Two new mixed finite elements for linear elasticity

Jay Gopalakrishnan

University of Florida

March 2010

The University of Texas at Austin

Thanks: NSF

References

- [G & Guzmán]: "A second elasticity element using the matrix bubble with tightened stress symmetry," preprint.
- [Cockburn, G & Guzmán]: "A new elasticity element made for enforcing weak stress symmetry," preprint.

The linear elasticity system

$$
\begin{aligned}
\underline{\boldsymbol{\sigma}} - C\underline{\boldsymbol{\epsilon}}(\mathbf{u}) &= \mathbf{0} && \text{in } \Omega, && \text{(constitutive law)}\\
\operatorname{div} \underline{\boldsymbol{\sigma}} &= \mathbf{f} && \text{in } \Omega, && \text{(equilibrium equation)}\\
\mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega, && \text{(kinematic boundary condition)}\\
\underline{\boldsymbol{\sigma}}^t - \underline{\boldsymbol{\sigma}} &= \mathbf{0} && \text{in } \Omega, && \text{(angular momentum conservation).}
\end{aligned}
$$

Notations:

$\underline{\boldsymbol{\sigma}}$	stress tensor (symmetric)
$\mathbf{u}$	displacement vector
$\underline{\boldsymbol{\epsilon}}(\mathbf{u})$	strain tensor $= (\operatorname{grad} \mathbf{u} + (\operatorname{grad} \mathbf{u})^t)/2$
C	elasticity tensor (fourth order)

(All differential operators are applied row-wise.)

Traditional and mixed approaches

- *Pure displacement formulation:* (Eliminate $\underline{\boldsymbol{\sigma}}$.)

$$\operatorname{div}\left(C\underline{\boldsymbol{\epsilon}}(\mathbf{u})\right) = \mathbf{f}$$

- *Hellinger-Reissner principle:* (Solve a system for $\underline{\boldsymbol{\sigma}}$ and $\mathbf{u}$.)

$$\begin{cases} D\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\epsilon}}(\mathbf{u}) = \mathbf{0} \\ \operatorname{div}\underline{\boldsymbol{\sigma}} = \mathbf{f} \end{cases}$$

Here $D = C^{-1}$. E.g., for an isotropic material $C^{-1}\underline{\boldsymbol{\sigma}} = \underline{\boldsymbol{\epsilon}}(\mathbf{u})$ reads as

$$\underline{\boldsymbol{\epsilon}}(\mathbf{u}) \;=\; \frac{1+\nu}{E}\underline{\boldsymbol{\sigma}} - \frac{\nu}{E}(\operatorname{tr}\underline{\boldsymbol{\sigma}})\delta \;\equiv\; D\underline{\boldsymbol{\sigma}}$$

where $\nu =$ Poisson ratio and $E =$ Young's modulus.

Why mixed approach?

There are several compelling reasons to pursue mixed methods:

- Gives a direct and hopefully more accurate approximation for the stress $\underline{\boldsymbol{\sigma}}$, which is often the quantity of interest.
- Works for materials near the incompressible limit (does not "lock"). [Stenberg, 1988], [Arnold, Douglas & Gupta, 1984]
- Suited for plasticity problems where elimination of stress variable is difficult. [Brezzi, Johnson & Mercier, 1977]

However, to quote from the recent review [Arnold, Falk & Winther, 2010], *mixed methods for linear elasticity*...

> *"...proved very elusive. Indeed, one of the motivations of the pioneering work of* [Raviart & Thomas, 1977] *on mixed finite elements for the Laplacian, was the hope that the solution to this easier problem would pave the way to such elements for elasticity, and there were many attempts to generalize their elements to the elasticity system..."*

$$
\begin{aligned}
D\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\epsilon}}(\mathbf{u}) &= \mathbf{0} && \text{in } \Omega, && \text{(constitutive law)}\\
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\underline{\boldsymbol{\sigma}}^t - \underline{\boldsymbol{\sigma}} &= \mathbf{0} && \text{in } \Omega, && \text{(angular momentum conservation)}
\end{aligned}
$$

where

$\underline{\boldsymbol{\sigma}}$	stress tensor
$\mathbf{u}$	displacement vector
$\underline{\boldsymbol{\epsilon}}(\mathbf{u})$	strain tensor $= (\operatorname{grad} \mathbf{u} + (\operatorname{grad} \mathbf{u})^t)/2$

$$
\begin{aligned}
D\underline{\boldsymbol{\sigma}} - \underline{\epsilon}(\mathbf{u}) &= \mathbf{0} && \text{in } \Omega, && \rightarrow\ D\underline{\boldsymbol{\sigma}} - \operatorname{grad}(\mathbf{u}) + \underline{\boldsymbol{\rho}} = \mathbf{0} \\
\operatorname{div}\underline{\boldsymbol{\sigma}} &= \mathbf{f} && \text{in } \Omega, && \text{(equilibrium equation)} \\
\mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega, && \text{(kinematic boundary condition)} \\
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$$

where

$\underline{\boldsymbol{\sigma}}$	stress tensor
$\mathbf{u}$	displacement vector
$\underline{\epsilon}(\mathbf{u})$	strain tensor $= (\operatorname{grad}\mathbf{u} + (\operatorname{grad}\mathbf{u})^t)/2$
$\underline{\boldsymbol{\rho}}$	rotation tensor $= (\operatorname{grad}\mathbf{u} - (\operatorname{grad}\mathbf{u})^t)/2$

Symmetry via a Lagrange Multiplier

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$$
\begin{aligned}
D\underline{\boldsymbol{\sigma}} - \underline{\epsilon}(\mathbf{u}) &= \mathbf{0} && \text{in } \Omega, && \rightarrow\ D\underline{\boldsymbol{\sigma}} - \operatorname{grad}(\mathbf{u}) + \underline{\boldsymbol{\rho}} = \mathbf{0} \\
\operatorname{div} \underline{\boldsymbol{\sigma}} &= \mathbf{f} && \text{in } \Omega, && \text{(equilibrium equation)} \\
\mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega, && \text{(kinematic boundary condition)} \\
\underline{\boldsymbol{\sigma}}^t - \underline{\boldsymbol{\sigma}} &= \mathbf{0} && \text{in } \Omega, && \text{(angular momentum conservation)}
\end{aligned}
$$

Variational problem with weakly imposed symmetry: Find $\underline{\boldsymbol{\sigma}}, \mathbf{u}, \underline{\boldsymbol{\rho}}$ such that

$$
\begin{aligned}
(D\underline{\boldsymbol{\sigma}}, \underline{\mathbf{v}}) + (\mathbf{u}, \operatorname{div} \underline{\mathbf{v}}) + (\underline{\boldsymbol{\rho}}, \underline{\mathbf{v}}) &= 0, \\
(\operatorname{div} \underline{\boldsymbol{\sigma}}, \boldsymbol{\omega}) &= (\mathbf{f}, \boldsymbol{\omega}), \\
(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\eta}}) &= 0,
\end{aligned}
\qquad
\left|
\begin{aligned}
\underline{\boldsymbol{\sigma}}, \underline{\mathbf{v}} &\in \underline{\mathbf{H}}(\operatorname{div}, \Omega) \\
\mathbf{u}, \boldsymbol{\omega} &\in \mathbf{L}^2(\Omega) \\
\underline{\boldsymbol{\rho}}, \underline{\boldsymbol{\eta}} &\in \operatorname{skw}(\underline{\mathbf{L}}^2(\Omega))
\end{aligned}
\right.
$$

for all $\underline{\mathbf{v}}, \boldsymbol{\omega}, \underline{\boldsymbol{\eta}}$ in the appropriate spaces.

$\operatorname{skw}(\cdot)$ = skew-symmetric part

$(\cdot, \cdot)$ = L^2-inner product

Two 2nd order systems

Laplace's equation	*Elasticity system*
$c\,\mathbf{q} + \operatorname{grad} p = 0$ $\operatorname{div}\mathbf{q} = f$	$D\underline{\boldsymbol{\sigma}} - \underline{\epsilon}(\mathbf{u}) = \mathbf{0}$ $\operatorname{div}\underline{\boldsymbol{\sigma}} = \mathbf{f}$ $\underline{\boldsymbol{\sigma}}^t - \underline{\boldsymbol{\sigma}} = \mathbf{0}$
$(c\,\mathbf{q}, \mathbf{v}) + (p, \operatorname{div}\mathbf{v}) = 0$ $(\operatorname{div}\mathbf{q}, w) = (f, w)$	$(D\underline{\boldsymbol{\sigma}}, \underline{\mathbf{v}}) + (\mathbf{u}, \operatorname{div}\underline{\mathbf{v}}) + (\underline{\boldsymbol{\rho}}, \underline{\mathbf{v}}) = 0$ $(\operatorname{div}\underline{\boldsymbol{\sigma}}, \boldsymbol{\omega}) = (\mathbf{f}, \boldsymbol{\omega})$ $(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\eta}}) = 0$
BDM element [Brezzi, Douglas & Marini, 1985] $\mathbf{q} \in \mathcal{P}_{k+1}$ $p \in \mathcal{P}_k$	AFW element [Arnold, Falk & Winther, 2007] $\underline{\boldsymbol{\sigma}} \in \underline{\boldsymbol{\mathcal{P}}}_{k+1}$ $\mathbf{u} \in \boldsymbol{\mathcal{P}}_k$ $\underline{\boldsymbol{\rho}} \in \operatorname{skw}(\underline{\boldsymbol{\mathcal{P}}}_k)$
($\mathcal{P}_k$ = polynomials of degree $\le k$)	($\operatorname{skw}(\cdot)$ = skew-symmetric part)

Two 2nd order systems

Laplace's equation	*Elasticity system*
$c\,\mathbf{q} + \operatorname{grad} p = 0$ $\operatorname{div}\mathbf{q} = f$	$D\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\epsilon}}(\mathbf{u}) = \mathbf{0}$ $\operatorname{div}\underline{\boldsymbol{\sigma}} = \mathbf{f}$ $\underline{\boldsymbol{\sigma}}^t - \underline{\boldsymbol{\sigma}} = \mathbf{0}$
$(c\,\mathbf{q},\mathbf{v}) + (p,\operatorname{div}\mathbf{v}) = 0$ $(\operatorname{div}\mathbf{q}, w) = (f, w)$	$(D\underline{\boldsymbol{\sigma}},\underline{\mathbf{v}}) + (\mathbf{u},\operatorname{div}\underline{\mathbf{v}}) + (\underline{\boldsymbol{\rho}},\underline{\mathbf{v}}) = 0$ $(\operatorname{div}\underline{\boldsymbol{\sigma}},\boldsymbol{\omega}) = (\mathbf{f},\boldsymbol{\omega})$ $(\underline{\boldsymbol{\sigma}},\underline{\boldsymbol{\eta}}) = 0$
BDM element [Brezzi, Douglas & Marini, 1985] $\mathbf{q} \in \boldsymbol{\mathcal{P}}_{k+1}$ $p \in \mathcal{P}_k$ RT element [Nédélec, 1980] $\mathbf{q} \in \boldsymbol{\mathcal{P}}_k + \mathbf{x}\mathcal{P}_k$ $p \in \mathcal{P}_k$	AFW element [Arnold, Falk & Winther, 2007] $\underline{\boldsymbol{\sigma}} \in \underline{\boldsymbol{\mathcal{P}}}_{k+1}$ $\mathbf{u} \in \boldsymbol{\mathcal{P}}_k$ $\underline{\boldsymbol{\rho}} \in \operatorname{skw}(\underline{\boldsymbol{\mathcal{P}}}_k)$??

Two 2nd order systems

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$c\,\mathbf{q} + \operatorname{grad} p = 0$ $\operatorname{div} \mathbf{q} = f$	$D\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\epsilon}}(\mathbf{u}) = \mathbf{0}$ $\operatorname{div} \underline{\boldsymbol{\sigma}} = \mathbf{f}$ $\underline{\boldsymbol{\sigma}}^t - \underline{\boldsymbol{\sigma}} = \mathbf{0}$
$(c\,\mathbf{q}, \mathbf{v}) + (p, \operatorname{div} \mathbf{v}) = 0$ $(\operatorname{div} \mathbf{q}, w) = (f, w)$	$(D\underline{\boldsymbol{\sigma}}, \underline{\mathbf{v}}) + (\mathbf{u}, \operatorname{div} \underline{\mathbf{v}}) + (\underline{\boldsymbol{\rho}}, \underline{\mathbf{v}}) = 0$ $(\operatorname{div} \underline{\boldsymbol{\sigma}}, \boldsymbol{\omega}) = (\mathbf{f}, \boldsymbol{\omega})$ $(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\eta}}) = 0$
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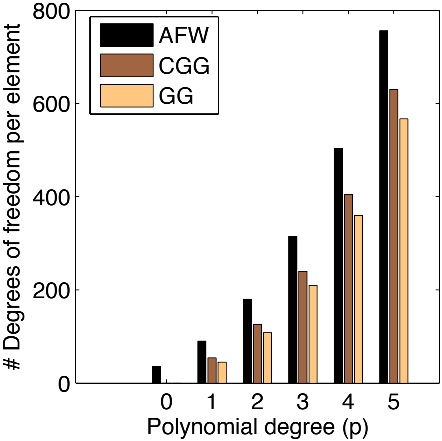
Two new stress elements

The purpose of this talk is to introduce two analogues of the Raviart-Thomas element in elasticity.



Our elements have less unknowns

The first new stress element

(Ciarlet-style) Definition: The finite element (K, V, Σ) is given by

$$
\begin{aligned}
K &= \text{tetrahedron}, && \text{(geometry)}\\
V &= \underline{\mathbf{RT}}_k + \operatorname{curl}\left(\operatorname{curl}\left(\underline{\tilde{\mathbf{A}}}_k\right)\underline{\mathbf{b}}_K\right), && \text{(space)}\\
\Sigma &= \{\ell_{\boldsymbol{\mu}}, \ell_{\underline{\boldsymbol{\nu}}}, \ell_{\underline{\boldsymbol{\eta}}}\}, && \text{(degrees of freedom)}
\end{aligned}
$$

where the degrees of freedom are:

$$
\begin{aligned}
\ell_{\boldsymbol{\mu}}(\underline{\boldsymbol{\sigma}}) &= \int_F \underline{\boldsymbol{\sigma}}\mathbf{n}\cdot\boldsymbol{\mu}, && \forall \boldsymbol{\mu} \in \boldsymbol{\mathcal{P}}_k(F), \quad \forall \text{ faces } F \text{ (unit normal } \mathbf{n}),\\
\ell_{\underline{\boldsymbol{\nu}}}(\underline{\boldsymbol{\sigma}}) &= \int_K \underline{\boldsymbol{\sigma}} : \underline{\boldsymbol{\nu}}, && \forall \underline{\boldsymbol{\nu}} \in \underline{\boldsymbol{\mathcal{P}}}_{k-1}(K),\\
\ell_{\underline{\boldsymbol{\eta}}}(\underline{\boldsymbol{\sigma}}) &= \int_K \underline{\boldsymbol{\sigma}} : \underline{\boldsymbol{\eta}}, && \forall \underline{\boldsymbol{\eta}} \in \underline{\tilde{\mathbf{A}}}_k(K), \qquad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
\underline{\mathbf{RT}}_k &= \text{matrices with rows in Raviart-Thomas space } \boldsymbol{\mathcal{P}}_k + \mathbf{x}\mathcal{P}_k,\\
\underline{\tilde{\mathbf{A}}}_k &= \operatorname{skw}(\underline{\boldsymbol{\mathcal{P}}}_k)/\operatorname{skw}(\underline{\boldsymbol{\mathcal{P}}}_{k-1}),\\
\underline{\mathbf{b}}_K &= \text{“bubble matrix” (not standard bubbles!) defined later.}
\end{aligned}
$$

Local results

Theorem (Unisolvency of the finite element)

The element (K, V, Σ) *is unisolvent.*

Theorem (Weakly symmetric commuting projection)

If $\underline{\boldsymbol{\Pi}}$ *denotes the interpolant of the new finite element, then it satisfies*

$$\operatorname{div} \underline{\boldsymbol{\Pi}}\, \underline{\boldsymbol{\sigma}} = \mathbf{P} \operatorname{div} \underline{\boldsymbol{\sigma}}$$

where $\mathbf{P}$ *is the* $L^2(K)$*-orthogonal projection into* $\mathcal{P}_k(K)$. *Additionally,*

$$(\underline{\boldsymbol{\Pi}}\, \underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\eta}}) = 0,$$

for all skew-symmetric matrices $\underline{\boldsymbol{\eta}}$ *in* $\mathrm{skw}\underline{\mathcal{P}}_k$.

Global error estimates

Let $\underline{\mathbf{P}}$ denote the L^2-orthogonal projection into $\underline{\mathcal{P}}_k$.

Theorem (Optimal rates of convergence)

The error in the discrete solution components are bounded by projection errors as follows:

$$\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\|_{L^2} + \|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h\|_{L^2} \le C\big(\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\Pi}}\,\underline{\boldsymbol{\sigma}}\|_{L^2} + \|\underline{\boldsymbol{\rho}} - \underline{\mathbf{P}}\underline{\boldsymbol{\rho}}\|_{L^2}\big).$$

- Compare with the AFW element:

$$\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\|_{L^2} = O(h^{k+1}) \qquad \leftarrow \text{ok, since}$$

$$\underline{\mathcal{P}}_k \subset \text{ our element } \subsetneq \underline{\mathcal{P}}_{k+1}.$$

$$\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h^{\mathrm{AFW}}\|_{L^2} = O(h^{k+1}) \qquad \leftarrow \text{suboptimal, as AFW uses full } \underline{\mathcal{P}}_{k+1}.$$

- We use AFW's breakthrough analysis in our proofs.

Let $\underline{\mathbf{P}}$ denote the L^2-orthogonal projection into $\underline{\mathcal{P}}_k$.

Theorem (Optimal rates of convergence)

The error in the discrete solution components are bounded by projection errors as follows:

$$\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\|_{L^2} + \|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h\|_{L^2} \le C\big(\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\Pi}}\,\underline{\boldsymbol{\sigma}}\|_{L^2} + \|\underline{\boldsymbol{\rho}} - \underline{\mathbf{P}}\underline{\boldsymbol{\rho}}\|_{L^2}\big).$$

Theorem (Superconvergence via duality)

If, for some $s > 0$, the regularity estimate

$$\|\underline{\boldsymbol{\sigma}}\|_{H^s} + \|\mathbf{u}\|_{H^{1+s}} \le C\|\mathbf{f}\|_{L^2}$$

holds for all $\mathbf{f}$ in $L^2(\Omega)$, then

$$\|\mathbf{P}\mathbf{u} - \mathbf{u}_h\|_{L^2} \le C\, h^s\,\big(\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\Pi}}\,\underline{\boldsymbol{\sigma}}\|_{L^2} + \|\underline{\boldsymbol{\rho}} - \underline{\mathbf{P}}\underline{\boldsymbol{\rho}}\|_{L^2}\big).$$

How did we find the element?

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- These degrees of freedom are essential:

$$(A)\begin{cases} \ell_{\boldsymbol{\mu}}(\underline{\boldsymbol{\sigma}}) = \displaystyle\int_F \underline{\boldsymbol{\sigma}}\mathbf{n}\cdot\boldsymbol{\mu}, & \forall \boldsymbol{\mu} \in \mathcal{P}_k(F), \quad \forall \text{ faces } F, \\ \ell_{\underline{\boldsymbol{\nu}}}(\underline{\boldsymbol{\sigma}}) = \displaystyle\int_K \underline{\boldsymbol{\sigma}} : \underline{\boldsymbol{\nu}}, & \forall \underline{\boldsymbol{\nu}} \in \underline{\mathcal{P}}_{k-1}(K), \end{cases}$$

$$(B)\left\{ \ell_{\underline{\boldsymbol{\eta}}}(\underline{\boldsymbol{\sigma}}) = \int_K \underline{\boldsymbol{\sigma}} : \underline{\boldsymbol{\eta}}, \qquad \forall \underline{\boldsymbol{\eta}} \in \underline{\tilde{\mathbf{A}}}_k(K). \right.$$

  - (A) is needed for commutativity.
  - (B) is needed for weak symmetry.

How did we find the element?

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$$
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$$

$$
(B)\left\{ \ell_{\underline{\boldsymbol{\eta}}}(\underline{\boldsymbol{\sigma}}) = \int_K \underline{\boldsymbol{\sigma}} : \underline{\boldsymbol{\eta}}, \qquad \forall \underline{\boldsymbol{\eta}} \in \underline{\tilde{\mathbf{A}}}_k(K).\right.
$$

  - (A) is needed for commutativity.
  - (B) is needed for weak symmetry.

- By standard arguments, (A) controls $\underline{\mathbf{RT}}_k$. So we are motivated to add *just enough* functions $\underline{\boldsymbol{\sigma}}$ with

$$
\operatorname{div}\underline{\boldsymbol{\sigma}} = 0, \quad \underline{\boldsymbol{\sigma}}\mathbf{n}|_F = 0,
$$

to allow d.o.f.s in (B). This is the "$\operatorname{curl}(\operatorname{curl}(\underline{\tilde{\mathbf{A}}}_k)\,\underline{\mathbf{b}}_K)$"-component.

The bubble matrix

Definition of the bubble matrix:

$$\underline{\mathbf{b}}_K = \sum_{\ell=0}^{3} \lambda_{\ell-3}\lambda_{\ell-2}\lambda_{\ell-1} \,(\nabla\lambda_\ell)\,(\nabla\lambda_\ell)^t.$$

- λ_ℓ denotes the barycentric coordinates.
- Indices on λ's are calculated mod 4.
- $\underline{\mathbf{b}}_K$ is the sum of four rank-one matrices.

Lemma

$$\forall \underline{\psi} \in \underline{\mathbf{RT}}^k(K): \quad \left.\begin{array}{c} \operatorname{div} \underline{\psi} = 0, \\ \underline{\psi}\, \mathbf{n}|_{\partial K} = 0 \end{array}\right\} \iff \left\{\begin{array}{c} \exists\, \underline{\mathbf{v}} \in \underline{\mathcal{P}}^{k-1}(K) \textit{ such that} \\ \underline{\psi} = \operatorname{curl}(\operatorname{curl}(\underline{\mathbf{v}})\underline{\mathbf{b}}_K). \end{array}\right.$$

Where did this $\underline{\mathbf{b}}_K$ come from. . . ? Motivation →

The bubble matrix

- Let $\mathbf{v} \in \mathcal{P}_{k+1}$. Then [G, García-Castillo & Demkowicz, 2005]

$$\operatorname{div} \mathbf{v} = 0, \quad \mathbf{v} \cdot \mathbf{n}|_{\partial K} = 0 \quad \iff \quad \mathbf{v} = \operatorname{curl}(p_{k-1} \mathbf{w}_{ij})$$

for some p_{k-1} in $\mathcal{P}_{k-1}$ and a higher order Whitney form

$$\mathbf{w}_{ij} = \lambda_{i-3}\lambda_{i-2}\lambda_{i-1}\nabla\lambda_i - \lambda_{j-3}\lambda_{j-2}\lambda_{j-1}\nabla\lambda_j.$$

The bubble matrix

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$$\mathbf{w}_{ij} = \lambda_{i-3}\lambda_{i-2}\lambda_{i-1}\nabla\lambda_i - \lambda_{j-3}\lambda_{j-2}\lambda_{j-1}\nabla\lambda_j.$$

- But $\mathbf{w}_{ij}$ can be generated as follows:

$$\mathbf{w}_{12} = \alpha\left(\quad \lambda_2\lambda_3\lambda_0\nabla\lambda_1(\nabla\lambda_1)^t + \lambda_3\lambda_0\lambda_1\nabla\lambda_2(\nabla\lambda_2)^t \quad \right) \cdot (\nabla\lambda_3 \times \nabla\lambda_0)$$

The bubble matrix

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The bubble matrix

- Let $\mathbf{v} \in \mathcal{P}_{k+1}$. Then [G, García-Castillo & Demkowicz, 2005]

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Terms in blue define the bubble matrix $\underline{\mathbf{b}}_K$.

The bubble matrix

- Let $\mathbf{v} \in \mathcal{P}_{k+1}$. Then [G, García-Castillo & Demkowicz, 2005]

$$\operatorname{div} \mathbf{v} = 0, \quad \mathbf{v} \cdot \mathbf{n}|_{\partial K} = 0 \quad \iff \quad \mathbf{v} = \operatorname{curl}(p_{k-1}\mathbf{w}_{ij})$$

for some p_{k-1} in $\mathcal{P}_{k-1}$ and a higher order Whitney form

$$\mathbf{w}_{ij} = \lambda_{i-3}\lambda_{i-2}\lambda_{i-1}\nabla\lambda_i - \lambda_{j-3}\lambda_{j-2}\lambda_{j-1}\nabla\lambda_j.$$

- But $\mathbf{w}_{ij}$ can be generated as follows:

$$\mathbf{w}_{12} = \alpha\Big(\lambda_2\lambda_3\lambda_0\nabla\lambda_1(\nabla\lambda_1)^t + \lambda_3\lambda_0\lambda_1\nabla\lambda_2(\nabla\lambda_2)^t + \lambda_0\lambda_1\lambda_2\nabla\lambda_3(\nabla\lambda_3)^t + \lambda_1\lambda_2\lambda_3\nabla\lambda_0(\nabla\lambda_0)^t \Big)\cdot(\nabla\lambda_3 \times \nabla\lambda_0)$$

Terms in blue define the bubble matrix $\underline{\mathbf{b}}_K$.

- Thus, $\mathbf{v} = \operatorname{curl}(\underline{\mathbf{b}}_K \mathbf{r}_{k-1})$ for some vector polynomial $\mathbf{r}_{k-1}$ in $\mathcal{P}_{k-1}$.

This motivates the introduction of the bubble matrix in the element.

Some implementation aspects

- The new element can be implemented using standard finite element techniques. (Mappings are available for affine elements.)
- Hybridization can be used to reduce to a small symmetric positive definite system.
- Postprocessing techniques can be used to enhance accuracy.

Hybridization

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- Standard implementation results in a large indefinite system.
- We recommend hybridized implementation:

$$
\begin{aligned}
(D\underline{\boldsymbol{\sigma}}^h, \underline{\mathbf{v}})_{\Omega_h} \;+ (\mathbf{u}^h, \operatorname{div} \underline{\mathbf{v}})_{\Omega_h} + (\underline{\boldsymbol{\rho}}^h, \underline{\mathbf{v}})_{\Omega_h} + \langle \boldsymbol{\lambda}^h, \underline{\mathbf{v}}\mathbf{n}\rangle_{\partial\Omega_h} &= 0,\\
(\operatorname{div} \underline{\boldsymbol{\sigma}}^h, \boldsymbol{\omega})_{\Omega_h} &= (\mathbf{f}, \boldsymbol{\omega})_{\Omega_h},\\
(\underline{\boldsymbol{\sigma}}^h, \underline{\boldsymbol{\eta}})_{\Omega_h} &= 0,\\
\langle \underline{\boldsymbol{\sigma}}^h \mathbf{n}, \boldsymbol{\mu}\rangle_{\partial\Omega_h} &= 0,
\end{aligned}
$$

Notations: Differential operators are applied piecewise, and

$$
(\omega, v)_{\Omega_h} = \sum_{K\in\Omega_h} \int_K \omega v, \qquad \langle \omega, v\rangle_{\partial\Omega_h} = \sum_{K\in\Omega_h} \int_{\partial K} \omega v.
$$

The new functions $\boldsymbol{\lambda}^h$ and $\boldsymbol{\mu}$ are on mesh *faces* and are in

$$
\mathbf{M}^h = \{\boldsymbol{\mu} : \boldsymbol{\mu}|_F \in \mathcal{P}^k(F),\ \forall \text{ mesh faces } F, \boldsymbol{\mu}|_{\partial\Omega} = 0\}.
$$

Hybridization

- Standard implementation results in a large indefinite system.
- We recommend hybridized implementation:

$$
\begin{aligned}
(D\underline{\boldsymbol{\sigma}}^h, \underline{\mathbf{v}})_{\Omega_h} + (\mathbf{u}^h, \operatorname{div} \underline{\mathbf{v}})_{\Omega_h} + (\underline{\boldsymbol{\rho}}^h, \underline{\mathbf{v}})_{\Omega_h} + \langle \boldsymbol{\lambda}^h, \underline{\mathbf{v}}\mathbf{n} \rangle_{\partial\Omega_h} &= 0, \\
(\operatorname{div} \underline{\boldsymbol{\sigma}}^h, \boldsymbol{\omega})_{\Omega_h} &= (\mathbf{f}, \boldsymbol{\omega})_{\Omega_h}, \\
(\underline{\boldsymbol{\sigma}}^h, \underline{\boldsymbol{\eta}})_{\Omega_h} &= 0, \\
\langle \underline{\boldsymbol{\sigma}}^h \mathbf{n}, \boldsymbol{\mu} \rangle_{\partial\Omega_h} &= 0,
\end{aligned}
$$

Theorem (Elimination of all interior element unknowns)

There are locally computable, mesh-dependent, forms $a_h(\boldsymbol{\mu}, \gamma)$ and $b_h(\boldsymbol{\mu})$ such that $\boldsymbol{\lambda}^h$ is the unique solution of

$$
a_h(\boldsymbol{\lambda}^h, \boldsymbol{\mu}) = b_h(\boldsymbol{\mu}), \qquad \text{for all } \boldsymbol{\mu} \in \mathbf{M}^h.
$$

The functions $\underline{\boldsymbol{\sigma}}^h, \mathbf{u}^h, \underline{\boldsymbol{\rho}}^h$ can be computed locally (element by element) once $\boldsymbol{\lambda}^h$ is found.

Postprocessing

We can use Stenberg's postprocessing: [Stenberg, 1988]

Compute $\mathbf{u}^{h,\star}$ (element by element) in $\mathcal{P}^{k+1}(K)$ satisfying

$$\begin{aligned} (\mathrm{grad}\,\mathbf{u}^{h,\star}, \mathrm{grad}\,\boldsymbol{\omega})_K &= (D\underline{\boldsymbol{\sigma}}^h + \underline{\boldsymbol{\rho}}^h, \mathrm{grad}\,\boldsymbol{\omega})_K, && \forall \boldsymbol{\omega} \in \mathcal{P}^{k+1}(K)/\mathcal{P}^k(K), \\ (\mathbf{u}^{h,\star}, \mathbf{w})_K &= (\mathbf{u}^h, \mathbf{w})_K, && \forall \mathbf{w} \in \mathcal{P}^k(K). \end{aligned}$$

Theorem (Enhanced accuracy by postprocessing)

Suppose $\mathbf{u}$ *is in* $\mathbf{H}^{k+2}(\Omega)$ *and the full regularity assumption holds. Then*

$$\|\mathbf{u} - \mathbf{u}^{h,\star}\|_{L^2(\Omega)} \le C\, h^{k+2}\, |\mathbf{u}|_{H^{k+2}(\Omega)}.$$

Second new stress element

$$K = \text{tetrahedron}, \qquad \text{(geometry)}$$

$$V = \underline{\mathcal{P}}^k + \operatorname{curl}(\operatorname{curl}(\underline{\tilde{\mathbf{A}}}_k)\,\underline{\mathbf{b}}_K), \qquad \text{(space)}$$

$$\Sigma = \{\ell_{\boldsymbol{\mu}}, \ell_{\underline{\boldsymbol{\nu}}}, \ell_{\underline{\boldsymbol{\eta}}}\}, \qquad \text{(degrees of freedom)}$$

where the degrees of freedom are:

$$\ell_{\boldsymbol{\mu}}(\underline{\boldsymbol{\sigma}}) = \int_F \underline{\boldsymbol{\sigma}}\mathbf{n}\cdot\boldsymbol{\mu}, \quad \forall \boldsymbol{\mu} \in \mathcal{P}_k(F), \quad \forall \text{ faces } F \text{ (unit normal } \mathbf{n}),$$

$$\ell_{\underline{\boldsymbol{\nu}}}(\underline{\boldsymbol{\sigma}}) = \int_K \underline{\boldsymbol{\sigma}} : \underline{\boldsymbol{\nu}}, \quad \forall \underline{\boldsymbol{\nu}} \in \underline{\mathbf{N}}^{k-1}(K) \equiv \text{Nédélec space of the first kind}$$

$$\equiv \underline{\mathcal{P}}^{\ell-1}(K) + \underline{\mathbf{S}}^{\ell}(K)$$

$$\ell_{\underline{\boldsymbol{\eta}}}(\underline{\boldsymbol{\sigma}}) = \int_K \underline{\boldsymbol{\sigma}} : \underline{\boldsymbol{\eta}}, \quad \forall \underline{\boldsymbol{\eta}} \in \underline{\tilde{\mathbf{A}}}_k(K).$$

Here, as before, $\underline{\tilde{\mathbf{A}}}_k = \operatorname{skw}(\underline{\mathcal{P}}_k)/\operatorname{skw}(\underline{\mathcal{P}}_{k-1})$ and $\underline{\mathbf{b}}_K$ = the bubble matrix.
Changes with the first element are highlighted in red.

Results for the second element

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- The second element is unisolvent.
- Like the first element, its interpolant $\underline{\boldsymbol{\Pi}}_2$ commutes:

$$\operatorname{div}(\underline{\boldsymbol{\Pi}}_2\ \underline{\boldsymbol{\sigma}}) = \mathbf{P}(\operatorname{div}\underline{\boldsymbol{\sigma}}).$$

- Unlike the first element, its interpolant is not weakly symmetric always.
- But when stress divergence is discrete, it has a weak symmetry.
- Thus we are able to push through an error analysis and obtain optimal error estimates:

$$\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\|_{L^2} + \|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h\|_{L^2} \le C\left(\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\Pi}}_2\,\underline{\boldsymbol{\sigma}}\|_{L^2} + \|\underline{\boldsymbol{\rho}} - \underline{\mathbf{P}}\,\underline{\boldsymbol{\rho}}\|_{L^2}\right).$$

- The second method can be hybridized, just like the first.
- A superconvergence estimate for a projection of displacement error can be proved.
- The postprocessing can be used to enhance displacement accuracy.

Comparison of some 3D mixed elements

	$\mathbf{W}(K)$ (displacements)	$\underline{\mathbf{V}}(K)$ (stresses)	$\underline{\mathbf{A}}(K)$ (rotations)	
[Stenberg 1988]				
	$\mathcal{P}^{k-1}(K)$	$\underline{\mathcal{P}}^k(K) + \operatorname{curl}(\underline{\mathcal{P}}^{k-1}(K)\lambda_0\lambda_1\lambda_2\lambda_3)$	$\operatorname{skw}\underline{\mathcal{P}}^k(K)$	$k \geq 1$
[Arnold, Brezzi & Douglas 1984]			continuous	
PEERS	$\mathcal{P}^0(K)$	$\underline{\mathbf{RT}}^0(K) + \operatorname{curl}(\underline{\mathcal{P}}^0(K)\lambda_0\lambda_1\lambda_2\lambda_3)$	$\operatorname{skw}\underline{\mathcal{P}}^1(K)$	$k = 0$
[Arnold, Falk & Winther 2007]				
AFW	$\mathcal{P}^k(K)$	$\underline{\mathcal{P}}^{k+1}(K)$	$\operatorname{skw}\underline{\mathcal{P}}^k(K)$	$k \geq 0$
Our first element				
	$\mathcal{P}^k(K)$	$\underline{\mathbf{RT}}^k(K) + \operatorname{curl}((\operatorname{curl}\underline{\tilde{\mathbf{A}}}^k(K))\underline{\mathbf{b}}_K)$	$\operatorname{skw}\underline{\mathcal{P}}^k(K)$	$k \geq 1$
Our second element				
	$\mathcal{P}^{k-1}(K)$	$\underline{\mathcal{P}}^k(K) + \operatorname{curl}((\operatorname{curl}\underline{\tilde{\mathbf{A}}}^k(K))\underline{\mathbf{b}}_K)$	$\operatorname{skw}\underline{\mathcal{P}}^k(K)$	$k \geq 1$

Comparison of some 3D mixed elements

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	$\mathbf{W}(K)$ (displacements)	$\underline{\mathbf{V}}(K)$ (stresses)	$\underline{\mathbf{A}}(K)$ (rotations)	
[Stenberg 1988]	$\mathcal{P}^{k-1}(K)$	$\underline{\mathcal{P}}^k(K) + \mathrm{curl}\,(\underline{\mathcal{P}}^{k-1}(K)\lambda_0\lambda_1\lambda_2\lambda_3)$	$\mathrm{skw}\underline{\mathcal{P}}^k(K)$	$k \geq 1$
[Arnold, Brezzi & Douglas 1984] PEERS	$\mathcal{P}^0(K)$	$\underline{\mathbf{RT}}^0(K) + \mathrm{curl}\,(\underline{\mathcal{P}}^0(K)\lambda_0\lambda_1\lambda_2\lambda_3)$	continuous $\mathrm{skw}\underline{\mathcal{P}}^1(K)$	$k = 0$
[Arnold, Falk & Winther 2007] AFW	$\mathcal{P}^k(K)$	$\underline{\mathcal{P}}^{k+1}(K)$	$\mathrm{skw}\underline{\mathcal{P}}^k(K)$	$k \geq 0$
Our first element	$\mathcal{P}^k(K)$	$\underline{\mathbf{RT}}^k(K) + \mathrm{curl}\,((\mathrm{curl}\,\underline{\tilde{\mathbf{A}}}^k(K))\underline{\mathbf{b}}_K)$	$\mathrm{skw}\underline{\mathcal{P}}^k(K)$	$k \geq 1$
Our second element	$\mathcal{P}^{k-1}(K)$	$\underline{\mathcal{P}}^k(K) + \mathrm{curl}\,((\mathrm{curl}\,\underline{\tilde{\mathbf{A}}}^k(K))\underline{\mathbf{b}}_K)$	$\mathrm{skw}\underline{\mathcal{P}}^k(K)$	$k \geq 1$
On special grids	$\mathcal{P}^{k-1}(K)$	$\underline{\mathcal{P}}^k(K)$	$\mathrm{skw}\underline{\mathcal{P}}^k(K)$	$k \geq 2$

Hseih-Clough-Toucher grids

Assumption (Hseih-Clough-Toucher grids)

In either the 2D (triangular) or the 3D (tetrahedral) case, assume:

1. *The mesh is obtained from a quasiuniform mesh after splitting each of its elements into four elements by connecting the vertices of the element to its barycenter.*
2. *In the 2D case, assume $k \geq 1$.*
3. *In the 3D case, assume $k \geq 2$.*

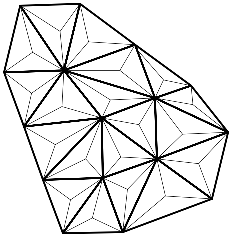

Figure from [Arnold & Qin 1992]

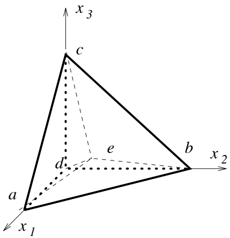

Figure from [Zhang 2005]

Serendipitous exact symmetry

Theorem (Weak symmetry can yield exact symmetry)

For such grids, the mixed method with weakly imposed symmetry using

$$\mathbf{W}(K) = \mathcal{P}^{k-1}(K), \qquad \underline{\mathbf{V}}(K) = \underline{\mathcal{P}}^{k}(K), \qquad \underline{\mathbf{A}}(K) = \mathrm{skw}\underline{\mathcal{P}}^{k}(K),$$

is stable for $k \geq 2$ (and yields optimal error estimates). Moreover, its stress approximation $\underline{\boldsymbol{\sigma}}^h$ is exactly symmetric.

Thanks: Professor Rick Falk

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is stable for $k \ge 2$ (and yields optimal error estimates). Moreover, its stress approximation $\underline{\boldsymbol{\sigma}}^h$ is exactly symmetric.

- We prove this theorem by exploiting a connection between the Stokes and elasticity systems.
- We use the fact that the "$\mathbf{P}_{k+1}$-P_k" element is a stable Stokes element, as proved by [Arnold & Qin 1992] (2D) and [Zhang 2005] (3D).
- Further examples of grids where this phenomena occur can be found: It is enough to avoid certain "singular" mesh objects identified in the early papers of [Scott & Vogelius 1985] and [Vogelius 1983].

Conclusion

- We presented new families of triangular and tetrahedral stress elements for mixed methods in linear elasticity.
- They have lesser degrees of freedom than other comparable elements.
- They have better symmetry properties.
- They yield optimal error estimates.
- The solution can be obtained by solving one global symmetric positive definite system and additional local operations (due to hybridizability).
- Postprocessing techniques are available to increase displacement accuracy (when solution is regular).
- There are simple meshes where weakly symmetric methods serendipitously yield exactly symmetric numerical stresses.