Hybridized DG methods

Jay Gopalakrishnan

University of Florida

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Collaborators: Bernardo Cockburn

University of Minnesota

Raytcho Lazarov

Texas A&M University

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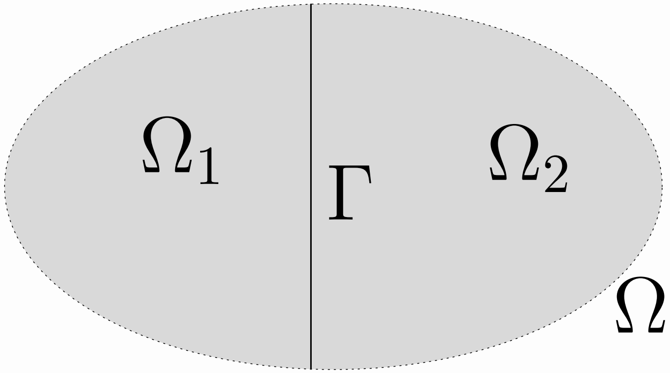
Outline

- A domain decomposition technique applied to CG method
- Hybridization as a domain decomposition technique.
- The hybridized RT method
- Developing hybridizable DG methods (H-DG methods)
- Previous DG methods that are H-DG methods
- New H-DG methods
- Coupling techniques

The interface problem

In standard domain decomposition with two subdomains, the interface problem solves for the trace of the solution:

$$\begin{aligned} -\Delta u &= f \qquad \text{on } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$



A function λ in $H^{1/2}_{00}(\Gamma)$ satisfies

$$\lambda = u|_\Gamma \iff (\boldsymbol{\nabla}\mathcal{H}\lambda, \boldsymbol{\nabla}\mathcal{H}\mu) = (f, \mathcal{H}\mu) \qquad \forall \mu$$

where $\mathcal{H}\mu$ denotes the harmonic extension of μ.

We call λ the “Lagrange multiplier” for reasons clear later.

The interface problem

In standard domain decomposition with two subdomains, the interface problem solves for the trace of the solution:

$$-\Delta u = f \quad \text{on } \Omega$$
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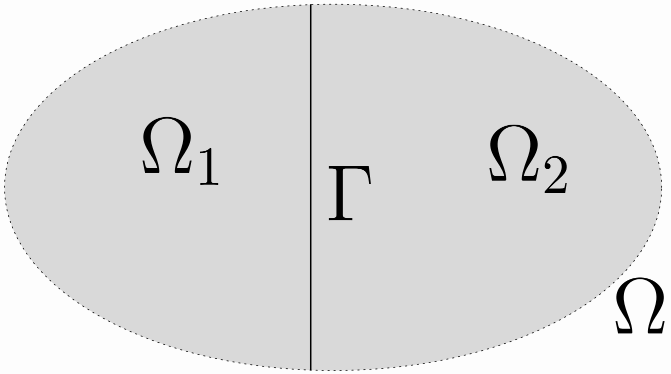


A function λ_h in M_h satisfies [Bramble, Pasciak & Schatz, 1986]

$$\lambda_h = u_h|_\Gamma \iff (\nabla \mathcal{H}_h \lambda_h, \nabla \mathcal{H}_h \mu) = (f, \mathcal{H}_h \mu) \qquad \forall \mu$$

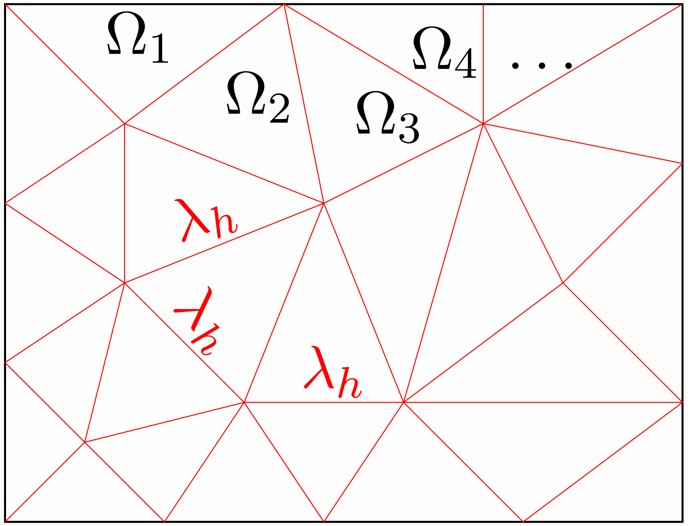
where $\mathcal{H}_h \mu$ denotes the discrete harmonic extension of μ.

This holds for the continuous Galerkin (CG) finite element method even when subdomains reduce to mesh elements.

Static condensation

When subdomains reduce to elements,

$$\underbrace{(\overbrace{\boldsymbol{\nabla}\mathcal{H}_h}^{\text{Local lifting}}\ \lambda_h, \overbrace{\boldsymbol{\nabla}\mathcal{H}_h}^{\boldsymbol{\mathcal{Q}}}\ \mu)}_{a_h(\lambda_h,\mu)} = \underbrace{(f,\mathcal{H}_h\mu)}_{b_h(\mu)}.$$



- λ_h satisfies a variational formulation involving the bilinear form

  $$a_h(\lambda_h,\mu) = (\boldsymbol{\mathcal{Q}}\lambda_h, \boldsymbol{\mathcal{Q}}\mu)$$

  with the *element by element* lifting $\boldsymbol{\mathcal{Q}}$.
- The matrix for λ_h is the same as what one obtains by *static condensation* of the CG method.

Do other methods have such Lagrange multipliers?

Mixed methods

Mixed methods for $-\Delta u = f$ give such λ_h after “hybridization”.

1st order reformulation:

$$\begin{aligned} \boldsymbol{q} + \boldsymbol{\nabla} u &= 0, && \text{on } \Omega \\ \operatorname{div} \boldsymbol{q} &= f, && \text{on } \Omega \\ u &= 0, && \text{on } \partial\Omega \end{aligned}$$

The (nonhybridized) mixed method:

$$\begin{aligned} (\boldsymbol{q}_h, \boldsymbol{v}) - (u_h, \operatorname{div} \boldsymbol{v}) &= 0, \\ (w, \operatorname{div} \boldsymbol{q}_h) &= (f, w). \end{aligned}$$

(The Raviart-Thomas (RT) method.)

Polynomial spaces:

$P_k(\tau) =$ set of all polynomials of degree $\le k$ on element τ,

$$V_h^{\text{RT}} = \{\boldsymbol{v} \in H(\text{div}) : \boldsymbol{v}|_\tau \in \boldsymbol{x} P_k(\tau) + \boldsymbol{P}_k(\tau) \quad \forall \tau\},$$

$$W_h = \{\boldsymbol{v} \in L^2(\Omega) : \boldsymbol{v}|_\tau \in P_k(\tau) \quad \forall \tau\}.$$

Then $\boldsymbol{q}_h$ and $\boldsymbol{v}$ are in V_h^{RT}, while u_h and w are in W_h.

Hybridized mixed method

recall

The non-hybridized mixed method:

$$(\boldsymbol{q}_h, \boldsymbol{v}) - (u_h, \operatorname{div} \boldsymbol{v}) = 0,$$
$$(w, \operatorname{div} \boldsymbol{q}_h) = (f, w).$$

$\boldsymbol{q}_h$ is in $\mathcal{V}_h^{\mathrm{RT}} \subseteq H(\mathrm{div})$, so

$$\underbrace{[\![\boldsymbol{q}_h \cdot \boldsymbol{n}]\!]}_{\text{Normal flux jump}} = 0.$$

Hybridization is the process of removing continuity constraints of finite element spaces without altering the solution:

$$\begin{aligned} (\boldsymbol{q}_h, \boldsymbol{v}) \quad & - (u_h, \operatorname{div} \boldsymbol{v}) + \langle \lambda_h, [\![\boldsymbol{v}_h \cdot \boldsymbol{n}]\!] \rangle && = 0, \\ (w, \operatorname{div} \boldsymbol{q}_h) & && = (f, w), \\ \langle [\![\boldsymbol{q}_h \cdot \boldsymbol{n}]\!], \mu \rangle & && = 0. \end{aligned}$$

Now $\boldsymbol{q}_h$ is not sought in $\mathcal{V}_h^{\mathrm{RT}} \subseteq H(\mathrm{div})$, but rather in $\boldsymbol{V}_h^{\mathrm{RT}} = \{\boldsymbol{v} : \boldsymbol{v}|_\tau \in \boldsymbol{x} P_k(\tau) + \boldsymbol{P}_k(\tau) \quad \forall \tau\}$.

Here,

$$\langle [\![\boldsymbol{q}_h \cdot \boldsymbol{n}]\!], \mu \rangle = \sum_e \int_e \mu \, [\![\boldsymbol{q}_h \cdot \boldsymbol{n}]\!].$$

Hybridized mixed method

recall

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The functions μ and λ_h are in *(compare with the CG case)*

$$M_h^{\mathrm{RT}} = \text{ space of the jumps } = \{\eta : \quad \eta|_e \in P_k(e), \ \forall \text{ interior mesh edges } e\}.$$

Hybridized mixed method

recall

The non-hybridized mixed method:

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- $\boldsymbol{q}_h$ and u_h of the hybridized method coincide with that of the mixed method.
- No longer the same as static condensation (extra information in λ_h).

A variational characterization

THEOREM. *Like in the CG case, the Lagrange multiplier of the RT method* λ_h *is the unique function in* M_h^{RT} *that satisfies*

$$a_h(\lambda_h, \mu) = b_h(\mu), \quad \text{for all } \mu \in M_h^{\mathrm{RT}},$$

where the bilinear form and the right hand side are defined by

$$a_h(\eta, \mu) = (\mathbf{\mathfrak{Q}}\eta, \mathbf{\mathfrak{Q}}\mu) \qquad \text{and}$$
$$b_h(\mu) = (f, \mathcal{U}\mu), \quad \text{for all } \eta, \mu \in M_h^{\mathrm{RT}}.$$

Here the lifting map $\mu \mapsto (\mathbf{\mathfrak{Q}}\mu, \mathcal{U}\mu)$ *is defined as follows:* ⟶

Compare with CG case: $\underbrace{(\overbrace{\nabla \mathcal{H}_h}^{\mathbf{\mathfrak{Q}}} \lambda_h, \overbrace{\nabla \mathcal{H}_h}^{\mathbf{\mathfrak{Q}}} \mu)}_{a_h(\lambda_h, \mu)} = \underbrace{(f, \mathcal{H}_h \mu)}_{b_h(\mu)}.$

Different spaces, different liftings, but same structure.

◯ ▷*DG case*

The lifting maps

Definition of $\mu \mapsto (\boldsymbol{\mathfrak{Q}}\mu,\, \mathcal{U}\mu)$ for RT: On each element τ of the mesh, $\boldsymbol{\mathfrak{Q}}\mu|_\tau \in \boldsymbol{x}P_k(\tau) + \boldsymbol{P}_k(\tau)$ and $\mathcal{U}\mu|_\tau \in P_k(\tau)$ form the unique solution of

$$\int_\tau \boldsymbol{\mathfrak{Q}}\mu \cdot \boldsymbol{r} - \int_\tau \mathcal{U}\mu \;\mathrm{div}\; \boldsymbol{r} = -\int_{\partial\tau\setminus\partial\Omega} \mu\, \boldsymbol{r}\cdot\boldsymbol{n},$$

$$\int_\tau w \;\mathrm{div}\, \boldsymbol{\mathfrak{Q}}\mu = 0,$$

for all $\boldsymbol{r} \in \boldsymbol{R}_k(\tau)$ and $w \in P_k(\tau)$.

Thus, $a_h(\eta, \mu)$ and $b_h(\mu)$ are locally computable.

Why hybridization?

Hybridization allows us to retain all the advantages of mixed methods, while removing its disadvantages:

- Results in a symmetric positive definite system, unlike mixed method equations.
- Results in a smaller system. (It involves just λ_h, instead of $\boldsymbol{q}_h, u_h,$ and λ_h.)
- Good for high order elements due to dimensional reduction.
- Once λ_h is found, $\boldsymbol{q}_h$ and u_h can be found locally (element by element).
- λ_h can be used to get a higher order solution by postprocessing [Arnold & Brezzi, 1985].

Hybridize DG?

- *Motivation:* Remove the criticism that discontinuous Galerkin (DG) methods have too many unknowns, via the reduction of unknowns by hybridization.

- *Difficulty:* Hybridization of mixed methods proceeded by relaxing the continuity constraints of finite element spaces.

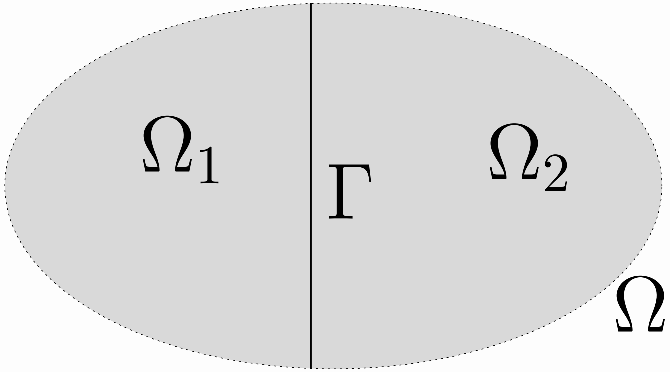
  But DG uses spaces with no continuity constraints, so no constraint to relax ?!

- *Solution:* Identify a “conservativity condition” . . .

Transmission conditions

$$-\Delta u = f \qquad \text{on } \Omega$$

$$u = 0 \qquad \text{on } \partial\Omega$$



Transmission conditions on Γ:

$$[\![u]\!] = 0$$

$$[\![\boldsymbol{q} \cdot \boldsymbol{n}]\!] = 0$$

← Conservativity condition.

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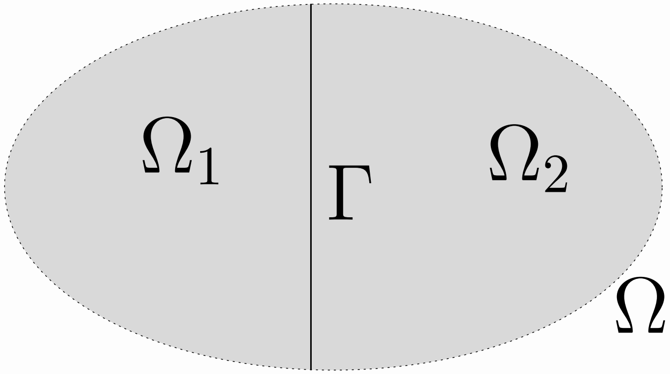


Transmission conditions on Γ:

$$[\![u]\!] = 0$$
$$[\![\boldsymbol{q}\cdot\boldsymbol{n}]\!] = 0$$

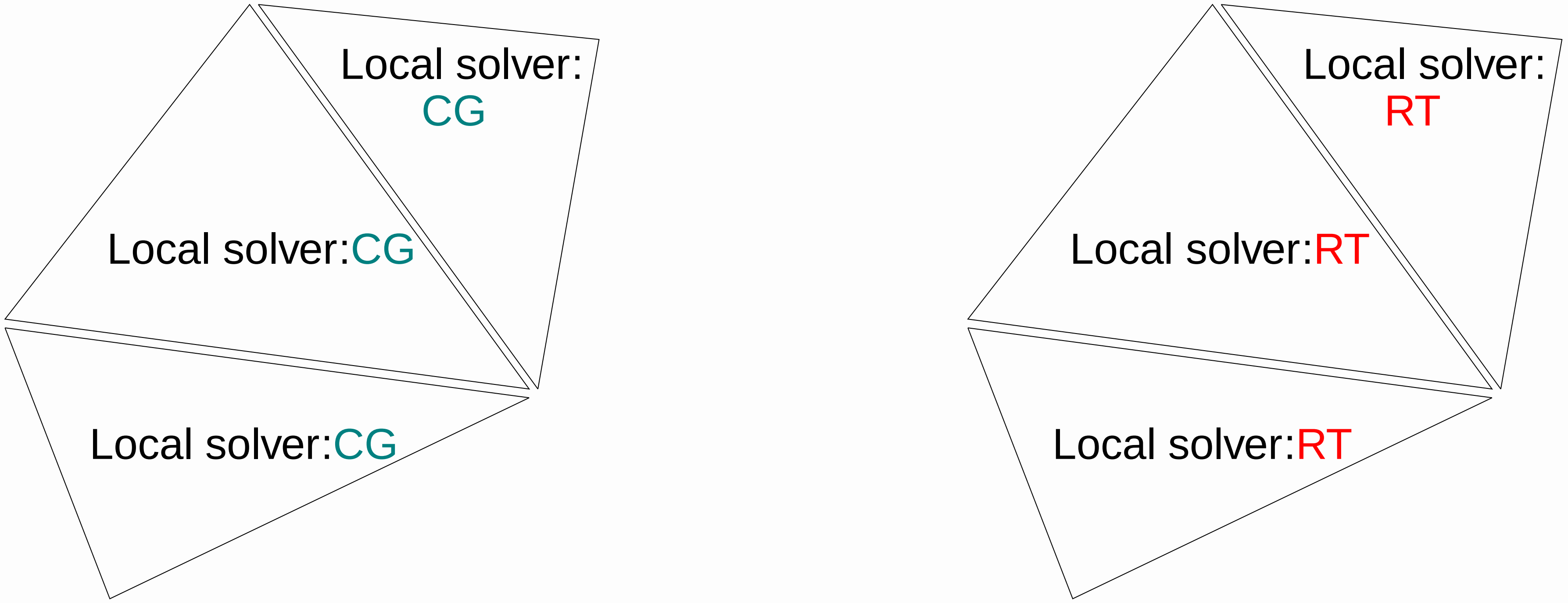
← Conservativity condition.

Every method has discrete versions of these conditions . . .

	CG method	RT method	DG method
$[\![u]\!] = 0$	✓	?	?
$[\![\boldsymbol{q}\cdot\boldsymbol{n}]\!] = 0$	?	✓	?

Local solvers & Conservativity

Elements are endowed with local solvers.



Local solver: CG

Local solver:CG

Local solver:CG

Local solver: RT

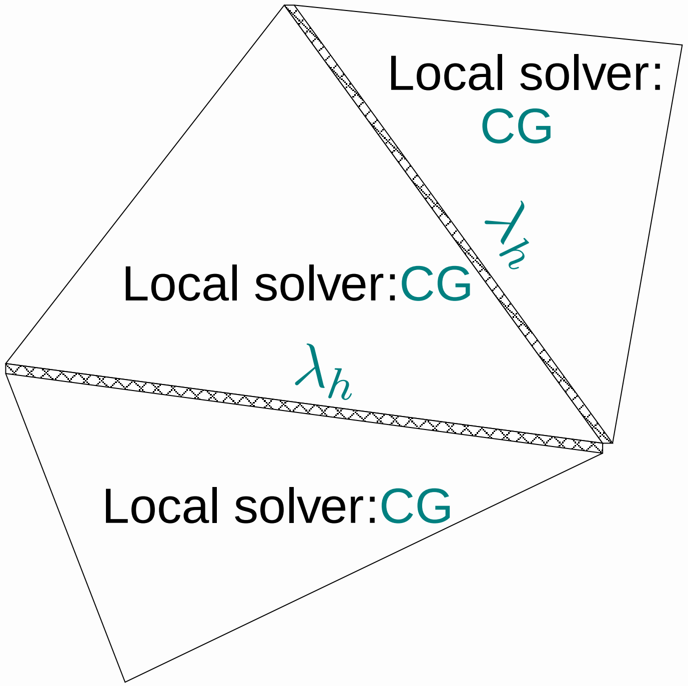
Local solver:RT

Local solver:RT

Local solvers & Conservativity

Elements are endowed with local solvers.

Global methods are obtained using Lagrange multipliers.



Local solver: CG

λ_h

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λ_h

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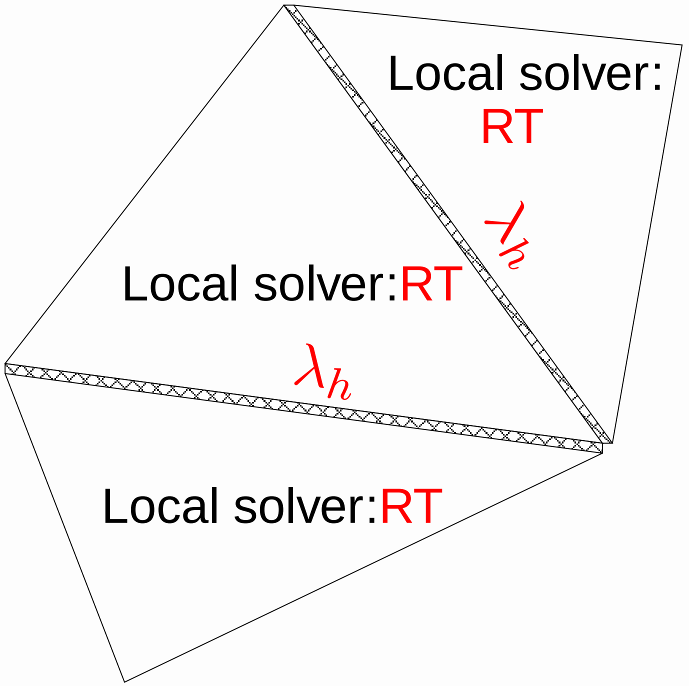

Local solver: RT

λ_h

Local solver:RT

λ_h

Local solver:RT

$$a_h^{\mathrm{CG}}(\lambda_h, \mu) = b_h^{\mathrm{CG}}(\mu)$$

$$\Updownarrow$$

$$[\![\text{(a flux approx)} \cdot \boldsymbol{n}]\!] \approx 0$$

$$a_h^{\mathrm{RT}}(\lambda_h, \mu) = b_h^{\mathrm{RT}}(\mu)$$

$$\Updownarrow$$

$$[\![\boldsymbol{q}_h \cdot \boldsymbol{n}]\!] = 0$$

Conservativity conditions

Derivation of DG methods

[Arnold, Brezzi, Cockburn & Marini, 2001]

$$\boldsymbol{q} + \boldsymbol{\nabla} u = 0 \implies$$

$$\int_{\mathcal{T}} \boldsymbol{q} \cdot \boldsymbol{v} - \int_{\mathcal{T}} u \,\mathrm{div}\, \boldsymbol{v} + \int_{\partial \mathcal{T}} u \,(\boldsymbol{v} \cdot \boldsymbol{n}) = 0$$

Derivation of DG methods

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$\boldsymbol{q} + \boldsymbol{\nabla} u = 0 \implies$

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$$\operatorname{div} \, \boldsymbol{q} = f \implies$$

$$-\int_\tau \boldsymbol{\nabla} w \cdot \boldsymbol{q} + \int_{\partial\tau} w \, \boldsymbol{q} \cdot \boldsymbol{n} = \int_\tau f \, w$$

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$$\operatorname{div}\ \boldsymbol{q} = f \implies$$

$$-\int_\tau \boldsymbol{\nabla} w \cdot \boldsymbol{q}_h + \int_{\partial\tau} w\,\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n} = \int_\tau f\,w$$

Various DG methods are obtained by setting different expressions for the *numerical fluxes* $\widehat{u}_h$ and $\widehat{\boldsymbol{q}}_h$.

The conservativity condition is implicit in the choice of $\widehat{\boldsymbol{q}}_h \ldots$

Derivation of DG methods

Equations of the scheme:

Lagrange multiplier

$$\sum_{\tau} \left(\int_{\tau} \boldsymbol{q}_h \cdot \boldsymbol{v} - \int_{\tau} u_h \,\mathrm{div}\, \boldsymbol{v} + \int_{\partial\tau\setminus\partial\Omega} \lambda_h \boldsymbol{v} \cdot \boldsymbol{n} \right) = 0$$

$$\sum_{\tau} \left(- \int_{\tau} \boldsymbol{\nabla} w \cdot \boldsymbol{q}_h + \int_{\partial\tau} w \, \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n} \right) = \int_{\Omega} f \, w$$

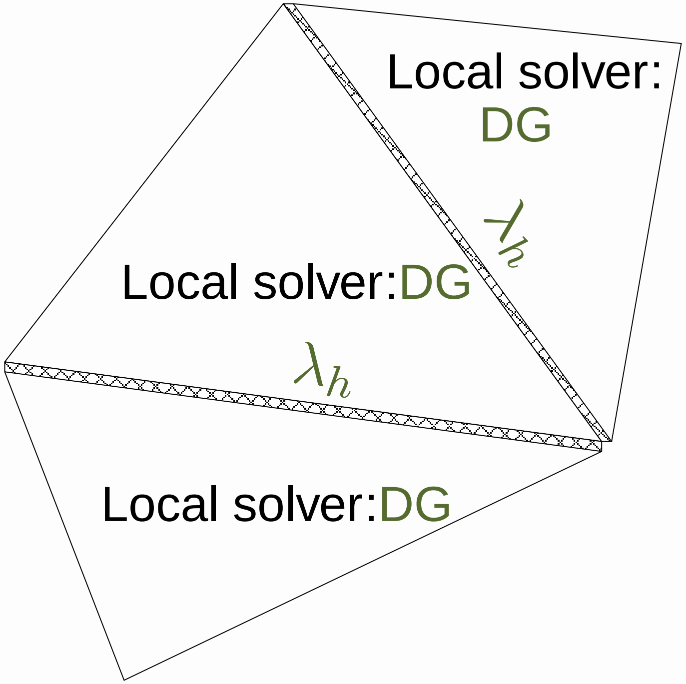
Add conservativity condition: $\displaystyle\sum_{\text{edges } e} \int_{e} [\![\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}]\!] \mu = 0$

- $\boldsymbol{q}_h$ and u_h are both in discontinuous spaces.
- $\widehat{\boldsymbol{q}}_h$ needs to be prescribed.
- $\lambda_h = \widehat{u}_h$ is now an unknown and no longer needs to be prescribed.

To hybridize DG. . .

To hybridize such a DG scheme we need

- Local DG solvers
- Conservativity condition (✓)
- Lagrange multiplier space (✓)



Let us define the local solvers (liftings) . . .

DG local solvers

The DG lifting $\mu \mapsto (\mathfrak{Q}\mu,\ \mathfrak{U}\mu)$ is defined by

$$\int_\tau \mathfrak{Q}\mu \cdot \boldsymbol{v} - \int_\tau (\mathfrak{U}\mu)\,\mathrm{div}\,\boldsymbol{v} = -\int_{\partial\tau} \mu\, \boldsymbol{v}\cdot\boldsymbol{n},$$

$$-\int_\tau \boldsymbol{\nabla} w \cdot \mathfrak{Q}\mu + \int_{\partial\tau} w\ \text{flux}\cdot\, \boldsymbol{n} = 0$$

(The functions are in the DG spaces on the element τ.)

Different prescriptions of flux give different DG local solvers.

DG local solvers

The DG lifting $\mu \mapsto (\boldsymbol{\mathcal{Q}}\mu,\, \mathcal{U}\mu)$ is defined by

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$$-\int_\tau \boldsymbol{\nabla} w \cdot \boldsymbol{\mathcal{Q}}\mu + \int_{\partial\tau} w\, \widehat{\boldsymbol{\mathcal{Q}}}\mu \cdot \boldsymbol{n} = 0$$

(The functions are in the DG spaces on the element τ.)

Different prescriptions of $\widehat{\boldsymbol{\mathcal{Q}}}\mu$ give different DG local solvers.

- $\widehat{\boldsymbol{\mathcal{Q}}}\mu = \boldsymbol{\mathcal{Q}}\mu + \gamma(\,\mathcal{U}\mu - \mu)\,\boldsymbol{n} \qquad \longrightarrow$ (LDG local solver)
- $\widehat{\boldsymbol{\mathcal{Q}}}\mu = -a\,\boldsymbol{\nabla}\,\mathcal{U}\mu + \gamma(\,\mathcal{U}\mu - \mu)\,\boldsymbol{n} \quad \longrightarrow$ (IP local solver)

etc.

Hybridized DG scheme

THEOREM. *The conservativity condition of the DG scheme characterizes the Lagrange multiplier (numerical trace):*

$$\sum_e \int_e [\![\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}]\!]\mu = 0 \quad \Longleftrightarrow \quad a_h(\lambda_h, \mu) = b_h(\mu)$$

where the bilinear form now involves the DG liftings,

$$a_h(\eta, \mu) = \int_\Omega \boldsymbol{\mathcal{Q}}\eta \cdot \boldsymbol{\mathcal{Q}}\mu + \sum_e \int_e [\![(\,\mathcal{U}\mu - \mu)(\widehat{\boldsymbol{\mathcal{Q}}}\eta - \boldsymbol{\mathcal{Q}}\eta) \cdot \boldsymbol{n}]\!],$$

and $b_h(\cdot)$ has a similar expression involving the local solvers.

Moreover, $\boldsymbol{q}_h$ and u_h can be recovered locally element by element once λ_h is computed.

⥀ ◁ RT case

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and $b_h(\cdot)$ *has a similar expression involving the local solvers.*

IMPORTANCE: Such hybridized DG schemes yield matrices of size and sparsity identical to that of mixed methods.

↺ ◁ *RT case*

Unified hybridization

The theorem holds for any method in the form

$$\sum_{\tau} \left(\int_{\tau} \boldsymbol{q}_h \cdot \boldsymbol{v} - \int_{\tau} u_h \operatorname{div} \boldsymbol{v} + \int_{\partial\tau \setminus \partial\Omega} \lambda_h \, \boldsymbol{v} \cdot \boldsymbol{n} \right) = 0$$

$$\sum_{\tau} \left(- \int_{\tau} \boldsymbol{\nabla} w \cdot \boldsymbol{q}_h + \int_{\partial\tau} w \, \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n} \right) = \int_{\Omega} f \, w$$

$$\sum_{e} \int_{e} [\![\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}]\!] \mu = 0.$$

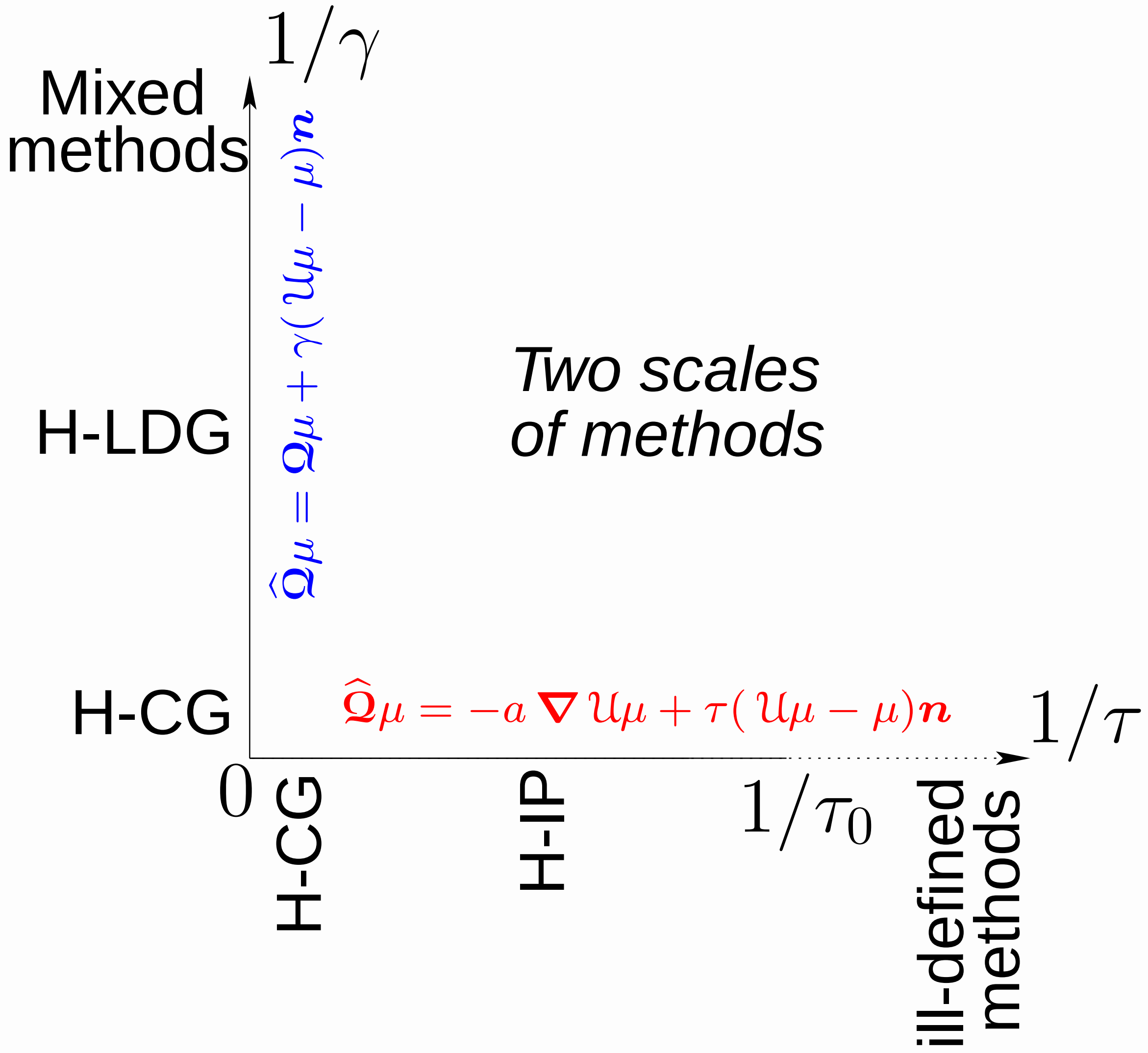
Thm: $\displaystyle \sum_{e} \int_{e} [\![\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}]\!] \mu = 0 \iff a_h(\lambda_h, \mu) = b_h(\mu).$

gives a unified framework for hybridizing CG, RT, and DG.

Examples

Different choices of $\widehat{\mathcal{Q}}\mu$ give different methods:

- Many DG schemes
- The CG method
- The RT method
- The BDM method
- P_1 & other non-conforming methods



(As penalty $\gamma \longrightarrow \infty$ we get the hybridized CG method.)

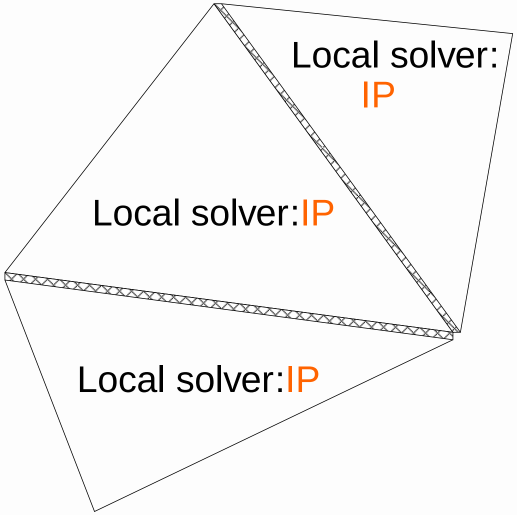
H-IP method $\neq$ IP method

Eg.1: *H-IP method* = the hybridized DG method with local solver set to IP.

PROPOSITION. *The numerical fluxes of the H-IP method on interior edges (with constant single valued penalty γ) are*

$$\lambda_h = \{\!\{u_h\}\!\} - \frac{1}{2\gamma}[\![a\,\boldsymbol{\nabla} u_h \cdot \boldsymbol{n}]\!],$$

$$\widehat{\boldsymbol{q}}_h = -\{\!\{a\,\boldsymbol{\nabla} u_h\}\!\} + \frac{\gamma}{2}[\![u_h\,\boldsymbol{n}]\!].$$



H-IP method

Cf. standard IP method:
$$\begin{cases} \widehat{u}_h = \{\!\{u_h\}\!\} \quad := (u^+ + u^-)/2 \\ \widehat{\boldsymbol{q}}_h = -\{\!\{a\,\boldsymbol{\nabla} u_h\}\!\} + \gamma\,[\![u_h\,\boldsymbol{n}]\!]. \end{cases}$$

Ewing-Wang-Yang method

Eg.2: The EWY method originally motivated by:

$$-\operatorname{div}(a\,\boldsymbol{\nabla} u) = f \qquad \Longrightarrow \begin{cases} (a\,\boldsymbol{\nabla} u, \boldsymbol{\nabla} v)_K - \langle a\nabla u \cdot \boldsymbol{n}, v\rangle_{\partial K} = (f, v)_K \\ [\![u]\!] = 0 \end{cases}$$

Ewing-Wang-Yang method

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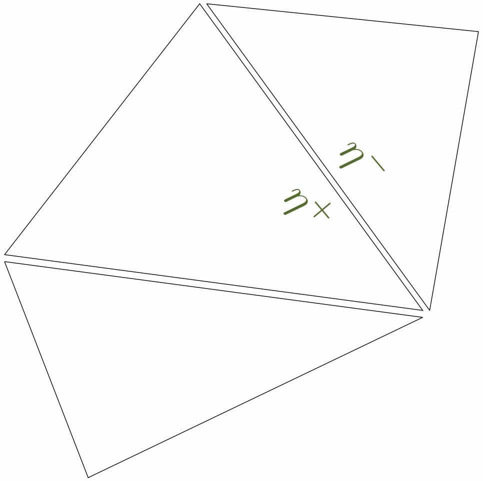
Now

$$[\![u]\!] = 0 \qquad \Longleftrightarrow \qquad \sum_K \langle u, \eta\rangle_{\partial K} = 0$$

for all double valued functions η on interior edges with $\eta^+ + \eta^- = 0$. This motivates:



Ewing-Wang-Yang method

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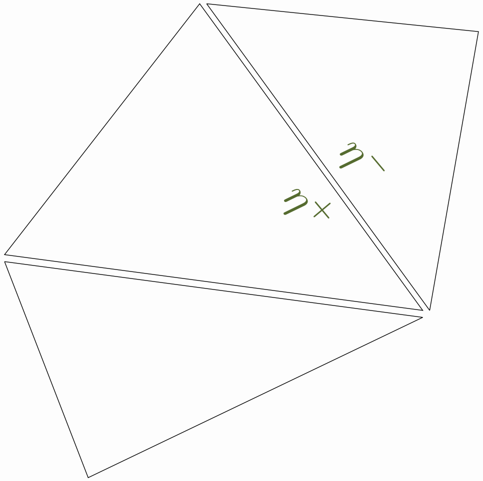
$$[\![u]\!] = 0 \quad \Longleftrightarrow \quad \sum_K \langle u, \eta\rangle_{\partial K} = 0$$

for all double valued functions η on interior edges with $\eta^+ + \eta^- = 0$. This motivates:



$$\left\{ \begin{array}{ll} \displaystyle\sum_K \Big((a\,\boldsymbol{\nabla} u, \boldsymbol{\nabla} v)_K - \langle \ell, v\rangle_{\partial K} \Big) = (f, v)_\Omega, & \forall v \\ \displaystyle\sum_K \langle u, \eta\rangle_{\partial K} = 0, & \forall \eta \end{array} \right\}$$

The EWY method

The equation of the EWY method: Find u_h and ℓ_h satisfying

$$\sum_K (a\,\boldsymbol{\nabla} u_h, \, \boldsymbol{\nabla} v)_K - \langle \ell_h \, , \, v\rangle_{\partial K} - \langle \eta \, , \, u_h \rangle_{\partial K}$$

$$- \alpha h \langle \, (\ell_h - a\,\boldsymbol{\nabla} u_h \cdot \boldsymbol{n}) \, , \, (\eta - a\,\boldsymbol{\nabla} v \cdot \boldsymbol{n}) \, \rangle_{\partial K} = (f, v)_\Omega.$$

- u_h and v are in $P_k(K)$ for all mesh elements K.
- ℓ_h is an approximation to $a\,\boldsymbol{\nabla} u \cdot \boldsymbol{n}$ from $P_k(e)$ on all mesh edges e. On every interior edge, it is a double-valued function whose branches from either side satisfy $\ell_h^+ + \ell_h^- = 0$. (Test function η is in the same space.)

The EWY method

The equation of the EWY method: Find u_h and ℓ_h satisfying

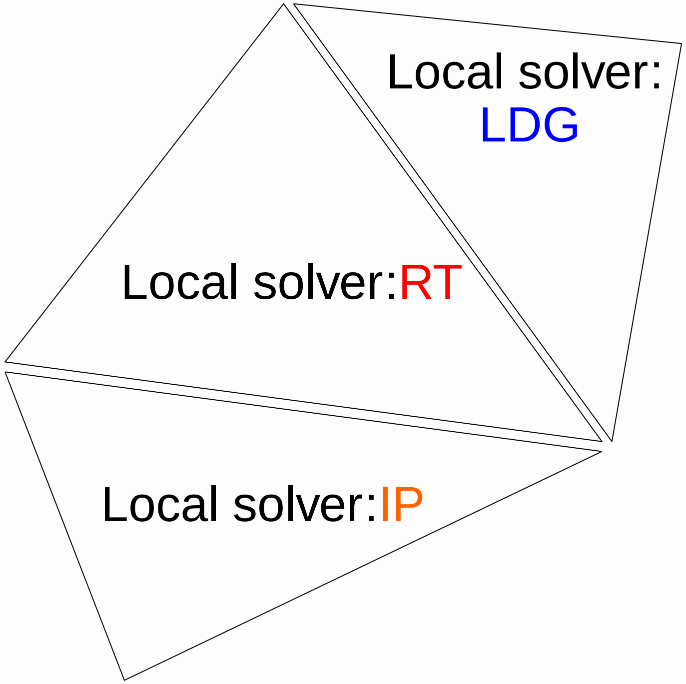
$$\sum_K (a\,\boldsymbol{\nabla} u_h, \boldsymbol{\nabla} v)_K - \langle \ell_h\,, v\rangle_{\partial K} - \langle \eta\,, u_h\rangle_{\partial K}$$
$$- \alpha h \langle\, (\ell_h - a\,\boldsymbol{\nabla} u_h \cdot \boldsymbol{n})\,, (\eta - a\,\boldsymbol{\nabla} v \cdot \boldsymbol{n})\,\rangle_{\partial K} = (f, v)_\Omega.$$

PROPOSITION. *The EWY method is a H-IP method. Specifically, the solution u_h of the EWY method coincides with that of an H-IP method when $\gamma^{-1} = \alpha h$. Moreover*

$$\ell_h = -\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}$$

where $\widehat{\boldsymbol{q}}_h$ is the numerical flux of this H-IP method.

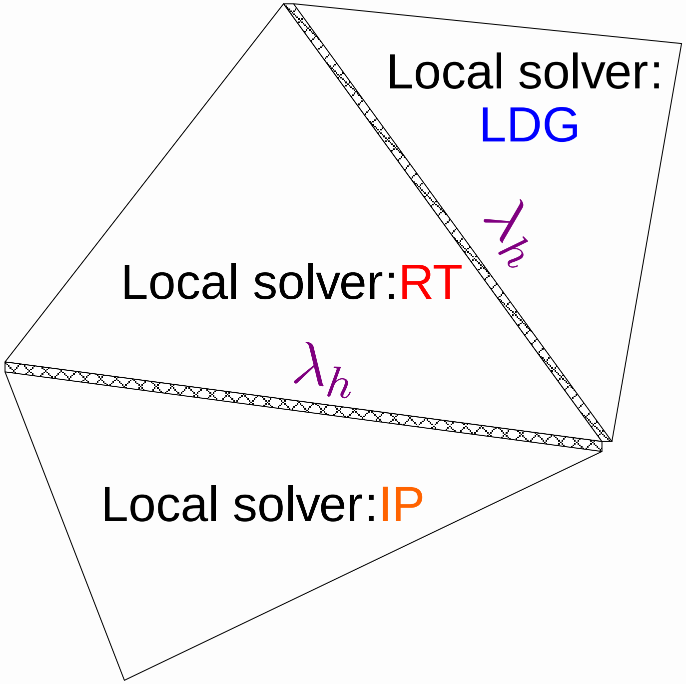
Coupling methods



Suppose we set different local solvers in different elements. How can we couple them?

Using the hybridization framework, we can generalize previous works on coupling methods [Perugia & Schötzau, 2001], [Cockburn & Dawson, 2002], and mortaring [Wheeler & Yotov, 1998].

Coupling methods



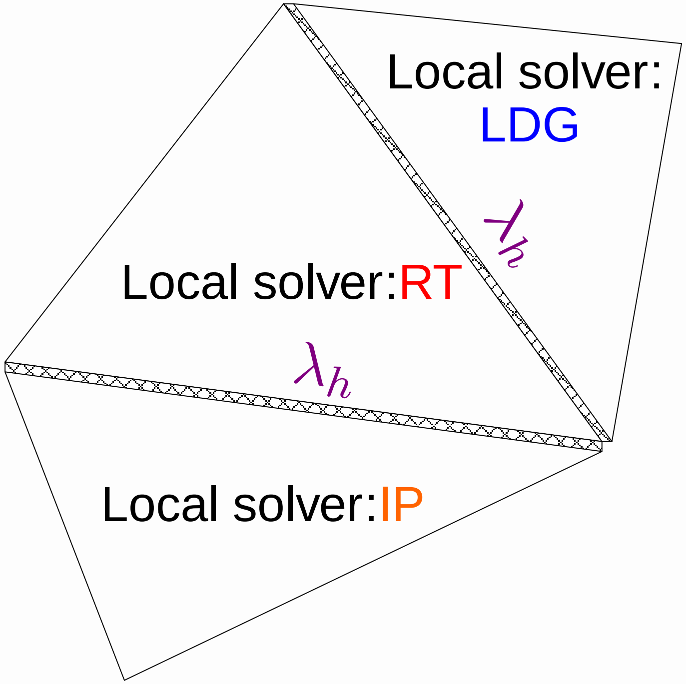
To couple methods, we only have to set the Lagrange multiplier space M_h. A choice of M_h implies a conservativity condition.

For all μ in M_h,

$$\sum_e \int_e [\![\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}]\!]\mu = 0 \qquad \iff \qquad \underbrace{a_h(\lambda_h,\mu)}_{(\boldsymbol{\Omega}\lambda_h,\,\boldsymbol{\Omega}\mu)} = b_h(\mu).$$

Here, on each element τ, the lifting $(\boldsymbol{\Omega}\lambda_h)\big|_\tau$ is defined using the local solver set on that element (e.g. LDG, RT, IP).

Coupling methods



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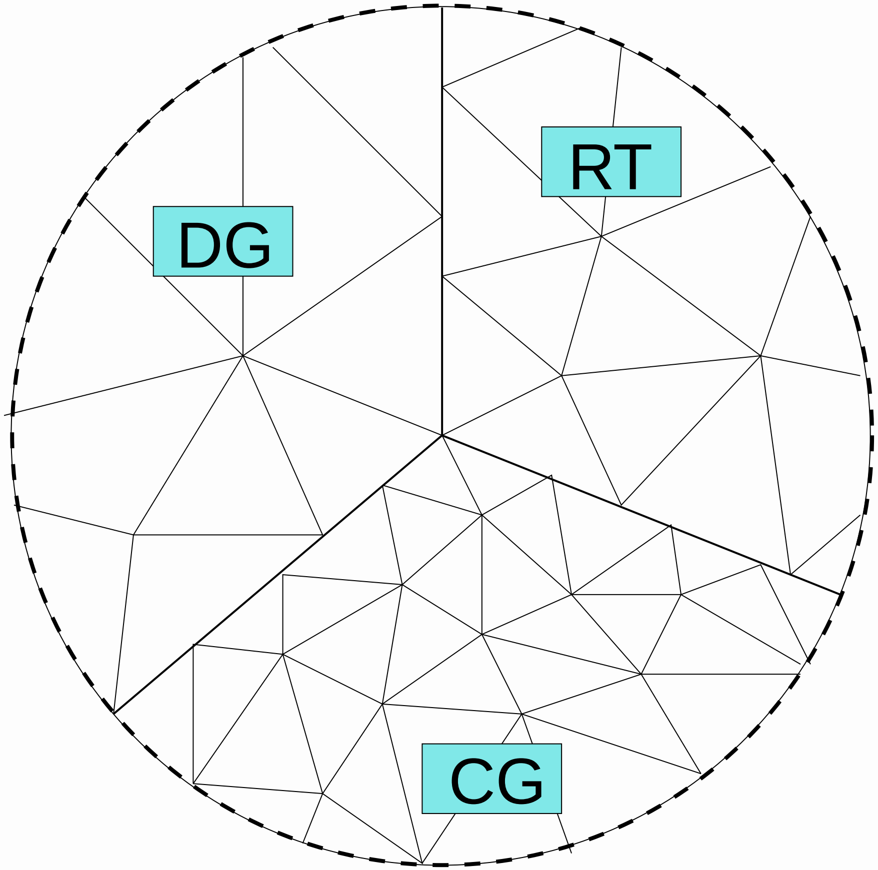
For all μ in M_h,

$$\sum_e \int_e [\![\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}]\!]\mu = 0 \qquad \Longleftrightarrow \qquad a_h(\lambda_h,\mu) = b_h(\mu).$$

- Weakly conservative coupling: $[\![\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}]\!] \notin M_h$.
- Strongly conservative coupling: $[\![\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}]\!] \in M_h$.

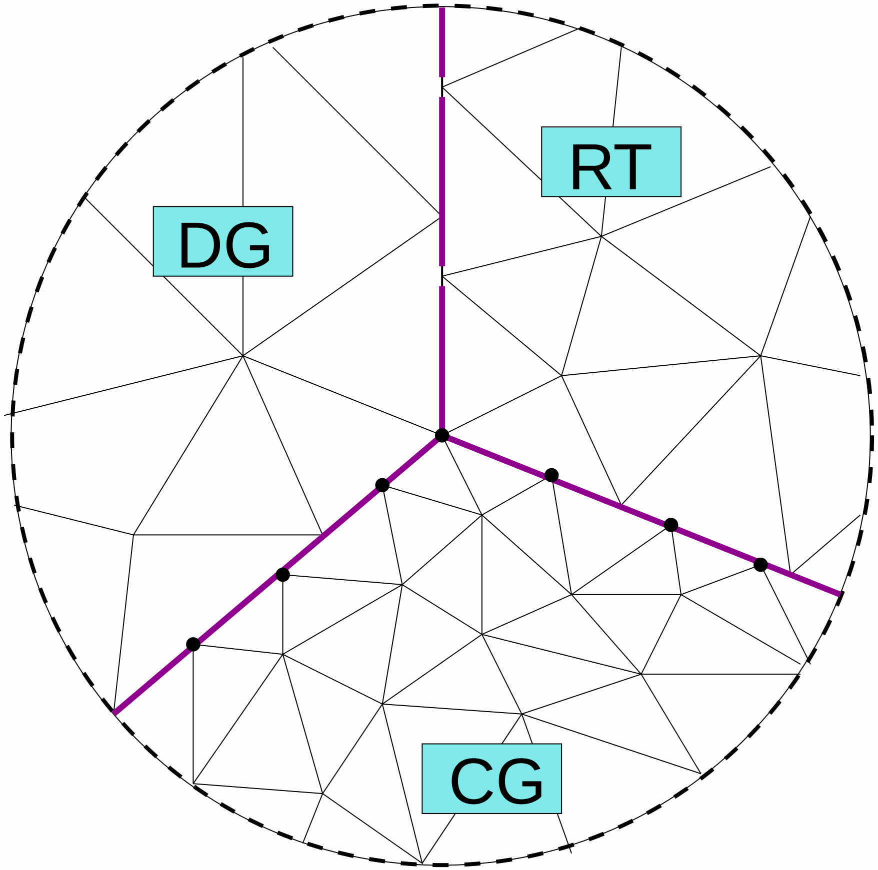
Mortar techniques

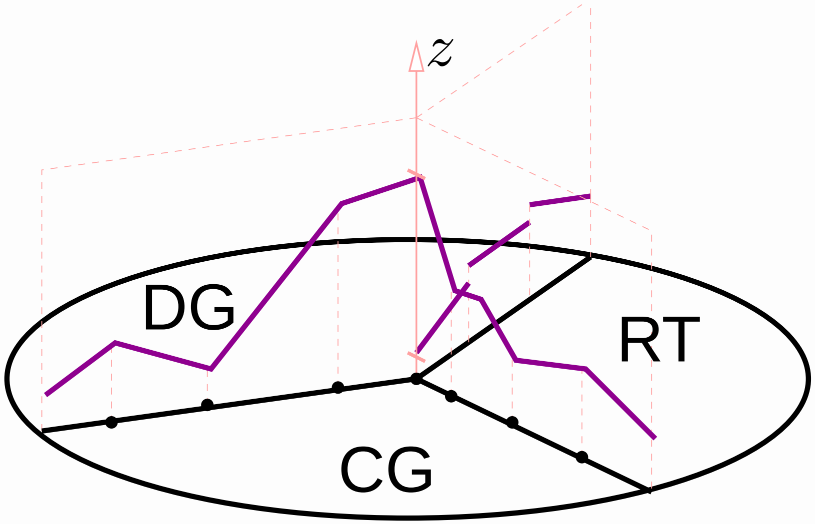


We can couple methods even across non-matching meshes.

We consider one example.

Mortar techniques



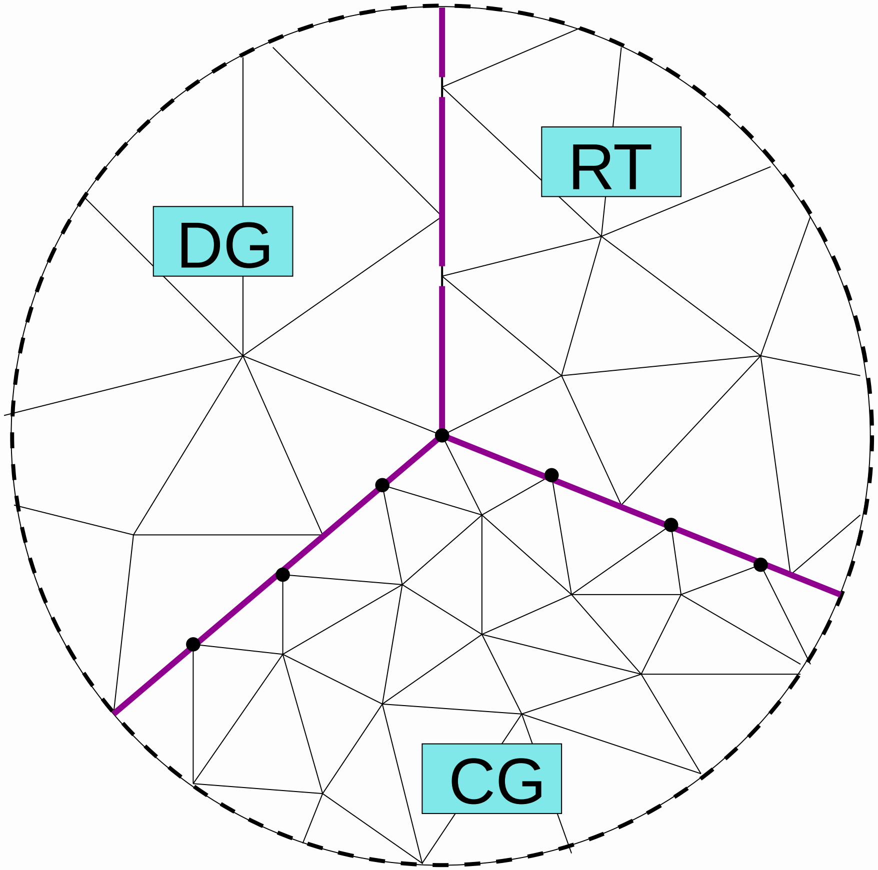


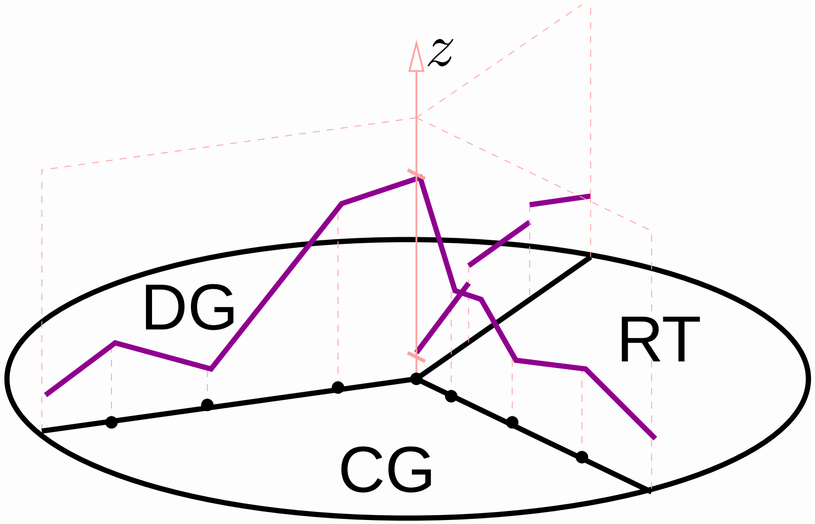
Graph of a typical function in M_h

Set M_h as follows:

- Choose M_h functions on a DG-RT interface as the traces from either the DG or the RT side (arbitrarily).
- On CG-DG or CG-RT interface, must choose M_h as the trace from CG side.

Mortar techniques



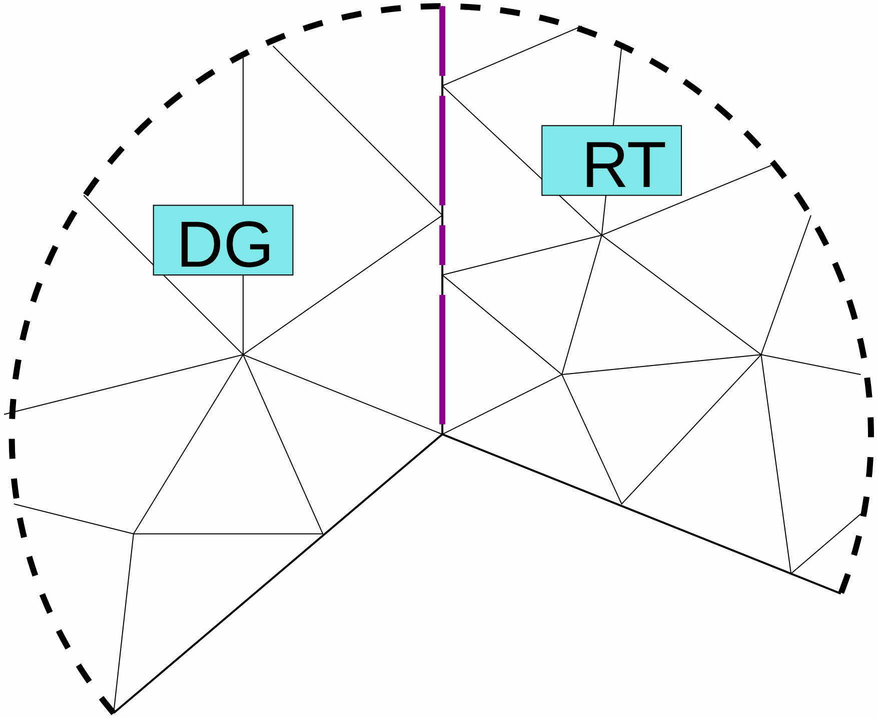


Graph of a typical function in M_h

THEOREM. *With this choice of* M_h *to couple CG, DG, and RT methods, there is a unique solution* $\lambda_h \in M_h$ *to the composite variational equation*

$$a_h(\lambda_h, \mu) = b_h(\mu), \quad \text{for all } \mu \in M_h.$$

Conservative mortars



Given two conservative methods, in order to couple them across a nonmatching interface *conservatively*, we should enforce

$$[\![\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}]\!] = 0.$$

Therefore, we should choose M_h such that

$$\sum_e \int_e [\![\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}]\!]\mu = 0, \quad \forall \mu \in M_h \quad \Longrightarrow \quad [\![\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}]\!] = 0.$$

(Must use the mesh formed by vertices from both sides: expect bad conditioning.)

Conclusion

- Hybridization brings a new point of view to FEM.
  - Traditional FEM construction: Design degrees of freedom suitable to enforce continuity across elements.
  - Hybridized FEM construction: Focus on solvers within an element without regard to continuity. Then choose “Lagrange multipliers” or a “conservativity condition”.
- It is possible to hybridize some DG methods.
  - Such DG methods are competitive with mixed methods in the number of unknowns.
- Can couple various methods
  - within a conforming mesh, or across non-matching mesh interfaces, in exactly conservative, or weakly conservative fashion.