Multigrid convergence for some axisymmetric problems

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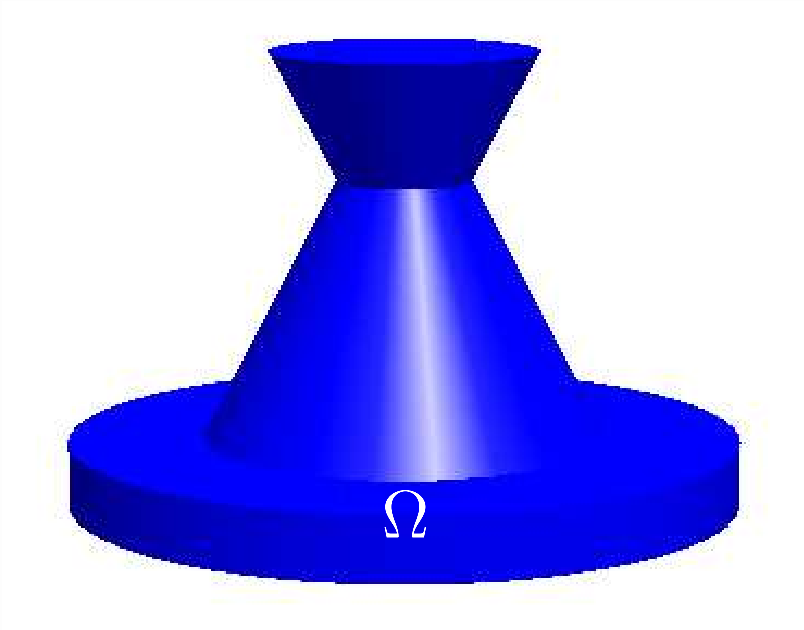
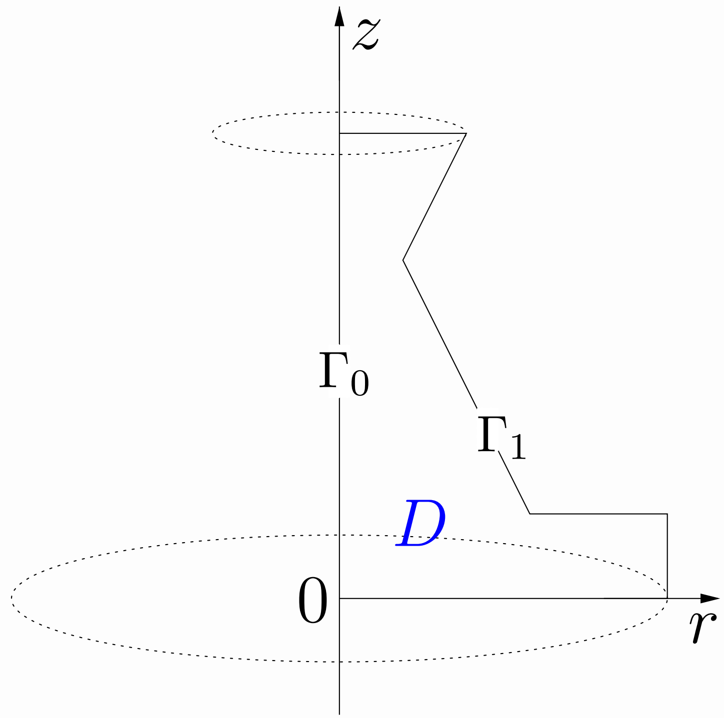
Thanks: NSF

Axial symmetry





$$\Omega \longrightarrow D$$

- Significant computational savings (3D to 2D domain).
- But numerical analyses face difficulties due to Γ_0.

Axisymmetric Laplace equation

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<u>Dirichlet problem on Ω</u>

$$-\Delta U = f, \quad \text{on } \Omega$$
$$U = 0, \quad \text{on } \partial\Omega$$

(f is axisymmetric.)

<u>Reduced problem on D</u>

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{\partial^2 u}{\partial z^2} = f, \quad \text{on } D$$
$$u = 0, \quad \text{on } \Gamma_1$$
$$??? \quad \text{on } \Gamma_0$$

Axisymmetric Laplace equation

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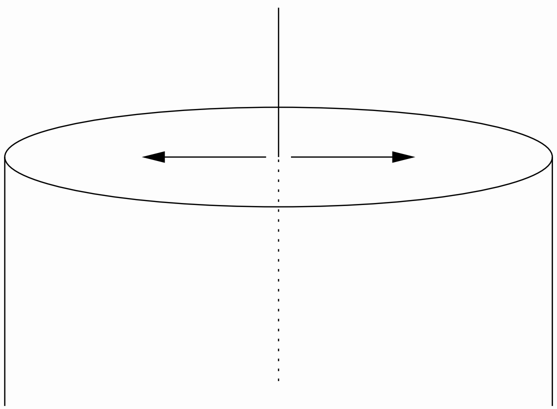
For smooth functions ϕ, since $\partial_r\phi$ is an even function of r,



$$\partial_r\phi|_{\Gamma_0} = 0$$

Axisymmetric Laplace equation

Dirichlet problem on Ω	Reduced problem on D
$$\begin{aligned} -\Delta U &= f, \quad \text{on } \Omega \\ U &= 0, \quad \text{on } \partial\Omega \end{aligned}$$ (f is axisymmetric.)	$$\begin{aligned} -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{\partial^2 u}{\partial z^2} &= f, \quad \text{on } D \\ u &= 0, \quad \text{on } \Gamma_1 \\ \partial_r u &= 0, \quad \text{on } \Gamma_0 \end{aligned}$$

Weak formulation: Find $u \in V$ such that

$$\int_D r(\partial_r u)(\partial_r v) + r(\partial_z u)(\partial_z v)\, \mathrm{d}r\mathrm{d}z = \int_D f v\, r\, \mathrm{d}r\mathrm{d}z$$

$\forall v \in V$.

Axisymmetric Laplace equation

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Weak formulation: Find $u \in V$ such that

$$\int_D r(\partial_r u)(\partial_r v) + r(\partial_z u)(\partial_z v)\,\mathrm{d}r\mathrm{d}z = \int_D f v\, r\,\mathrm{d}r\mathrm{d}z$$

$$\forall v \in V \equiv \{\underbrace{w \in L^2_r(D) : \partial_r w, \partial_z w \in L^2_r(D)}_{H^1_r(D)}, w|_{\Gamma_1} = 0\}.$$

Bilinear elements

Mesh D by square elements. V_h = bilinear F.E. subspace of V.
Exact solution $u \in V$:

$$a_r(u, v) = (f, v)_r, \qquad \forall v \in V.$$

Finite element approximation $u_h \in V_h$:

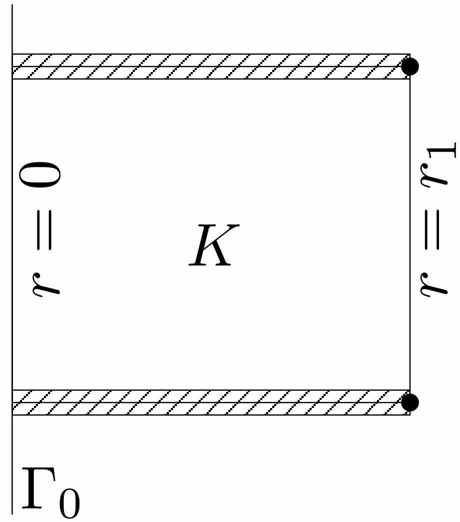
$$a_r(u_h, v_h) = (f, v_h)_r, \qquad \forall v_h \in V_h.$$

Error analysis:

$$
\begin{aligned}
|u - u_h|_{H^1_r(D)} &\le \inf_{v_h \in V_h} |u - v_h|_{H^1_r(D)} && \text{(standard)} \\
&\le |u - \Pi_h u|_{H^1_r(D)} && \text{(non-standard } \Pi_h\text{)} \\
&\le Ch|u|_{H^2_r(D)} \le Ch\|f\|_{L^2_r(D)} && \text{(regularity).}
\end{aligned}
$$

The interpolation operator

To overcome the problem that standard nodal interpolation operator is ill-defined on $H^2_r(D)$, we used the following Π_h:



On elements K intersecting Γ_0, $(\Pi_h v)|_K$ is defined to be the unique bilinear function satisfying

$$(\Pi_h v)(r_1, z_0) = v(r_1, z_0),$$

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$$\int_0^{r_1} \rho^{1/2} (\partial_r \Pi_h v)(\rho, z_0)\, \mathrm{d}\rho = \int_0^{r_1} \rho^{1/2} \partial_r v(\rho, z_0)\, \mathrm{d}\rho,$$

$$\int_0^{r_1} \rho^{1/2} (\partial_r \Pi_h v)(\rho, z_1)\, \mathrm{d}\rho = \int_0^{r_1} \rho^{1/2} \partial_r v(\rho, z_1)\, \mathrm{d}\rho.$$

Background

To solve the resulting linear system, consider using multigrid methods. Multigrid methods for problems with singular coefficients have been studied previously:

- For $$-\frac{\partial}{\partial x}\left(a(x,y)\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(b(x,y)\frac{\partial u}{\partial y}\right) = f,$$ with $a, b > 0$, $a = O(1)$, and b arbitrarily close to zero, V-cycle with line relaxations converges at a uniform rate [Bramble & Zhang, 2000], [Neuss, 1998].

- Our study is motivated by [Börm & Hiptmair, 2001]: They analyzed multigrid using both line relaxations and semicoarsening for the Laplace equation in polar coordinates on planar domains.

Convergence of V-cycle

Because of the singular coefficients in our axisymmetric equation

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{\partial^2 u}{\partial z^2} = f$$

multigrid algorithms with *line relaxations* and *semicoarsening* are usually suggested.

We prove that *neither is necessary*:

THEOREM. *Under the regularity assumption, the standard multigrid V-cycle (with point Gauß-Seidel or Jacobi smoother) converges at a rate independent of mesh sizes.*

The proof proceeds by using our previous fi nite element estimate in weighted norms together with the abstract regularity based multigrid theory.

Azimuthal Maxwell equations

3-D problem on Ω

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} = \boldsymbol{f}, \text{ on } \Omega,$$
$$\operatorname{div} \boldsymbol{u} = 0, \text{ on } \Omega,$$
$$\boldsymbol{n} \times \boldsymbol{u} = 0, \text{ on } \partial\Omega.$$

Azimuthal case:

$$\boldsymbol{f} = f(r, z)\boldsymbol{e}_\theta.$$

Reduced problem on D

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Reduced problem on D

$$\langle \boldsymbol{e}_\theta \rangle \xrightarrow{\mathbf{curl}} \langle \boldsymbol{e}_r, \boldsymbol{e}_z \rangle \xrightarrow{\mathbf{curl}} \langle \boldsymbol{e}_\theta \rangle$$

Hence we seek $\boldsymbol{u}$ in the form

$$u(r,z)\boldsymbol{e}_\theta.$$

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Azimuthal case:

$$\boldsymbol{f} = f(r,z)\boldsymbol{e}_\theta.$$

Reduced problem on D

$$-\partial_r\left(\frac{1}{r}\partial_r(ru)\right) - \partial_{zz}u = f, \text{ on } D$$
$$u = 0, \text{ on } \Gamma_1$$
$$u = 0, \text{ on } \Gamma_0$$

$$\langle \boldsymbol{e}_\theta\rangle \xrightarrow{\mathbf{curl}} \langle \boldsymbol{e}_r, \boldsymbol{e}_z\rangle \xrightarrow{\mathbf{curl}} \langle \boldsymbol{e}_\theta\rangle$$

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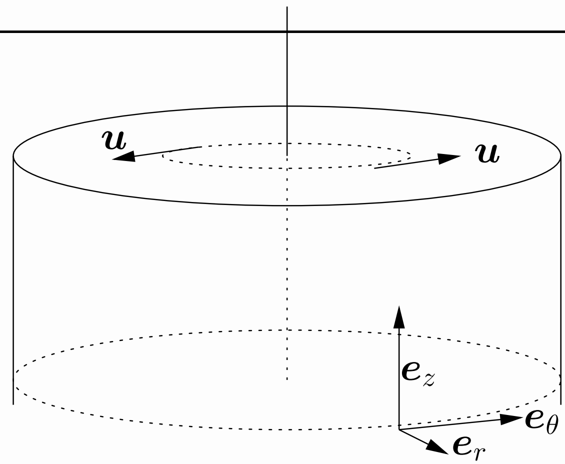

Weak formulation

Multiply by r and a test function v and integrate by parts:

$$\underbrace{\int_D \frac{1}{r}\partial_r(ru)\partial_r(rv) + r(\partial_z u)(\partial_z v)\, \mathrm{d}r\mathrm{d}z}_{a_\theta(u,v)} = \int_D f v\, r\, \mathrm{d}r\mathrm{d}z$$

THEOREM. *The form $a_\theta(u,v)$ is coercive and continuous on the space*

$$V^\theta = \{v \in H^1_r(D) \cap L^2_{1/r}(D) :\; v|_{\partial D} = 0\}.$$

Hence there is a unique solution to the weak formulation.

Note that

$$\frac{1}{r}\partial_r(ru) = \partial_r u + \frac{u}{r}$$

exists in $L^2_r(D)$ if both $\partial_r u$ and u/r are in $L^2_r(D)$. Hence the space V^θ.

Try bilinear elements

Mesh D by squares. Set $V_h^\theta =$ bilinear finite element subspace of V^θ. Let Π_h denote the nodal interpolant.

THEOREM. *For all smooth v,*

$$\|v - \Pi_h v\|_{a_\theta} \le Ch \left[\|\partial_{rr} v\|_{L^2_r(D)} + |\partial_z v|_{H^1_r(D)} + \left| \frac{1}{r} \partial_r (rv) \right|_{H^1_r(D)} \right]$$

Try bilinear elements

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But can we control the r.h.s. by data ?!

Even with full regularity, the Maxwell solution $\boldsymbol{u}$ is only so regular:

$$\boldsymbol{u} \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$$

$$\Longrightarrow \quad \partial_z v \quad \text{and} \quad \frac{1}{r}\partial_r(rv) \in H^1_r(D).$$

Non-polynomial elements

Hence, we consider an alternate finite element space:

$$\widetilde{V}_h^\theta = \{v_h \in V^\theta : v_h|_K \in \langle r, \frac{1}{r}, zr, \frac{z}{r}\rangle, \forall \text{elements } K\}.$$

The nodal interpolant $\widetilde{\Pi}_h^\theta$ of this space now satisfies an estimate without the offending term $\|\partial_{rr} v\|_{L^2_r(D)}$:

THEOREM. *For all smooth v,*

$$\|v - \widetilde{\Pi}_h^\theta v\|_{a_\theta} \le Ch\left[|\partial_z v|_{H^1_r(D)} + \left|\frac{1}{r}\partial_r(rv)\right|_{H^1_r(D)} \right].$$

COROLLARY. *The finite element approximation $\widetilde{u}_h \in \widetilde{V}_h^\theta$ satisfies the error estimate* $\|u - \widetilde{u}_h\|_{a_\theta} \le Ch\|f\|_{L^2_r(D)}$.

Return to bilinear elements

Nonetheless, in all our numerical experiments, the bilinear elements performed as well as the non-polynomial elements:

- It gave optimal finite element convergence rates.
- It gave optimal V-cycle convergence rates.

But a theory was missing . . .

Since we know that for the nodal bilinear interpolant Π_h,

$$\|u-\Pi_h u\|_{a_\theta} \le Ch\left[\|\partial_{rr}u\|_{L^2_r(D)} + |\partial_z u|_{H^1_r(D)} + \left|\frac{1}{r}\partial_r(ru)\right|_{H^1_r(D)}\right]$$

perhaps this term is controllable after all . . .

A regularity estimate

THEOREM. *If u solves the weak formulation of the azimuthal Maxwell equation, then*

$$\|\partial_{rr} u\|_{L^2_r(D)} + \|\partial_r(\frac{u}{r})\|_{L^2_r(D)} \le C\|f\|_{L^2_r(D)}.$$

The proof proceeds by applying a Hardy inequality. One delicate issue is that although

$$\partial_r(\frac{u}{r}) = \frac{\partial_r u}{r} - \frac{u}{r^2},$$

is in $L^2_r(D)$, the two terms on the right are not separately in $L^2_r(D)$ in general.

COROLLARY. *If u_h is the bilinear finite element approximation of the solution u of the azimuthal Maxwell problem, then*

$$\|u - u_h\|_{a_\theta} \le Ch\|f\|_{L^2_r(D)}.$$

Multigrid convergence

The only known previous results on multigrid convergence for the azimuthal Maxwell equation was for algorithms using line smoothings and semicoarsening.

Using our new approximation and regularity results, we show that neither is necessary:

THEOREM. *For the azimuthal Maxwell equation, under the regularity assumption, the standard multigrid V-cycle (with point Gauß-Seidel or Jacobi smoother) converges at a rate independent of mesh sizes for both*

- *the bilinear finite element discretization and*
- *the non-polynomial finite element discretization.*

Numerical results

J	Bilinear elements				$\langle r, 1/r, z, z/r\rangle$-elements	
	Laplace equation		Maxwell equation		Maxwell equation	
	$\kappa(B_J A_J)$	$\|I - B_J A_J\|_{a_r}$	$\kappa(B_J A_J)$	$\|I - B_J A_J\|_{a_\theta}$	$\kappa(B_J A_J)$	$\|I - B_J A_J\|_{a_\theta}$
2	1.13	0.12	1.08	0.08	1.08	0.08
3	1.19	0.16	1.17	0.14	1.17	0.14
4	1.20	0.17	1.20	0.17	1.20	0.17
5	1.21	0.17	1.21	0.17	1.21	0.17
6	1.21	0.17	1.21	0.17	1.21	0.17
7	1.21	0.17	1.21	0.17	1.21	0.17
8	1.21	0.17	1.21	0.17	1.21	0.17
9	1.21	0.17	1.21	0.17	1.21	0.17
10	1.21	0.17	1.21	0.17	1.21	0.17

Conclusion

- We proved simple finite element convergence estimates for two boundary values problems:
  - axisymmetric Laplace equation (bilinear elements),
  - azimuthal Maxwell equation (two different elements).
- We proved uniform convergence of V-cycle multgrid algorithm for both these problems.
  - Only point smoothings are needed, line smoothings are not necessary.
- Work in progress:
  - The meridian Maxwell problem.
  - Axisymmetric PML.
  - Multigrid algorithms for these.