Can the CG method yield conservative fluxes?

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Collaborators: B. Cockburn, H. Wang

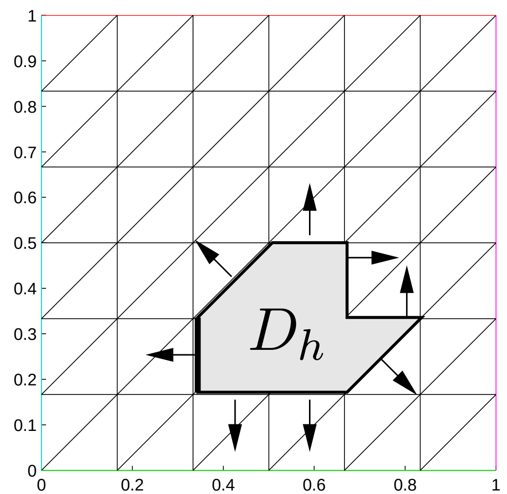
Thanks: NSF

Conservation

We say that a discrete flux $\boldsymbol{q}_h$ approximating the exact flux $\boldsymbol{q}$ is conservative if the total *outward flux* as measured by $\boldsymbol{q}$ and $\boldsymbol{q}_h$ coincides, i.e.,

$$\int_{\partial D_h} \boldsymbol{q} \cdot \boldsymbol{n} \, ds = \int_{\partial D_h} \boldsymbol{q}_h \cdot \boldsymbol{n} \, ds,$$

for any subdomain D_h formed by the union of some mesh elements.

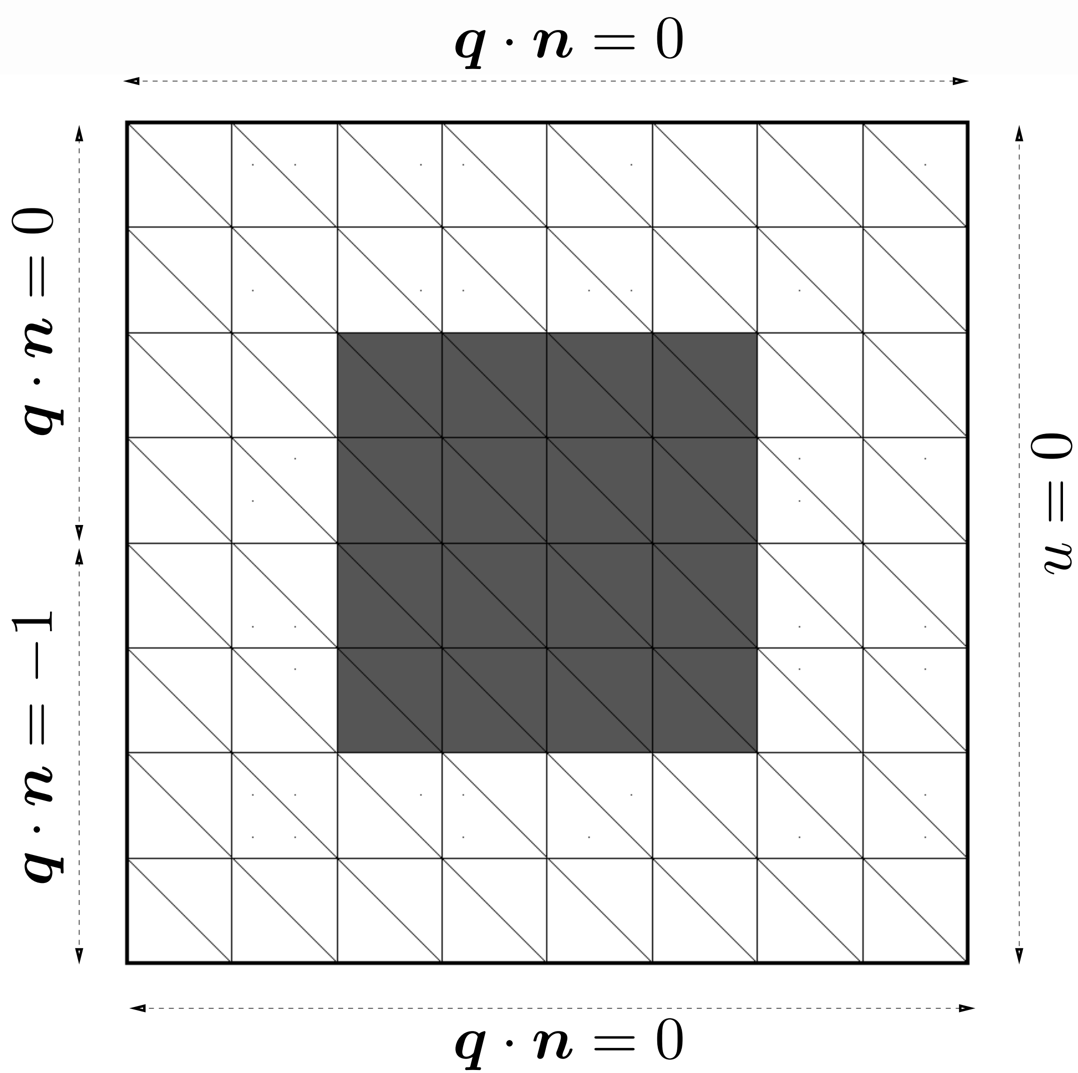


Traditional comparisons:

	CG vs. RT	CG vs. DG
Conservation :	✗ ✓	✗ ✓

(RT = Raviart-Thomas mixed method, DG = a discontinuous Galerkin method like LDG.)

Example



$\Omega=(0,1)^2$

$$a=\begin{cases}10^{-3}, & \text{in shaded area}\\ 1, & \text{elsewhere.}\end{cases}$$

(A simple model of a steady state porous media flow around an impermeable rock.)

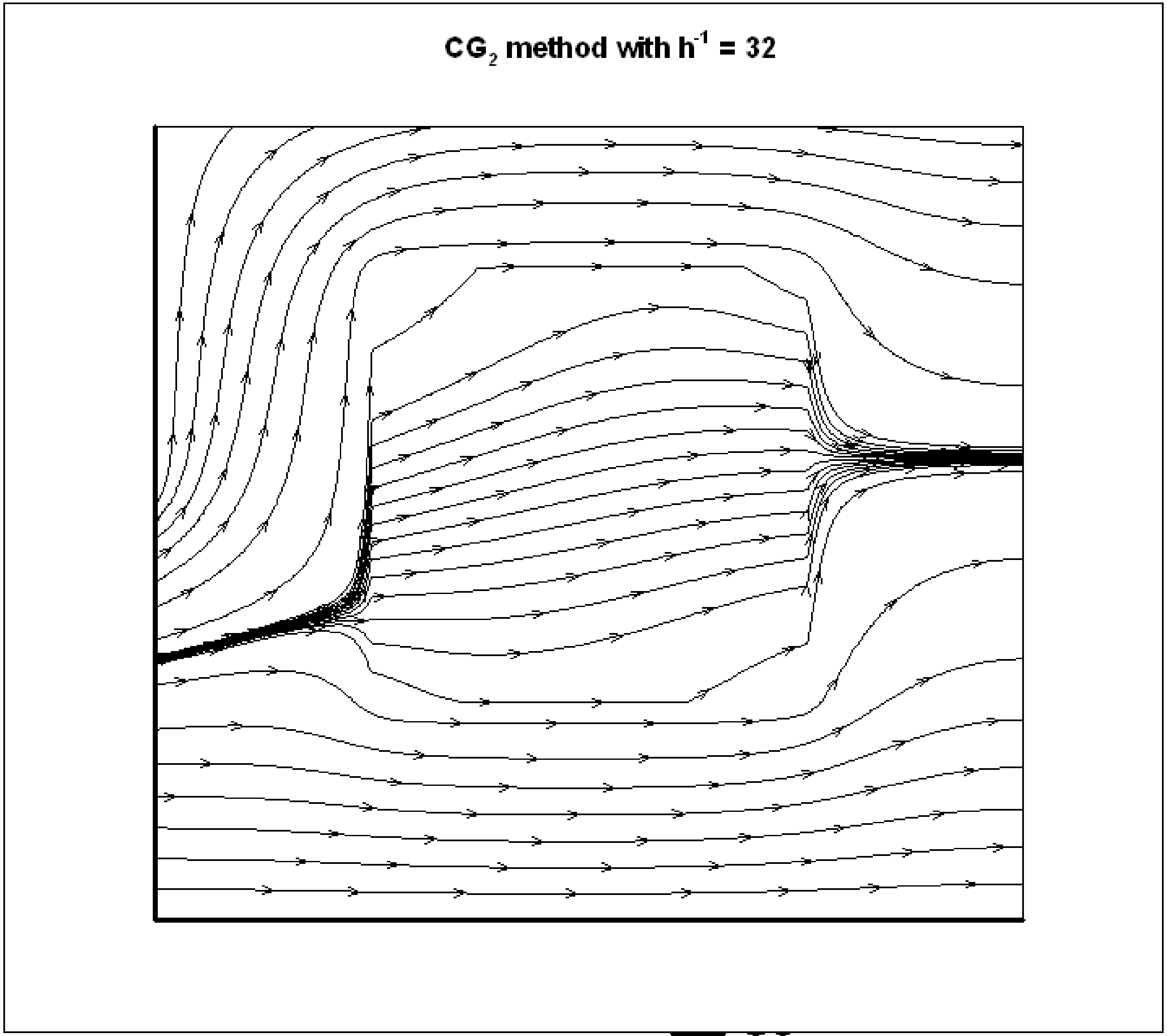
Find flux $\boldsymbol{q}$ and u :

$$\boldsymbol{q}+a\,\boldsymbol{\nabla}\,u=0, \quad \text{on } \Omega,$$

$$\operatorname{div}\,\boldsymbol{q}=f, \quad \text{on } \Omega,$$

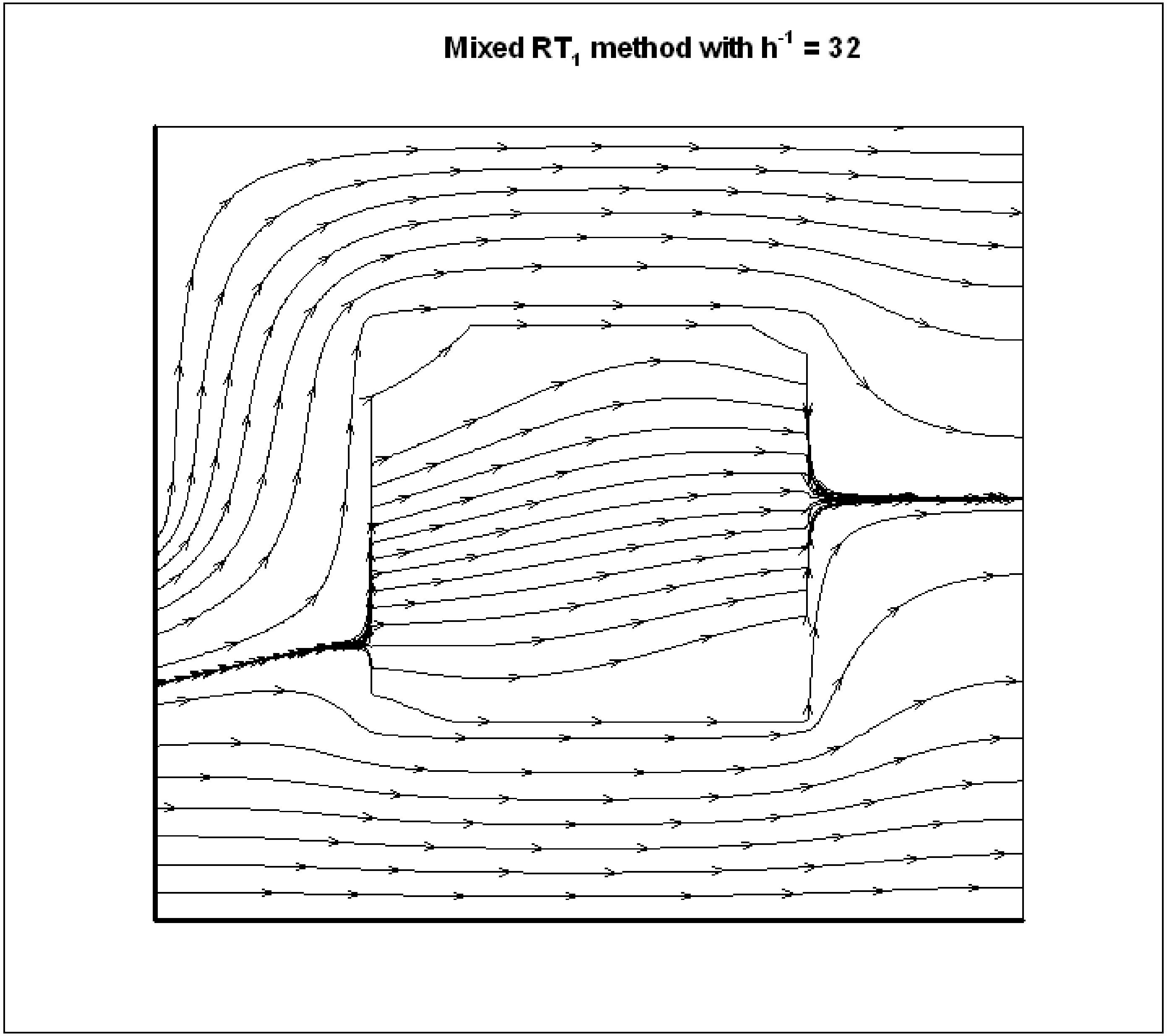
$$u=0, \quad \text{on } \partial\Omega.$$

Example



(CG flux: $-a \nabla U_h$)

Example



(RT flux: $\boldsymbol{q}_h$)

Background

- [Bastian & Riviere, 2003] (local postprocessing for DG solutions)
- [Hughes & Wells, 2005]
  [Hughes, Engel, Mazzei & Larson, 2000] (fluxes with a conservation property)
  [Brezzi, Hughes & Suli, 2001]
- [Carey, 2002] (superconvergent flux postprocessing formula)
  [Pehlivanov, Lazarov, Carey & Chow, 1992]
  [Chow, Carey & Lazarov, 1991]
- [J. Wheeler, 1973] (superconvergent fluxes in 1-D)
  [M. Wheeler, 1974]
  [Douglas, Dupont & Wheeler, 1974]

Why is RT conservative?

From the second equation of the RT method

$$\begin{aligned}(a^{-1}\boldsymbol{q}_h, \boldsymbol{v}) - (u_h, \operatorname{div} \boldsymbol{v}) &= 0,\\ (w, \operatorname{div} \boldsymbol{q}_h) &= (f, w),\end{aligned}$$

setting $w =$ characteristic function of D_h, we find that

$$\begin{aligned}\int_{D_h} \operatorname{div} \boldsymbol{q}_h &= \int_{D_h} \operatorname{div} \boldsymbol{q}\\ \implies \quad \int_{\partial D_h} \boldsymbol{q} \cdot \boldsymbol{n} &= \int_{\partial D_h} \boldsymbol{q}_h \cdot \boldsymbol{n}\end{aligned}$$

For methods like RT and DG which discretize "$\operatorname{div} \boldsymbol{q} = f$" directly, conservation comes easy, but not for CG.

Fluxes from CG method

recall

$$\left.\begin{aligned} &\text{Find } U : \\ -\operatorname{div}(a\,\nabla\,U) = f \quad &\text{on } \Omega, \\ U = 0 \quad &\text{on } \partial\Omega. \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} &\text{Find } U_h \in W_h^{\mathrm{CG}} \text{ satisfying} \\ &(a\,\nabla\,U_h, \nabla\,v) = (f, v), \\ &\text{for all } v \in W_h^{\mathrm{CG}}. \end{aligned}\right.$$

Here W_h^{CG} is the CG finite element space (of continuous functions).

If v is a continuous test function in W_h^{CG}, then

$$0 = (f, v) - (a\nabla U_h, \nabla v)$$

Fluxes from CG method

recall

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On one element K, the CG solution U_h satisfies

$$\langle q_{n,h}^K, v\rangle_{\partial K} = (f, v)_K - (a\nabla U_h, \nabla v)_K$$

with some boundary "flux" (approximating $\boldsymbol{q} \cdot \boldsymbol{n}|_{\partial K}$)

$$q_{n,h}^K \in \{z|_{\partial K} : z \in P_k(K)\}.$$

Fluxes from CG method

recall

Find U :

$$-\operatorname{div}(a \nabla U) = f \quad \text{on } \Omega,$$

$$U = 0 \quad \text{on } \partial\Omega.$$

$\Longrightarrow$

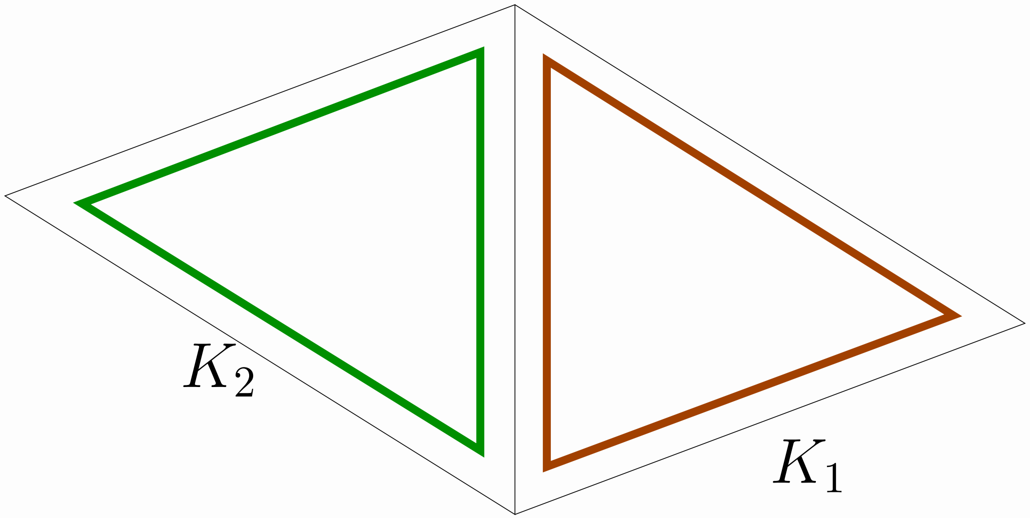
Find $U_h \in W_h^{\mathrm{CG}}$ satisfying

$$(a \nabla U_h, \nabla v) = (f, v),$$

for all $v \in W_h^{\mathrm{CG}}$.

Here W_h^{CG} is the CG finite element space (of continuous functions).

$$\langle q_{n,h}^{K_i}, v \rangle_{\partial K_i} = (f, v)_{K_i} - (a \nabla U_h, \nabla v)_{K_i}$$



In general, $q_{n,h}^{K_1} \neq q_{n,h}^{K_2}$ on the shared edge.

Constructing a good flux

Idea: **?**

- If we could find a $\boldsymbol{q}_h$ such that

$$\underbrace{(a\nabla U_h, \nabla v)_K}_{(-\boldsymbol{q}_h, \nabla v)_K} + \underbrace{\langle q^K_{n,h}, v\rangle_{\partial K}}_{\langle \boldsymbol{q}_h \cdot \boldsymbol{n},\, v\rangle_{\partial K}} = (f, v)_K,$$

then

$$(\mathrm{div}\, \boldsymbol{q}_h, v)_K = (f, v)_K,$$

and conservativity will follow, provided $[\![\boldsymbol{q}_h \cdot \boldsymbol{n}]\!] = 0$.

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Such a $\boldsymbol{q}_h$ can be constructed by a **local postprocessing**:

$$\boldsymbol{q}_h \in \underbrace{\boldsymbol{x} P_{k-1} + \boldsymbol{P}_{k-1}}_{\text{RT space}} \quad : \quad \begin{cases} \langle \boldsymbol{q}_h \cdot \boldsymbol{n}, v\rangle_e = \langle q^K_{n,h}, v\rangle_e, \\ (-\boldsymbol{q}_h, \boldsymbol{r})_K = (a\nabla U_h, \boldsymbol{r})_K, \end{cases}$$

for all $v \in P_{k-1}(e)$ and $\boldsymbol{r}$ in $\boldsymbol{P}_{k-2}(K)$.

Constructing a good flux

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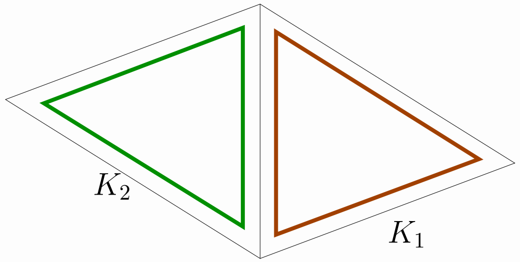
then

$$(\operatorname{div} \boldsymbol{q}_h, v)_K = (f, v)_K,$$

and conservativity will follow, provided $[\![\boldsymbol{q}_h \cdot \boldsymbol{n}]\!] = 0$.

But, $\langle \boldsymbol{q}_h \cdot \boldsymbol{n},\ v\rangle_e = \langle q^K_{n,h}, v\rangle_e \quad \not\Longrightarrow \quad [\![\boldsymbol{q}_h \cdot \boldsymbol{n}]\!]\Big|_e = 0,$



as $q^K_{n,h}$ from adjacent triangles of e do not generally coincide!

Single valued fluxes

Idea: ✗ ?

- Is it possible to construct a single valued flux function $\widehat{\boldsymbol{q}}_h$ on mesh edges such that

$$\langle \widehat{\boldsymbol{q}}_h \,,\ [\![v\boldsymbol{n}]\!] \rangle_{\mathcal{E}_h} = \sum_{K\in\mathfrak{T}_h} \langle q^K_{n,h}, v\rangle_{\partial K}, \qquad \forall v \in W_h^{\mathrm{DG}} \ ?$$

If such a $\widehat{\boldsymbol{q}}_h$ can be found, then we can get a good $\boldsymbol{q}_h$ by the same local postprocessing, but now using $\widehat{\boldsymbol{q}}_h$ in place of the "bad" multivalued $q^K_{n,h}$:

$$\boldsymbol{q}_h \in \boldsymbol{x}P_{k-1} + \boldsymbol{P}_{k-1} \quad : \quad \begin{cases} \langle \boldsymbol{q}_h \cdot \boldsymbol{n}, v\rangle_e = \langle \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, v\rangle_e, \\ (-\boldsymbol{q}_h, \boldsymbol{r})_K = (a\nabla U_h, \boldsymbol{r})_K, \end{cases}$$

for all $v \in P_{k-1}(e)$ and $\boldsymbol{r}$ in $\boldsymbol{P}_{k-2}(K)$.

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$\gg$ Why not choose $\widehat{\boldsymbol{q}}_h$ in the space of "jumps", namely in

$$\mathfrak{J}_h = \{\, [\![v\boldsymbol{n}]\!] : \quad v \in W_h^{\mathrm{DG}} \} \ ?$$

Then obviously we can solve uniquely for $\widehat{\boldsymbol{q}}_h \ldots$

Single valued fluxes

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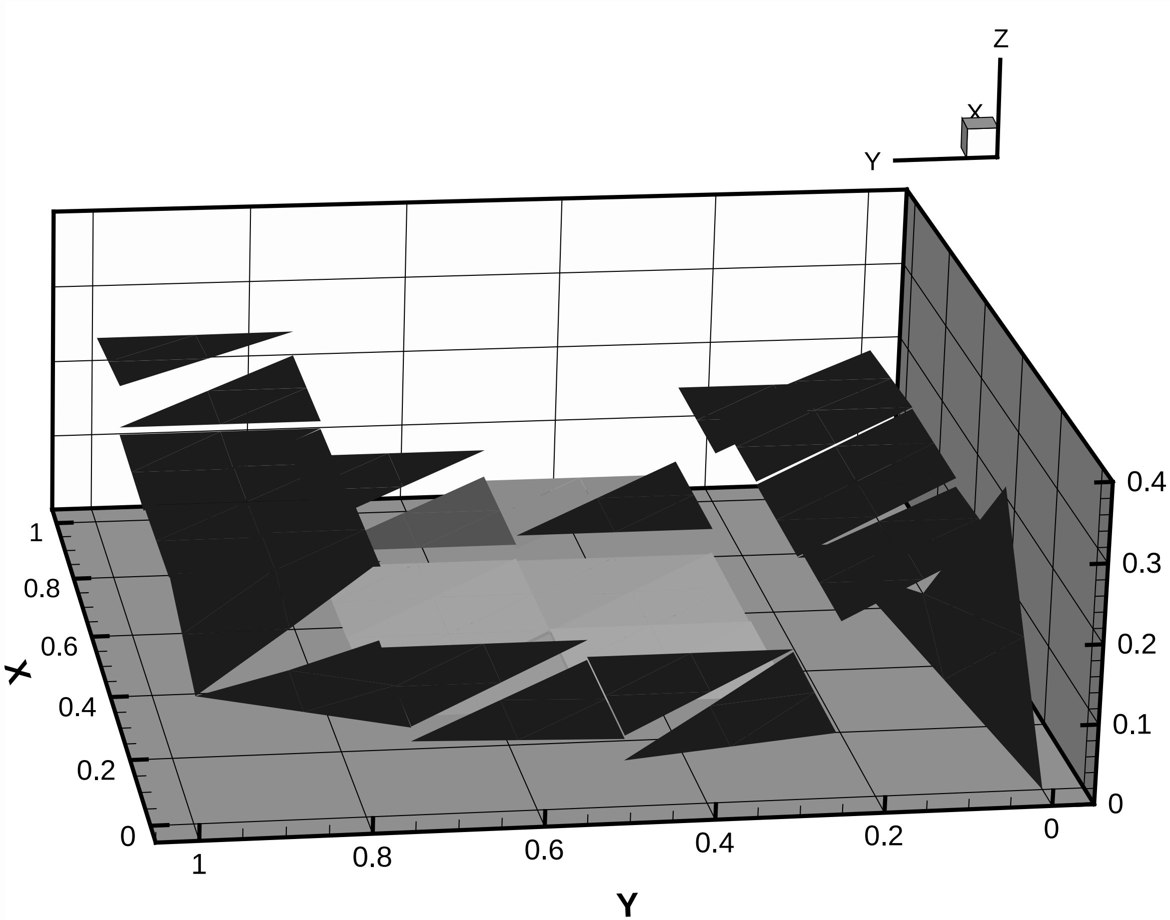
$\gg$ However, when we did that, we got garbage . . .

Boundary layers

If we choose $\widehat{\boldsymbol{q}}_h$ in $\mathfrak{I}_h$ and construct the flux $\boldsymbol{q}_h$ then we observe high errors $\left|\boldsymbol{q} - \boldsymbol{q}_h\right|$ near the boundary, as seen in the following plots. . .

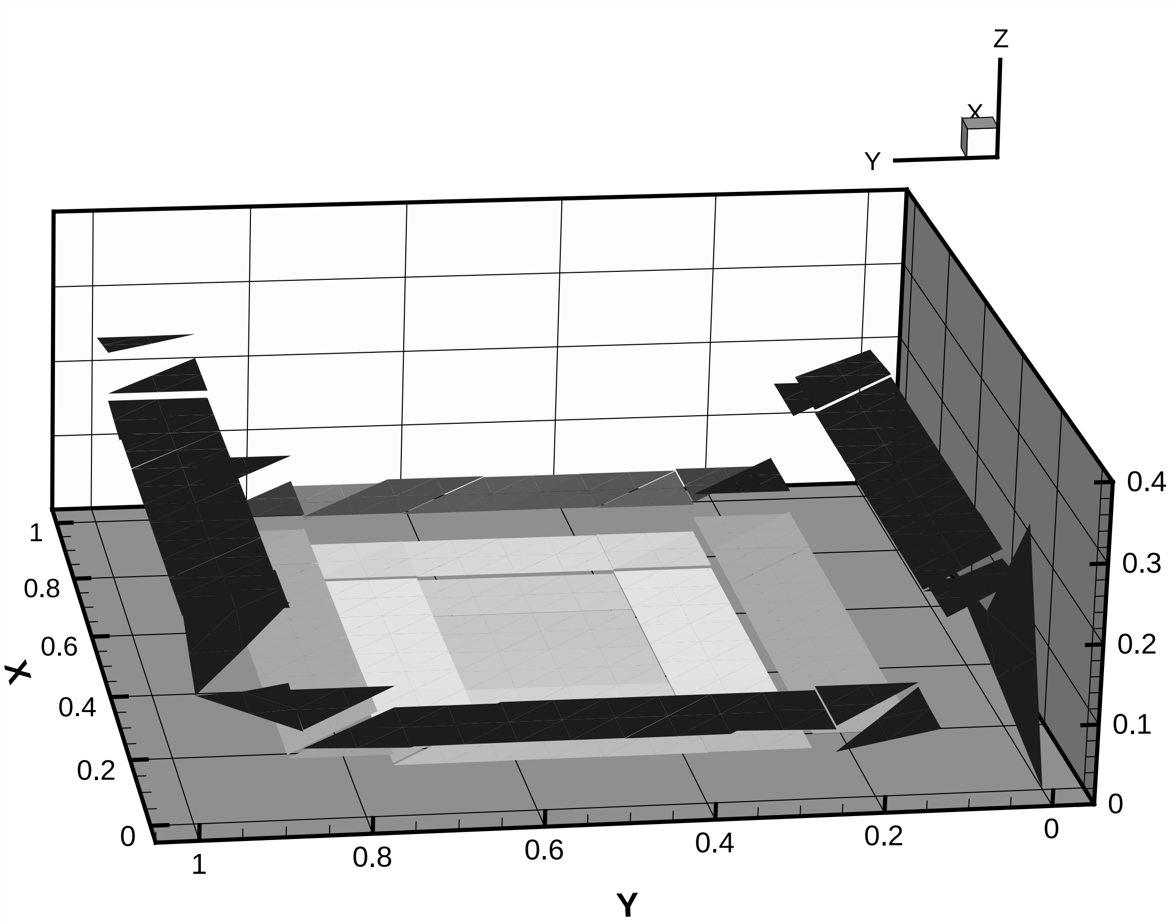
Run parameters are $a = 1$, $f = 0$, $\Omega = (0,1) \times (0,1)$, $\Gamma_D = \{0\} \times (0,1)$, the polynomial degrees are $k = 1$ and $\ell = k - 1$ (for postprocessing), and the boundary conditions are set in such a way that the exact solution is $u(x, y) = 1 + x$.

Boundary layers



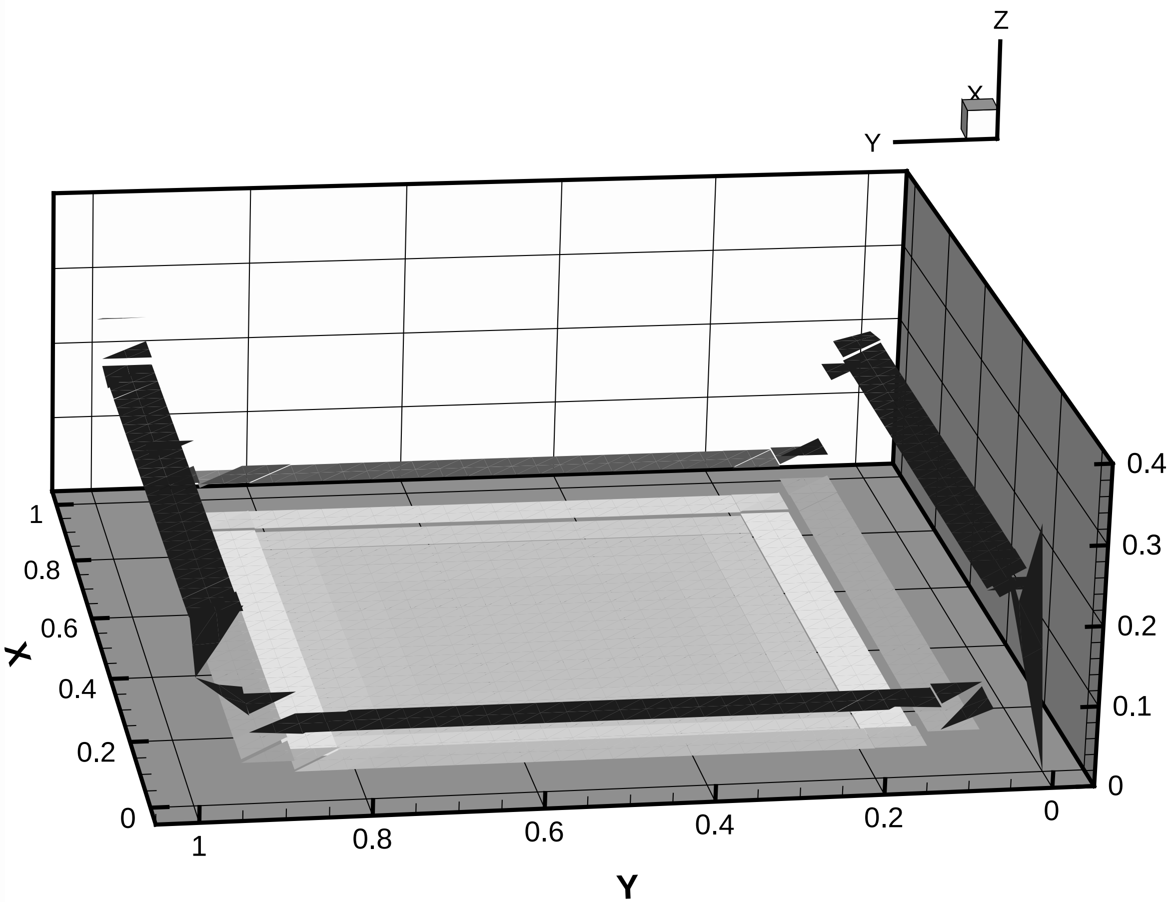
$|\boldsymbol{q} - \boldsymbol{q}_h|$ Level 1

Boundary layers



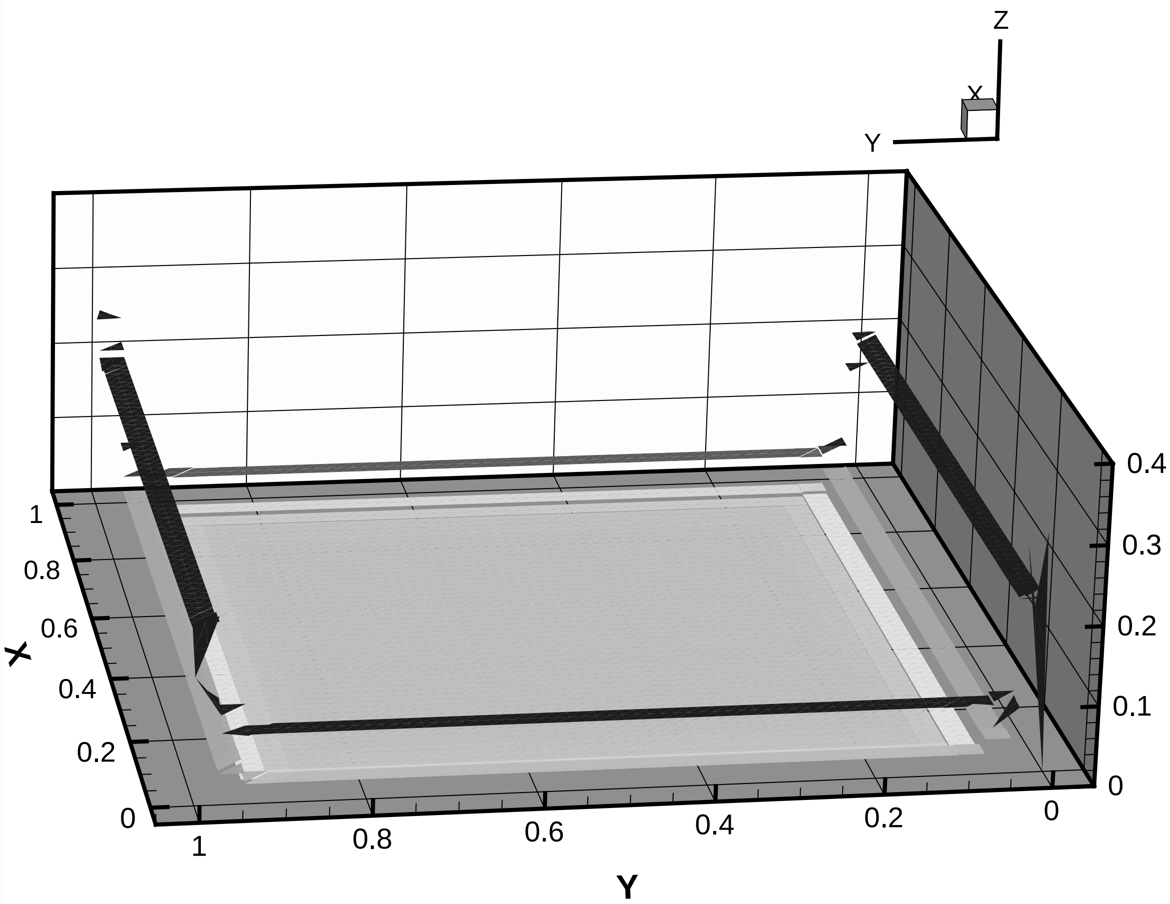
$|\boldsymbol{q} - \boldsymbol{q}_h|$ Level 2

Boundary layers



$$|\boldsymbol{q} - \boldsymbol{q}_h|$$ Level 3

Boundary layers



$|\boldsymbol{q} - \boldsymbol{q}_h|$ Level 4

Boundary layers

We saw that while the error is small far from the boundary, near the boundary it remains of order one. Therefore, we expect to see an order of convergence of $1/2$ in the L^2-norm.

	$k = 1$		$k = 2$		$k = 3$	
h	error	order	error	order	error	order
$1/8$	0.11E+00	0.46	0.62E-01	0.41	0.40E-01	0.46
$1/16$	0.77E-01	0.48	0.45E-01	0.46	0.29E-01	0.48
$1/32$	0.55E-01	0.49	0.32E-01	0.48	0.21E-01	0.49
$1/64$	0.39E-01	0.50	0.23E-01	0.49	0.15E-01	0.50

We believe that such difficulties arise because the space of jumps do not have constant functions. (2nd attempt dashed !#&!)

Fix?

Idea: ✗ ✗ ? (3rd attempt)

Inspired by the form of DG fluxes, we set

$$\widehat{\boldsymbol{q}}_h = \begin{cases} -a\nabla U_h + \alpha\, \boldsymbol{J}_h, & \text{on } \partial\Omega \\ & \text{(on Dirichlet boundary),} \\ -\{\!\{a\nabla U_h\}\!\} - \boldsymbol{\beta}\, [\![a\nabla U_h \cdot \boldsymbol{n}]\!] + \alpha\, \boldsymbol{J}_h, & \text{on } \mathcal{E}_h^\circ, \\ & \text{(on interior mesh edges).} \end{cases}$$

where $\boldsymbol{J}_h$ in $\mathfrak{J}_h$ is an unknown function to be determined by

$$\langle \widehat{\boldsymbol{q}}_h\,,\ [\![v\boldsymbol{n}]\!]\rangle_{\mathcal{E}_h} = \sum_{K\in\mathcal{T}_h} \langle q^K_{n,h}, v\rangle_{\partial K}, \qquad \forall v \in W_h^{\mathrm{DG}}.$$

Here α and $\boldsymbol{\beta}$ are some parameters (typically $\boldsymbol{\beta}\equiv\boldsymbol{0}$ and $\alpha\equiv 1$), and

$$\{\!\{v\}\!\} = \begin{cases} \frac{1}{2}\left(v^+ + v^-\right) & \text{on } \mathcal{E}_h^\circ, \\ v & \text{on } \partial\Omega. \end{cases}$$

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Recall the space of "jumps" $\mathfrak{J}_h = \{\, [\![v\boldsymbol{n}]\!] : \quad v \in W_h^{\mathrm{DG}}\}.$

$\gg$ We need to solve a global problem on $\mathfrak{J}_h \ldots$

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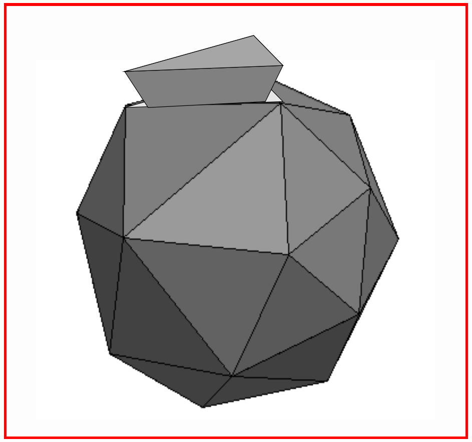
Spaces of jumps

Idea: ✗ ✗ ? (3rd attempt)

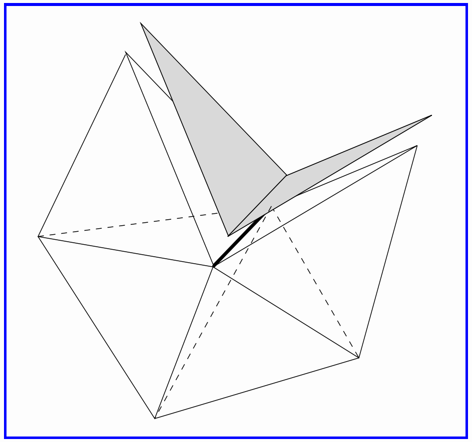
$$H^1(\Omega)/\mathbb{R} \xrightarrow{\textbf{grad}} H(\mathbf{curl}) \xrightarrow{\textbf{curl}} H(\mathrm{div})$$

$$[\![v\boldsymbol{n}]\!] = 0 \qquad [\![\boldsymbol{v} \times \boldsymbol{n}]\!] = 0 \qquad [\![\boldsymbol{v} \cdot \boldsymbol{n}]\!] = 0$$

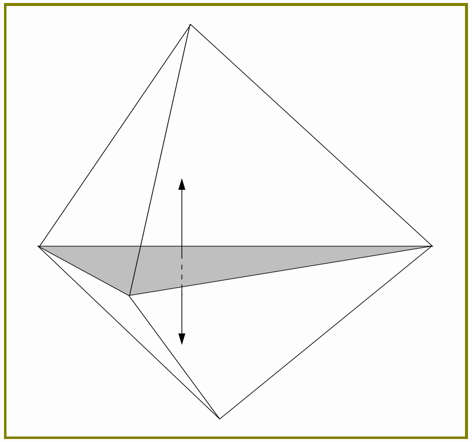
Basis for jumps of corresponding FE spaces without the continuity constraints:



Cone basis



Wedge basis



Face basis

Spaces of jumps

Idea: ✗ ✗ ✓

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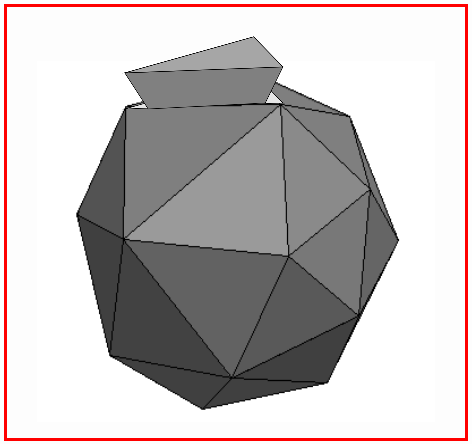

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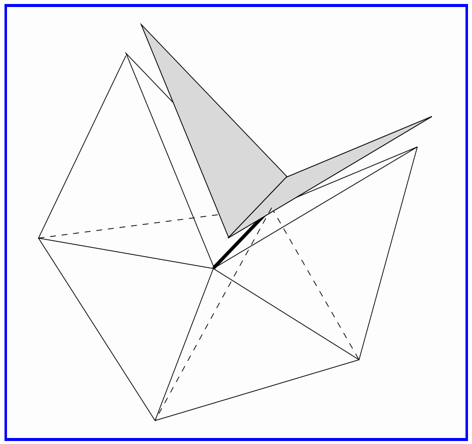
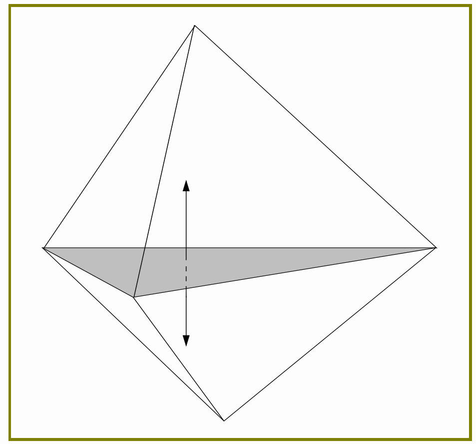


Wedge basis



Face basis

Using the cone basis for $\mathfrak{J}_h$, we can solve a **well-conditioned** global problem for $\boldsymbol{J}_h$.

Conservative flux

THEOREM. [Cockburn, G., & Wang] *The flux $\boldsymbol{q}_h$ obtained by our postprocessing of the CG solution has the following properties:*

1. *$\boldsymbol{q}_h$ is conservative.*
2. $[\![\boldsymbol{q}_h \cdot \boldsymbol{n}]\!] = 0.$
3. $(\operatorname{div} \boldsymbol{q}_h, v)_K = (f, v)_K$ *for all* $v \in P_{k-1}(K)$.
4. $\| \operatorname{div}(\boldsymbol{q} - \boldsymbol{q}_h)\|_{L^2(\Omega)} \le C h^k \,|\, f \,|_{H^k(\Omega)}.$
5. *If $a(\boldsymbol{x})$ is piecewise smooth and mesh is quasiuniform,*

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{L^2(\Omega)} \le C h^k \big(|\,\boldsymbol{q}\,|_{H^k(\Omega)} + |u|_{H^k(\Omega)}\big).$$

Furthermore, it is possible to compute the flux $\boldsymbol{q}_h$ in asymptotically optimal computational complexity.