

Q: Why is the trial space P_{p+1}?

$$b((u_p, \hat{u}_1), v) = -\int_0^1 u_p v' + \hat{u}_1 v(1)$$

A: Because inf-sup condition is then satisfied.

In more detail:

A simple DPG method

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In more detail:

- Choose a test space norm, say $\|v\|_V^2 = \|v'\|_{L^2}^2 + |v(1)|^2$.

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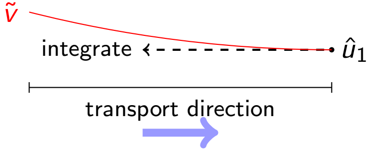
- Then, $\displaystyle\sup_{v\in H^1} \frac{b(\ (u_p, \hat{u}_1),\ v)}{\|v\|_V}$ is attained by $\displaystyle \tilde{v} = \hat{u}_1 + \int_x^1 u_p(s)\, ds.$



- This maximizing $\tilde{v}$ is the optimal test function.

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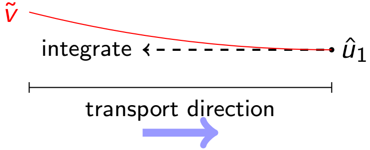
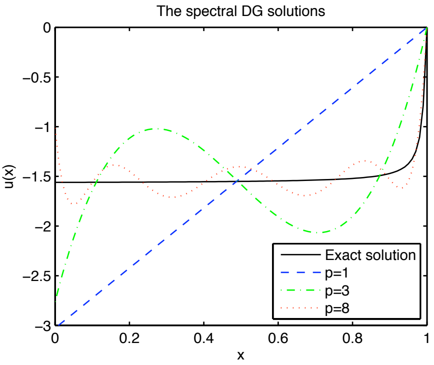


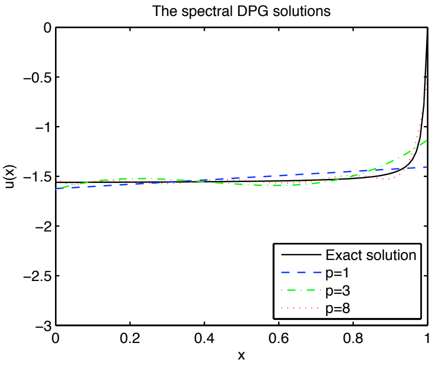
- This maximizing $\tilde{v}$ is the optimal test function.
- If $u_p \in P_p$, then $\tilde{v}$ is in P_{p+1}. Hence our trial space choice.

What have we gained?

Even in the simplest 1D 1-element case, we see that DPG makes a difference. Recall the initial results:



DG



DPG

DPG exhibits enhanced stability.

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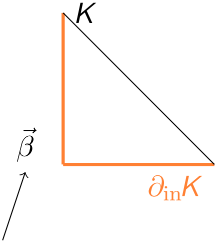
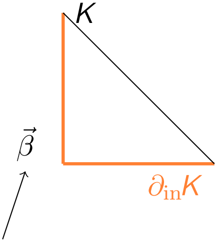


The 2D transport equation on one element K:

$$\left[\begin{array}{ll} \vec{\beta}\cdot\vec{\nabla}\, u = f & \text{on } K, \\ u = g & \text{on } \partial_{\text{in}}K \text{ (inflow boundary).} \end{array}\right.$$

$$\Longrightarrow \quad -\int_K u\,\vec{\beta}\cdot\vec{\nabla}\, v + \int_{\partial_{\text{out}}K} \vec{\beta}\cdot\vec{n} u v + \int_{\partial_{\text{in}}K} \vec{\beta}\cdot\vec{n} u v \quad = \int_K f\, v$$



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$$\implies \quad -\int_K u\, \vec\beta \cdot \vec\nabla\, v + \int_{\partial_{\rm out} K} \phi \qquad v + \int_{\partial_{\rm in} K} \vec\beta \cdot \vec n g\, v \quad = \int_K f\, v$$

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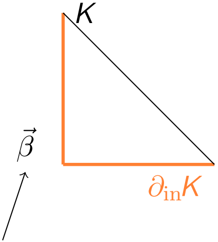

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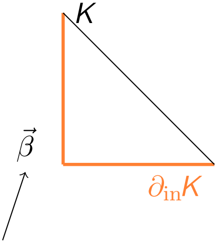
$$\Longrightarrow \quad \underbrace{-\int_K u\, \vec\beta \cdot \vec\nabla v + \int_{\partial_{\rm out} K} \phi \quad v}_{b(\,(u,\phi),\, v)} + \int_{\partial_{\rm in} K} \vec\beta \cdot \vec n g\, v \; = \int_K f\, v$$

The 2D, one element case



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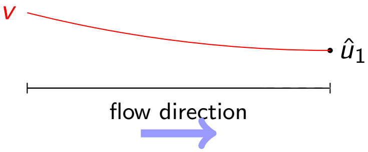
Variational formulation

Find <u>solution</u> $u \in L^2(K)$ and "<u>outflux</u>" $\phi \in L^2(\partial_{\mathrm{out}} K)$ satisfying

$$b(\,(u,\phi), v) = l(v), \qquad \text{for all } v \in L^2(K) \text{ with } \vec{\beta}\cdot\vec{\nabla}\, v \in L^2(K).$$

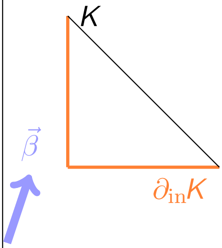
How to construct 2D test space?

1D case: $$v = \hat{u}_1 + \int_x^1 u(s)\, ds$$



The optimal test function v

2D case:



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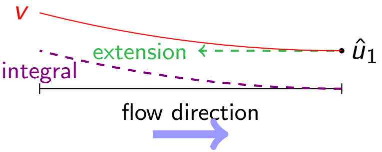
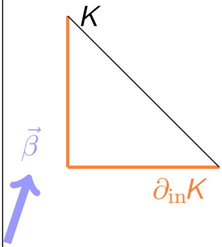


v = extension + integral

$= \mathcal{E}_{\text{out}}(\hat{u}_1)$ + higher degree

2D case:



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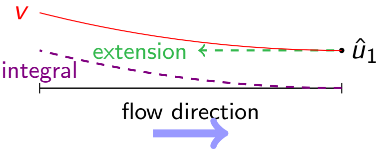

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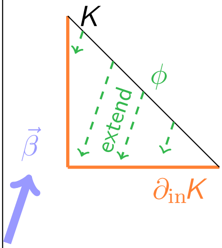
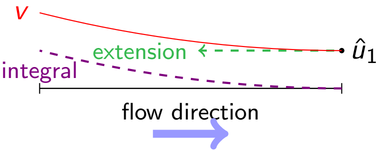


$v = \mathcal{E}_{\text{out}}(\phi)$ + higher degree in $\vec{\beta}$-direction

- $\mathcal{E}_{\text{out}}$ extends from outflow boundary, constantly along streamlines.

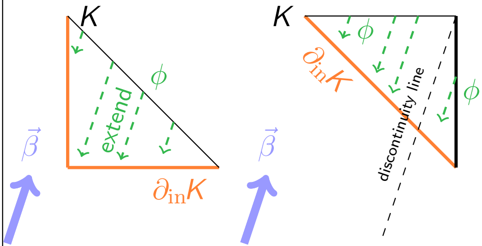
How to construct 2D test space?

1D case: $v = \hat{u}_1 + \int_x^1 u(s)\,ds$



$$v = \text{extension} + \text{integral}$$
$$= \mathcal{E}_{\text{out}}(\hat{u}_1) + \text{higher degree}$$

2D case:



$$v = \mathcal{E}_{\text{out}}(\phi) + \text{higher degree in } \vec{\beta}\text{-direction}$$

- $\mathcal{E}_{\text{out}}$ extends from outflow boundary, constantly along streamlines.
- Even if ϕ is polynomial on each edge, $\mathcal{E}_{\text{out}}(\phi)$ need not be! $\mathcal{E}_{\text{out}}(\phi)$ can be discontinuous inside K.

A new test space

The new finite element that forms the test space is composed of:

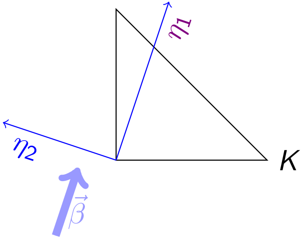
$$K = \text{interval/triangle/tetrahedron}, \qquad \text{(geometry)},$$
$$V_p(K) = \mathcal{E}_{\text{out}}(M_{p+1}(\partial_{\text{out}}K)) \oplus \eta_1 P_p(K) \qquad \text{(space)},$$
$$\Sigma = \text{the following set of moments:} \qquad \text{(degrees of freedom)},$$

$$\left[\begin{array}{ll} \displaystyle\int_K (\vec{\beta}\cdot\vec{\nabla} v) q & \text{for all } q \in P_p(K), \\ \displaystyle\int_F v\mu & \text{for all } \mu \in P_{p+1}(F) \text{ for all faces of } K. \end{array} \right.$$

Possible to implement with standard finite element technology.

Note:

- $M_{p+1}(\partial_{\text{out}}K)$ = set of functions that are polynomials of degree $\leq p+1$ on each edge of $\partial_{\text{out}}K$.
- η_1 = streamline coordinate.



The 2D spectral method

Trial space $= X_p(K) = \quad P_p(K) \times M_{p+1}(\partial_{\text{out}} K)$,

- solution u approximated in $P_p(K)$,
- outflux ϕ approximated in $M_{p+1}(\partial_{\text{out}} K)$.

Test space $= V_p(K)$, introduced in the previous slide.

The spectral method on one element

Find $(u_p, \phi_{p+1}) \in X_p(K)$ satisfying

$$-\int_K u_p \vec{\beta} \cdot \vec{\nabla}\, v + \int_{\partial_{\text{out}} K} \phi_{p+1} v = \int_K f v - \int_{\partial_{\text{in}} K} \vec{\beta} \cdot \vec{n}\, g, v,$$

for all $v \in V_p(K)$.

Theorem

The solution of the method (both u_p and ϕ_{p+1}) coincides with the (L^2) best possible approximations of the exact solution in the trial space.

The composite DPG method

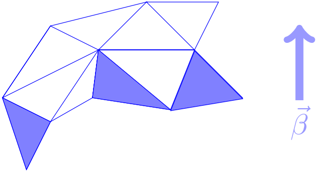
On a mesh of triangles, construct the composite method as follows:

- On each triangle K, set test and trial space to $X_p(K)$ and $V_p(K)$ (no interelement continuity).
- Elements are coupled through single-valued outflux ϕ_h in

$$M_h = \{\mu : \mu|_E \in P_{p+1}(E) \text{ for all mesh edges } E \text{ not on } \partial_{\text{in}}\Omega\},$$

The DPG-1 method

$$\sum_K \left(-\int_K u_h \vec{\beta} \cdot \vec{\nabla}\, v_h + \int_{\partial_{\text{out}} K} \phi_h v_h - \int_{\partial_{\text{in}} K \setminus \partial_{\text{in}}\Omega} \phi_h v_h \right) = \int_\Omega f\, v_h - \int_{\partial_{\text{in}}\Omega} \vec{\beta} \cdot \vec{n}\, g v_h.$$



We can solve the system by marching from the inflow boundary.

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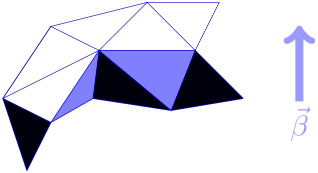

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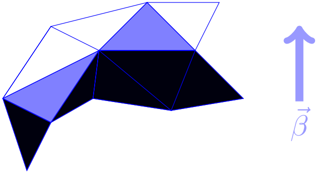

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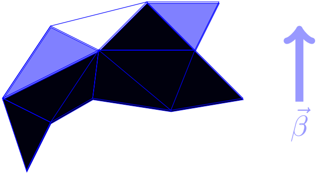

We can solve the system by marching from the inflow boundary.

Discretization errors of DPG-1

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Theorem (Optimal error estimates)

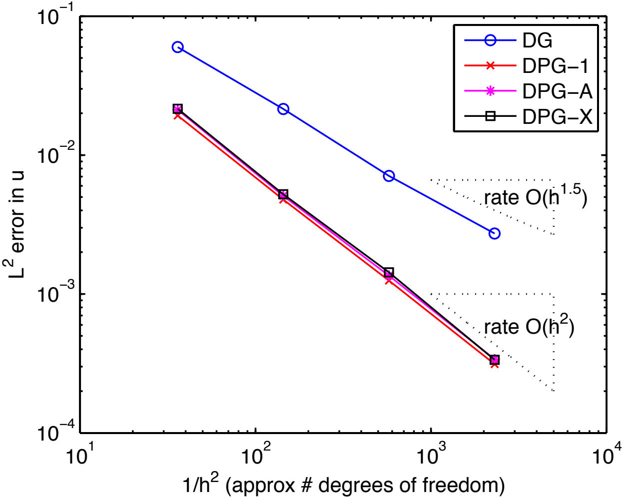
There is a constant C independent of h and p such that

$$\|u - u_h\|_{L^2(\Omega)} \leq C \frac{h^s}{p^s} \|u\|_{H^{s+1}(\Omega)}$$

for all $0 \leq s \leq p + 1$.

- This is the first known error estimate (for any FEM) for the transport equation that is optimal in h and p on general meshes.
- Yet, our techniques of proof need improvement:
  - We did not obtain estimates with the usual *regularity* assumption on u.
  - We could prove only suboptimal estimates for ϕ_h (although all our numerical experiments indicate that ϕ_h converges optimally).

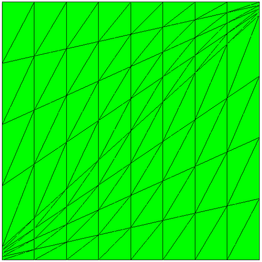
Recall Peterson's example



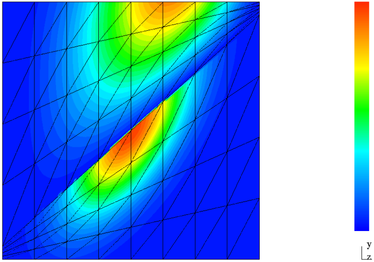
Experiment: Apply DG and three different DPG methods (with $p = 1$) to Peterson's transport example.

Example with a discontinuous solution

- We consider an example of [Houston, Schwab & Süli 2000]. (They used it to show that DG methods work better than SUPG in the presence of shock-like discontinuities when mesh is aligned with shocks.)



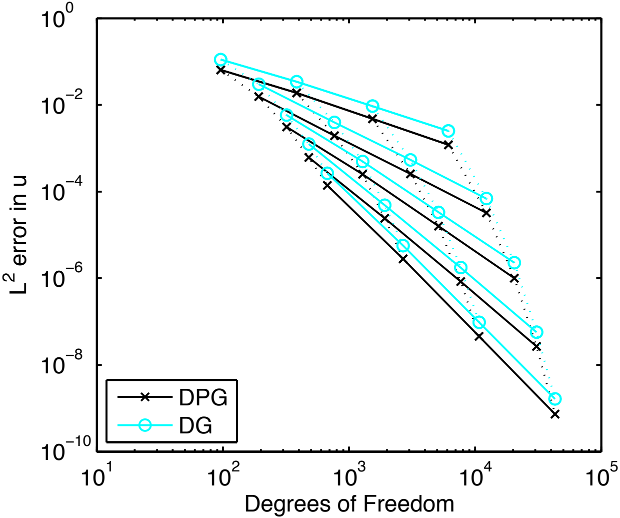
Mesh



Exact solution

- Experiment: Compare DPG and DG applied to this example.

Results:



DPG outperforms DG.

Solid lines indicate *h*-refinement.

Dotted lines indicate *p*-refinement.

hp optimal convergence rates are observed.

Next

1. How does the new compare to the old?
   - Sample comparisons between DPG and DG results.
2. Elements of design of schemes.
   - The example of simple 1D transport equation.
3. DPG method for the transport equation.
   - Extension of the 1D idea to 2D.
   - The spectral DPG method.
   - The composite DPG method on a mesh.

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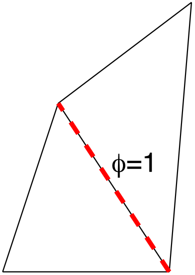
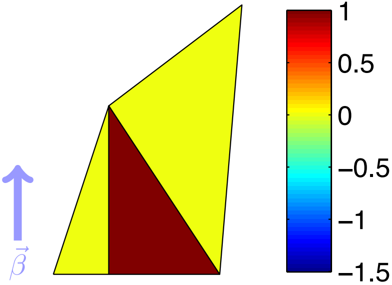
4. Extensions.
   - The DPG-X method.
   - Optimal test functions.
   - A method for all seasons?

The optimal test functions in 2D

- We constructed the test functions of the DPG-1 method heuristically (by simply generalizing the form of the optimal expression in 1D).



An outflux trial function on an edge.

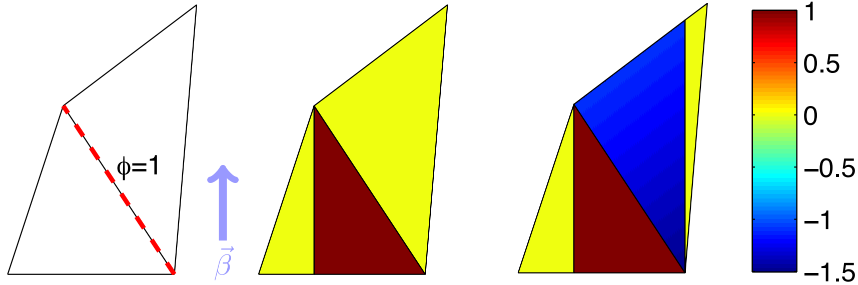


The corresponding test function $\mathcal{E}_{\text{out}}(\phi)$ in DPG-1.

The optimal test functions in 2D

- We constructed the test functions of the DPG-1 method heuristically (by simply generalizing the form of the optimal expression in 1D).

- But, they turn out to be *not* the optimal test functions in 2D ...



An outflux trial function on an edge.

The corresponding test function $\mathcal{E}_{\rm out}(\phi)$ in DPG-1.

The actual optimal test function.

Calculating the optimal test function

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Recall the variational formulation for the transport equation:

$$\underbrace{\sum_K \left(-\int_K u\vec{\beta}\cdot\vec{\nabla}\, v + \int_{\partial_{\mathrm{out}}K} \phi v - \int_{\partial_{\mathrm{in}}K\setminus\partial_{\mathrm{in}}\Omega} \phi v \right)}_{b(\,(u,\phi),\ v)} = \int_\Omega fv - \int_{\partial_{\mathrm{in}}\Omega} \vec{\beta}\cdot\vec{n}\, g v.$$

To maximize $\dfrac{b(\,(u,\phi),\ v)}{\|v\|_V}$,

- first set $\|\cdot\|_V$-norm by $\quad \|v\|_V^2 = \displaystyle\sum_K \left(\int_K |\vec{\beta}\cdot\vec{\nabla}\, v|^2 + \int_{\partial_{\mathrm{out}}K} |v|^2 \right),$

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- and then solve a local problem for the optimal test function v:

$$\text{Find } v: \qquad (v,\delta_v)_V = b(\,(u,\phi),\ \delta_v), \qquad \forall\ \delta_v.$$

Calculating the optimal test function

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$$\underbrace{\sum_K \left(-\int_K u\vec{\beta}\cdot\vec{\nabla}\, v + \int_{\partial_{\text{out}}K} \phi v - \int_{\partial_{\text{in}}K\setminus\partial_{\text{in}}\Omega} \phi v \right)}_{b(\,(u,\phi),\; v)} = \int_\Omega fv - \int_{\partial_{\text{in}}\Omega} \vec{\beta}\cdot\vec{n}\, gv.$$

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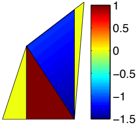
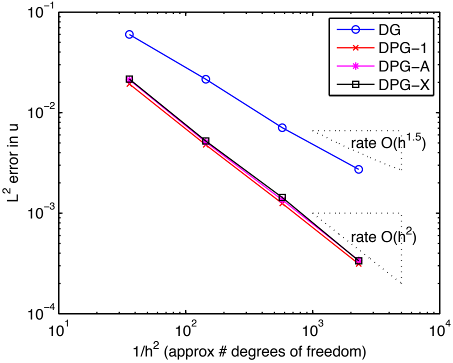
- The hand-calculated solution with $u = 0$,

  and $\phi =$indicator function of an edge, was shown on the previous slide:



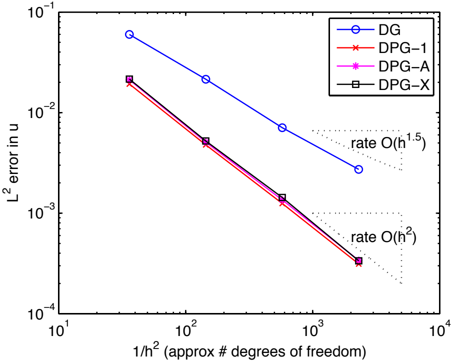
- The use of the exactly optimal test functions leads to a new method, which we call the DPG-X method.
- Its performance is comparable to DPG-1 method.



DG & DPG on Peterson's mesh

DPG-X

- The use of the exactly optimal test functions leads to a new method, which we call the DPG-X method.
- Its performance is comparable to DPG-1 method.
- While DPG-1 can be solved by marching from the inflow, DPG-X requires the solution of a *symmetric positive definite* system!



DG & DPG on Peterson's mesh

- For any bilinear form $b(u, v)$ in the DPG setting, the optimal test functions can be *locally* computed:

$$v_i = Tu_i : \qquad (Tu_i, \delta_v)_V = b(u_i, \delta_v), \qquad \forall\, \delta_v.$$

- This idea is not restricted to the transport equation. Methods now immediately generalize to
  - variable $\vec{\beta}$,
  - convection-diffusion,
  - and all other problems which can be formulated in DPG form!

  We only need to approximate the optimal test function problem.

- Stiffness matrix is symmetric (even for the pure transport problem).

$$\begin{aligned} B_{ij} = b(u_j, v_i) = (Tu_j, v_i)_V &= (Tu_j, Tu_i)_V \\ = (Tu_i, Tu_j)_V = b(v_i, u_j) &= B_{ji}. \end{aligned}$$

Stability

- The method is of least squares type. The novelty is in the potential for local computation of optimal test functions.
- With the optimal test space, inf-sup condition is obvious in the norm

$$\|u\|_E = \sup_{v \in V} \frac{b(u, v)}{\|v\|_V}.$$

- Error estimates follow immediately in $\|\cdot\|_E$.
- It can be a theoretically difficult problem to obtain error estimates in other norms.
- However, *hp*-adaptivity can proceed by estimators in the $\|\cdot\|_E$-norm.
- All our numerical experiments show extraordinary stability with h and p variations.

Conclusions

- We presented a DPG method for transport equation.
- The DPG method outperforms DG in computations.
- We proved optimal theoretical convergence estimates.
- The concept of optimal test functions leads to a new paradigm in designing numerical schemes. Methods are waiting to be discovered.

Full manuscripts:

1. L. Demkowicz and J. Gopalakrishnan, *A class of discontinuous Petrov-Galerkin methods. Part I: The transport equation*, Submitted, (2009).
2. L. Demkowicz and J. Gopalakrishnan, *A class of discontinuous Petrov-Galerkin methods. Part II: Optimal test functions*, Submitted, (2009).

☞Preprints available online.