A TENT PITCHING SCHEME
MOTIVATED BY FRIEDRICHS THEORY

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Abstract. Certain Friedrichs systems can be posed on Hilbert spaces normed with a graph norm. Functions in such spaces arising from advective problems are found to have traces with a weak continuity property at points where the inflow and outflow boundaries meet. Motivated by this continuity property, an explicit space-time finite element scheme of the tent pitching type, with spaces that conform to the continuity property, is designed. Numerical results for a model one-dimensional wave propagation problem are presented.

1. Introduction

A commonly used approach for constructing numerical methods to solve time-dependent problems is based on the method of lines, where a discretization of all space derivatives is followed by a discretization of time derivatives. The resulting methods are called implicit or explicit depending on whether one can advance in time with or without solving a spatially global problem. The study in this paper targets a different class of methods referred to as *locally implicit* space-time finite element methods, which advance in time using calculations that are local within space-time regions of simulation. Examples of such methods are provided by "tent pitching" schemes, which mesh the space-time region using tent-shaped subdomains and advance in time by varying amounts at different points in space.

Ideas to advance a numerical solution in time by local operations in space time regions were explored even as early as [20]. Recurrence relations on multiple slabs of rectangular space-time elements were considered in [14], whose ideas were generalized to non-rectangular space-time elements for beams and plates in [2]. These works are not so related to the current work as some of the more modern references. Closest in ancestry to the method we shall consider is found in [22] where it was called explicit space-time elements. The space-time discontinuous Galerkin (SDG) method was announced almost at the same time in [17] and continues to see active development [18, 21, 28]. Against this backdrop, we highlight two papers that brought tent pitching ideas into the numerical analysis community [10, 19]. The questions we choose to ask in this work have been heavily influenced by these two works. We should note that the name "tent pitching"

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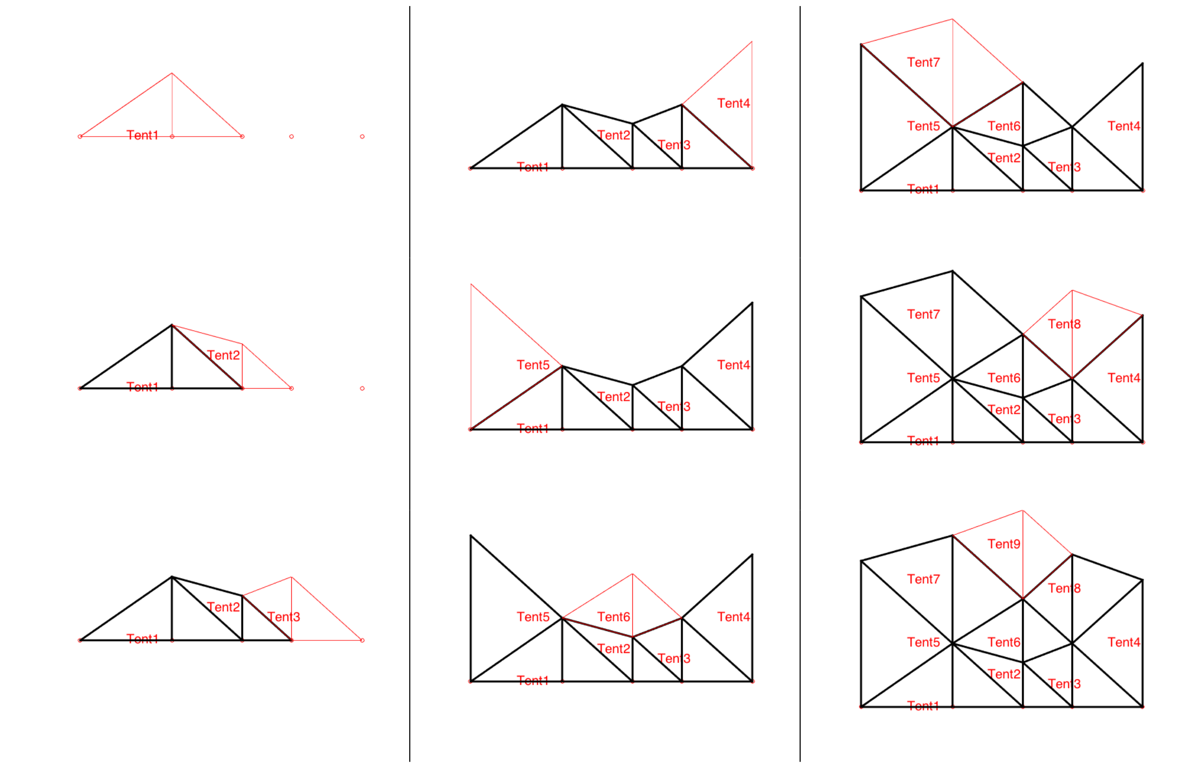


FIGURE 1. Tent pitching (read column by column)

has been traditionally used for meshing schemes that advance a space-time front [6, 25], but in this paper tent pitching refers to the discretization scheme together with all the required meshing.

To give an overview of what is involved in a tent pitching scheme, consider the case of a hyperbolic problem posed in one space dimension with time as the second dimension. Given a spatial mesh, we pitch a tent by erecting a tent pole (vertically in time) at a vertex, as in Figure 1. (Precise definitions of "tents" etc. appear later – see Definition 4.8.) In the plots of Figure 1, the horizontal and vertical dimensions are space and time, respectively. The height of the tent pole must be chosen small enough in relation to the hyperbolic propagation speed, so that the domain of dependence of all points in the tent remains within the tent's footprint. We then use the given initial data to solve, by some numerical scheme, the hyperbolic problem restricted to the tent. Proceeding to the next vertex where the second tent is pitched in Figure 1, we find that the initial data combined with the solution in the previous tent, provides inflow data to solve the hyperbolic problem there. Solution on the newer tents proceeds similarly. This shows the sense in which tent pitching schemes are locally implicit: they only involve solving local problems tent by tent.

Having explained tent pitching schemes in general, we should now emphasize that the main result of this paper is not a new tent pitching scheme (although one is included to show relevance). Rather, this paper is mainly concerned with answering a few theoretical questions *motivated* by tent pitching schemes. Indeed, our main result is a characterization of traces of a Friedrichs space on a tent-shaped domain and builds on the recent advances in Friedrichs theory [1, 7, 9, 11]. To explain the Friedrichs connection, we should first note that all the previous tent pitching schemes use non-conforming space-time discontinuous Galerkin discretizations. Design of tent pitching methods within a conforming setting, while holding the promise of locally adaptive time marching with fewer unknowns, pose interesting questions: What is the weak formulation that the tent pitching scheme should conform to? What are the spaces? What are the finite element subspaces one should use? These questions form the motivation for this study and while attempting to answer them, Friedrichs spaces and their traces appear naturally, as we shall see. While we are far from answering the above questions for a general Friedrichs system, our modest aim in this paper is to provide some answers for a few simple problems in one space dimension.

Accordingly, there are two parts to this paper. The first and the main part of the paper consists of Sections 2–4. While results of Sections 2 and 3 are applicable to any abstract Friedrichs system, Section 4 focusses mainly on an advection example and its implications for hyperbolic systems. This leads to observations on the traces of certain Friedrichs spaces. The theory clarifies a weak continuity property of the traces at the points where inflow and outflow part of boundaries (defined precisely later) meet. It is relevant in the tent pitching context because in the tent-shaped domains used in tent pitching schemes, inflow and outflow boundaries always meet. The second part of the paper, consisting of Sections 5 and 6, designs an explicit space-time finite element scheme of the tent pitching type using the spaces and weak formulations motivated by the first part. The method we construct is a low order method that works on unstructured grids. On uniform grids, comparison with a standard low order finite difference method does not reveal any striking advantages for the new method, as we will see in Section 7. Yet we hope that this study will pave the way to a better understanding of conforming tent pitching discretizations, the spaces involved, and eventually lead to high order methods on unstructured grids for multidimensional problems. We begin with some preliminaries on Friedrichs systems in the next section.

2. Friedrichs systems

Our approach is influenced by the modern take on the classical work of Friedrichs [11], as presented in [7, 8, 9]. Let L be a Hilbert space over $\mathbb{R}$ with inner product $(\cdot,\cdot)_L$ and norm $\|\cdot\|_L$, and let $\mathcal{D}$ be a dense subspace of L. Suppose A and $\tilde{A}$ are linear maps from $\mathcal{D}$ into L satisfying

$$(A\phi, \psi)_L = (\phi, \tilde{A}\psi)_L, \qquad \forall \phi, \psi \in \mathcal{D}, \tag{2.1a}$$

$$\exists\, c > 0 : \quad \|(A + \tilde{A})\phi\|_L \le c\|\phi\|_L, \qquad \forall \phi \in \mathcal{D}. \tag{2.1b}$$

Let W_0 be the completion of $\mathcal{D}$ in the norm $\|\phi\|_W = (\|\phi\|_L^2 + \|A\phi\|_L^2)^{1/2}$. Then, with L as a pivot Hilbert space, identified to be the same as its dual L', we have $\mathcal{D} \subseteq W_0 \subseteq L \equiv L' \subseteq W_0'$. It is now standard to extend A and $\tilde{A}$ as bounded linear operators from W_0 into L, i.e., $A, \tilde{A} \in \mathcal{L}(W_0, L)$. Moreover, it is also well-known that Assumption (2.1) implies that they can be further extended to $A, \tilde{A} \in \mathcal{L}(L, W_0')$ via

$$\langle A\ell, w_0\rangle_{W_0} = (\ell, \tilde{A}w_0)_L, \qquad \langle \tilde{A}\ell, w_0\rangle_{W_0} = (\ell, Aw_0)_L, \qquad \forall \ell \in L,\ w_0 \in W_0. \tag{2.2}$$

Here and throughout, we use $\langle\cdot,\cdot\rangle_X$ to denote the duality pairing in X. Next, defining $W = \{v \in L : Av \in L\}$, we observe that $W_0 \subseteq W$ and that W normed with $\|\cdot\|_W$ defined above is a Hilbert space. Hereon, the norm on any normed linear space X will be denoted by $\|\cdot\|_X$.

An important ingredient in Friedrichs theory is the "boundary" operator D in $\mathcal{L}(W, W')$ defined by

$$\langle Du, v\rangle_W = (Au, v)_L - (u, \tilde{A}v)_L \qquad \forall u, v \in W. \tag{2.3}$$

This is an abstraction of an integration by parts identity. For any operator $B \in \mathcal{L}(W, W')$, we define $B^* \in \mathcal{L}(W, W')$ by $\langle B^*u, v\rangle_W = \langle Bv, u\rangle_W$ for all $u, v \in W$. For subspaces $S \subseteq W$ and $R \subseteq W'$, define the right and left annihilators by

$$S^\perp = \{w' \in W' : \langle w', s\rangle_W = 0 \text{ for all } s \in S\},$$
$${}^\perp R = \{w \in W : \langle s', w\rangle_W = 0 \text{ for all } s' \in R\}.$$

These results are well known [9]:

Proposition 2.1. *The following are consequences of Assumption* (2.1)*:*

(a) $D^* = D$.
(b) The norm $\|w\|_{\tilde{W}} = (\|w\|_L^2 + \|\tilde{A}w\|_L^2)^{1/2}$ *is equivalent to* $\|w\|_W$ *for all* $w \in W$.
(c) $\ker D = W_0$.
(d) $\operatorname{ran} D = W_0^\perp$.

We will henceforth tacitly assume (2.1) throughout this section. In the traditional Friedrichs theory, another "boundary operator" M, also in $\mathcal{L}(W, W')$, plays a leading role. This is a generalization of certain matrices used by Friedrichs [11] to impose boundary conditions. In the generalization of Friedrichs theory to the Hilbert space setting, as described in [9], the operator M is assumed to satisfy

$$\langle Mw, w\rangle_W \geq 0, \quad \forall w \in W, \tag{2.4a}$$

$$W = \ker(D - M) + \ker(D + M). \tag{2.4b}$$

The theory in [9] addresses the unique solvability of two problems: The first is to find a $u \in W$, given any $f \in L$, satisfying $Au = f$ (typically a partial differential equation), and $(D - M)u = 0$ (typically a boundary condition). The second problem is the "dual" problem of solving $\tilde{A}u = f$ satisfying $(D + M^*)u = 0$. These two problems are uniquely solvable if and only if the following two conditions hold, respectively:

$$A : \ker(D - M) \to L \ \text{ is a bijection, and} \tag{2.5a}$$

$$\tilde{A} : \ker(D + M^*) \to L \ \text{ is a bijection.} \tag{2.5b}$$

Some sufficient conditions for (2.5) to hold can be found in [7, 9].

In [9], an intrinsic approach without the operator M was discovered. It uses the double cones

$$C^+ = \{w \in W : \langle Dw, w\rangle_W \ge 0\},$$
$$C^- = \{w \in W : \langle Dw, w\rangle_W \le 0\}.$$

The intrinsic approach replaces (2.4) by the following assumption on two subspaces of W denoted by V and V^*:

$$V \subseteq C^+, \qquad V^* \subseteq C^-, \tag{2.6a}$$

$$V = {}^\perp D(V^*), \qquad V^* = {}^\perp D(V)\,. \tag{2.6b}$$

Clearly, (2.6b) implies that both V and V^* are closed, and moreover,

$$\ker D = W_0 \subseteq V \cap V^*. \tag{2.7}$$

Note that the reflexivity of Hilbert spaces and (2.6b) imply that

$$(V^*)^\perp = \left({}^\perp D(V)\right)^\perp = D(V), \tag{2.8}$$
$$V^\perp = \left({}^\perp D(V^*)\right)^\perp = D(V^*).$$

The theory in [9] provides sufficient conditions for unique solvability of two problems: The first is to find a $u \in W$, given any $f \in L$, satisfying

$$Au = f \qquad \text{(typically a partial differential equation)}, \tag{2.9a}$$

$$u \in V \qquad \text{(typically a boundary condition)}. \tag{2.9b}$$

The second is the "dual" problem of solving for a $u \in V^*$ satisfying $\tilde{A}u = f$. These two problems are uniquely solvable if and only if

$$A : V \to L \ \text{ is a bijection, and} \tag{2.10a}$$

$$\tilde{A} : V^* \to L \ \text{ is a bijection.} \tag{2.10b}$$

A coercivity condition on $A + \tilde{A}$ is sufficient for (2.10) to hold, as proved in [9]. However, for operators like the transient wave operator considered later, $A + \tilde{A}$ is zero and cannot be coercive.

Hence, our first point of departure from [9] is the introduction of another simple sufficient condition for unique solvability. It requires that the operator A be bounded from below, a condition which is often easy to verify for time-dependent problems (see e.g. [26]). Although the next theorem requires both A and $\tilde{A}$ to be bounded below, one of these conditions can be easily removed in most applications, as detailed in Remark 2.3.

Theorem 2.2. *Suppose* (2.1) *and* (2.6) *hold. If there is a constant $c > 0$ such that $A : V \to L$ satisfies*

$$\|Au\|_L \ge c\|u\|_W \qquad \forall u \in V, \ \textit{and} \tag{2.11a}$$

$$\|\tilde{A}u\|_L \ge c\|u\|_W \qquad \forall u \in V^*, \tag{2.11b}$$

then (2.10) *holds.*

Proof. Inequality (2.11a) implies that $A : V \to L$ is injective and has closed range. Hence A is a bijection if its adjoint is injective, i.e., if

$$\{\ell \in L : \ (Av, \ell)_L = 0 \text{ for all } v \in V\} = \{0\}. \tag{2.12}$$

To prove (2.12), consider an ℓ satisfying $(Av, \ell)_L = 0$ for all $v \in V$. Then, for all $w_0 \in W_0 \subseteq V$, we have by (2.2) that $\langle \tilde{A}\ell, w_0\rangle_{W_0} = 0$, from which it follows, by the density of W_0 in L, that $\tilde{A}\ell = 0$ and $\ell \in W$. Hence we may apply (2.3), which yields

$$\langle Dv, \ell\rangle_W = (Av, \ell)_L - (v, \tilde{A}\ell)_L = 0, \quad \forall v \in V. \tag{2.13}$$

Thus $\ell \in {}^{\perp}D(V) = V^*$, and so (2.11b) implies $\ell = 0$. This proves (2.12).

That $\tilde{A}$ is a bijection is proved similarly. □

Remark 2.3. Note that under the assumptions of Theorem 2.2, if (2.11a) holds and $\tilde{A}$ is injective, then (2.11b) holds with the same constant c. This is most easily seen by viewing A as a closed (possibly unbounded) operator on L with $\mathrm{dom}(A) = V$. From (2.7), we know that $\mathcal{D} \subset W_0 \subset V \subset L$. Since $\mathcal{D}$ is dense in L, the domain of A is dense in L. Hence the adjoint A' is a well defined closed operator on L satisfying $(Av, s)_L = (v, A's)_L$ for all $v \in \mathrm{dom}(A)$ and $s \in \mathrm{dom}(A')$. By definition, $\mathrm{dom}(A')$ consists of all $s \in L$ for which there exists an $\ell \in L$ with the property $(s, Av)_L = (\ell, v)_L$ for all $v \in \mathrm{dom}(A) = V$. In particular, whenever $s \in \mathrm{dom}(A')$, by (2.2), $(s, Aw_0)_L = \langle \tilde{A}s, w_0\rangle_{W_0} = (\ell, w_0)_L$ for all $w_0 \in W_0 \subset V$, so $\tilde{A}s = \ell$ and consequently $s \in W$. Thus, whenever $s \in \mathrm{dom}(A')$, both $(s, Av)_L$ and $(\tilde{A}s, v)_L$ coincide with $(\ell, v)_L$ for all $v \in V$, which by (2.3), implies that $\langle Dv, s\rangle_W = 0$, which by (2.6), implies that $s \in V^* = {}^{\perp}D(V)$, i.e., $\mathrm{dom}(A') \subseteq V^*$. Combining with the easily provable reverse inclusion, $\mathrm{dom}(A') = V^*$. Next, we claim that A' is the same as $\tilde{A}$: Indeed, $(Av, s)_L - (v, A's)_L = \langle Dv, s\rangle_W = 0$ for all $v \in \mathrm{dom}(A) = V$ and $s \in \mathrm{dom}(A') = V^* = {}^{\perp}D(V)$, thus proving the claim. Now, since (2.11a) implies that the $\mathrm{ran}(A)$ is closed, by the Closed Range Theorem for closed operators [15], we conclude that $\mathrm{ran}(A') = \mathrm{ran}(\tilde{A})$ is closed. Hence if $\tilde{A}$ is also injective, then by standard arguments, (2.11b) follows.

To summarize, we have discussed two known approaches to abstract Friedrichs systems and introduced a new sufficient condition for unique solvability of Friedrichs problems. The first approach via (2.4) is closer to the classical theory (the M-approach) while the second is the approach via (2.6) (the V-approach). Whether these two approaches are equivalent is a natural question. It was shown in [9] that if an operator M exists that satisfies (2.4), then $V = \ker(D - M)$ and $V^* = \ker(D + M^*)$ satisfies (2.6). The converse remained unknown until it was proven in [1]. In the remainder of this paper, we will use only the V-approach.

3. A weak formulation with boundary fluxes

Consider the following abstract boundary value problem: Given $f \in L$ and $g \in W$, find $u \in W$ satisfying

$$Au = f, \tag{3.1a}$$

$$u - g \in V. \tag{3.1b}$$

Space-time Friedrichs systems with non-homogeneous conditions on space-time boundaries (which includes initial conditions) can be abstracted into this form.

To derive a weak formulation, we multiply (3.1a) by a test function $v \in W$ and use (2.3), to obtain $(u, \tilde{A}v)_L + \langle Du, v\rangle_W = (f, v)_L$. This implies

$$(u, \tilde{A}v)_L + \langle D(u - g), v\rangle_W = F(v), \tag{3.2}$$

where

$$F(v) = (f, v)_L - \langle Dg, v\rangle_W. \tag{3.3}$$

Now, we let $D(u - g)$ in (3.2) be an independent "flux" variable q. This leads us to formulate the following variational problem:

$$\begin{aligned} &\textit{Find } u \in L \textit{ and } q \in (V^*)^\perp \textit{ such that} \\ &\qquad (u, \tilde{A}v)_L + \langle q, v\rangle_W = F(v), \qquad \forall v \in W. \end{aligned} \tag{3.4}$$

The bilinear form on the left hand side will be denoted by $b((u, q), v)$. Our approach to the construction and analysis of this weak formulation is close (but not identical) to the approach in [3].

A similar derivation for the adjoint problem of finding a $\tilde{u} \in W$, given $f \in L$ and $g \in W$, such that

$$\tilde{A}\tilde{u} = f, \tag{3.5a}$$

$$\tilde{u} - g \in V^*, \tag{3.5b}$$

suggests the following dual weak formulation:

$$\begin{aligned} &\textit{Find } \tilde{u} \in L \textit{ and } \tilde{q} \in V^\perp \textit{ such that} \\ &\qquad (\tilde{u}, Av)_L - \langle \tilde{q}, v\rangle_W = \tilde{F}(v), \qquad \forall v \in W, \end{aligned} \tag{3.6}$$

where

$$\tilde{F}(v) = (f, v)_L + \langle Dg, v\rangle_W. \tag{3.7}$$

The bilinear form on the left hand side will now be denoted by $\tilde{b}((\tilde{u}, \tilde{q}), v)$.

In applications, the $\langle \cdot, \cdot\rangle_W$ terms can typically be identified as boundary terms, so q and $\tilde{q}$ can be interpreted as boundary fluxes. Finally, note that by virtue of (2.8), we can equivalently use $D(V)$ and $D(V^*)$ as the flux spaces in (3.4) and (3.6), respectively.

3.1. **Wellposedness.** Next, we prove that the new weak formulation is well posed and is equivalent to the classical formulation (3.1) in the following sense.

Theorem 3.1. *Suppose* (2.6) *and* (2.10) *hold. Then the following statements hold:*

(a) Given any $F \in W'$, there is a unique $(u, q) \in L \times (V^*)^\perp$ that solves (3.4)*. Moreover, if F is as in* (3.3) *for some given $f \in L$ and $g \in W$, then the solution (u, q) of* (3.4) *satisfies*

$$Au = f, \qquad u - g \in V, \qquad q = D(u - g). \tag{3.8}$$

(b) Given any $\tilde{F} \in W'$, there is a unique $(\tilde{u}, \tilde{q}) \in L \times V^\perp$ that solves (3.6)*. Moreover, if $\tilde{F}$ is as in* (3.7) *for some given $f \in L$ and $g \in W$, then the solution $(\tilde{u}, \tilde{q})$ of* (3.6) *satisfies*

$$\tilde{A}\tilde{u} = f, \qquad \tilde{u} - g \in V^*, \qquad \tilde{q} = D(\tilde{u} - g).$$

To prove this theorem, we will verify a uniqueness and an inf-sup condition in the following lemmas.

Lemma 3.2 (Uniqueness)**.** *Suppose* (2.6) *and* (2.10a) *hold. Then, whenever $u \in L$ and $q \in (V^*)^\perp$ satisfies $b((u, q), v) = 0$ for all $v \in W$, we have $(u, q) = 0$.*

Proof. Suppose

$$(u, \tilde{A}v)_L + \langle q, v\rangle_W = 0 \qquad \forall v \in W. \tag{3.9}$$

Since $q \in (V^*)^\perp$, we have $\langle q, v\rangle_W = 0$ for all $v \in W_0$ due to (2.7). Hence, choosing $v = v_0 \in W_0$ in (3.9), we conclude that $(u, \tilde{A}v_0)_L = 0$. Hence, using (2.2), we have $\langle Au, v_0\rangle_{W_0} = 0$ for all $v_0 \in W_0$, which implies, by density, that $Au = 0$ in L. In particular, this shows that u is in W. We may therefore apply (2.3) to (3.9) to get $(Au, v)_L - \langle Du, v\rangle_W + \langle q, v\rangle_W = 0$, for all $v \in W$. Since $Au = 0$,

$$\langle Dv, u\rangle_W = \langle q, v\rangle_W \quad \forall v \in W.$$

Since $q \in (V^*)^\perp$, the right hand side vanishes for all $v \in V^*$, so $u \in {}^\perp D(V^*)$. Hence by assumption (2.6), $u \in V$. By (2.10a), $u = 0$. Using this in (3.9), it also follows that $q = 0$. □

Lemma 3.3 (Inf-sup condition)**.** *Suppose* (2.6) *and* (2.10a) *hold. Then, there is a $C > 0$ such that for all $v \in W$,*

$$C\|v\|_W \le \sup_{(u,q)\in L\times(V^*)^\perp} \frac{|b((u, q), v)|}{\|(u, q)\|_{L\times W'}}, \qquad \forall v \in W.$$

Proof. By (2.10a), there is a $c > 0$ such that given any $v \in W$, there is a unique $w \in W$ satisfying

$$Aw = v, \tag{3.10a}$$

$$w \in V, \tag{3.10b}$$

$$\|w\|_W \le c\|v\|_L. \tag{3.10c}$$

Then, since $w \in V = {}^{\perp}D(V^*)$, we have $\langle Dv^*, w\rangle_W = 0$ for any $v^* \in V^*$. Therefore, $q = Dw$ is in $(V^*)^{\perp}$. Moreover,

$$\begin{aligned}
\|v\|_L^2 + \|\tilde{A}v\|_L^2 &= (Aw, v)_L + (\tilde{A}v, \tilde{A}v)_L && \text{by (3.10a)}\\
&= (w + \tilde{A}v, \tilde{A}v)_L + \langle Dw, v\rangle_W && \text{by (2.3)}
\end{aligned}$$

Note that the right hand side equals $b((w + \tilde{A}v, q), v)$ as $q = Dw$. Continuing,

$$\begin{aligned}
\|v\|_L^2 + \|\tilde{A}v\|_L^2 &= \frac{b((w + \tilde{A}v, q), v)}{\|(w + \tilde{A}v, q)\|_{L\times W'}} \|(w + \tilde{A}v, q)\|_{L\times W'}\\
&\le \left(\sup_{(z,r)\in L\times (V^*)^{\perp}} \frac{|b((z, r), v)|}{\|(z, r)\|_{L\times W'}} \right) \|(w + \tilde{A}v, q)\|_{L\times W'}.
\end{aligned}$$

By Proposition 2.1(b) and (3.10c), $\|w + \tilde{A}v\|_L \le \|w\|_L + \|v\|_{\tilde{W}} \le C\|v\|_W$ for a $C > 0$ depending on c. Moreover, if d denotes the norm of D, then $\|q\|_{W'} \le d\|w\|_W \le cd\|v\|_W$. Using these estimates to bound $\|(w + \tilde{A}v, q)\|_{L\times W'}$, the lemma is proved. □

Proof of Theorem 3.1. Lemmas 3.2 and 3.3 verify the conditions of the Babuška-Brezzi theory, from which the stated unique solvability follows.

Now suppose F is expressed in terms of f and g as in (3.3). Then choosing $v = v_0 \in W_0$ within the weak formulation,

$$(u, \tilde{A}v_0)_L + \langle q, v_0\rangle_W = (f, v_0)_L - \langle Dg, v_0\rangle_W = (f, v_0)_L$$

we obtain $(u, \tilde{A}v_0)_L = \langle Au, v_0\rangle_W = (f, v_0)_L$. This proves, by density, that $Au = f$ in L, and consequently $u \in W$. Then, returning to (3.4) and using (2.3) together with $Au = f$, we obtain $\langle q, v\rangle_W = \langle D(u - g), v\rangle_W$ for all $v \in W$, i.e., $q = D(u - g)$. Finally to show that $u - g \in V = {}^{\perp}D(V^*)$, consider an arbitrary $v^* \in V^*$. Then note that $q \in (V^*)^{\perp}$, so

$$0 = \langle q, v^*\rangle_W = \langle D(u - g), v^*\rangle_W = \langle Dv^*, u - g\rangle_W.$$

This proves (3.8). The remaining statements are proved similarly using (2.10b) in place of (2.10a). □

4. Examples

The assumptions on which the previous theory is based can be verified for several examples. We begin with the simplest example in one space dimension in § 4.1 where all the ideas are transparent. We then generalize to the example of multidimensional advection in § 4.2 and establish a new trace theorem for the associated graph space. The final example in § 4.3 considers a general symmetric hyperbolic system in one space dimension and leads into the discussion on the wave equation in the subsequent section.

4.1. **An example with no space derivatives.** We begin with a simple example in one space dimension that illustrates the essential points. Let K denote the open triangle in space-time $(x, t) \in \mathbb{R} \times \mathbb{R}$, with vertices at $(x, t) = (0, 0), (1, 0)$, and $(1, 1)$. Set

$$L = L^2(K), \quad \mathcal{D} = \mathcal{D}(K), \quad Au = \frac{\partial u}{\partial t} \tag{4.1}$$

(where $\mathcal{D}(K)$ denotes the set of compactly supported infinitely differentiable functions on K). Obviously, $\tilde{A} = -\partial_t$, so (2.1) is satisfied. We split the boundary of K into an inflow, outflow, and a characteristic part:

$$\partial_{\mathrm{i}} K = \{(x,t) \in \partial K : t = 0\}, \qquad \partial_{\mathrm{o}} K = \{(x,t) \in \partial K : x = t\},$$
$$\partial_{\mathrm{c}} K = \{(x,t) \in \partial K : x = 1\}.$$

Because $\operatorname{dist}(\partial_{\mathrm{i}} K, \partial_{\mathrm{o}} K) = 0$, although the operator D is defined on all W, we must be careful in speaking of traces of functions in W on these boundary parts. Indeed, $w(x,t) = x^{-1/2}$ is in W, but its restriction to $\partial_{\mathrm{i}} K$ is not in $L^2(\partial_{\mathrm{i}} K)$.

To study this further, define the maps

$$\tau_{\mathrm{i}} : v(x,t) \mapsto v(x,0) \quad \text{and} \quad \tau_{\mathrm{o}} : v(x,t) \mapsto v(x,x),$$

whose application to any function gives its traces on $\partial_{\mathrm{i}} K$ and $\partial_{\mathrm{o}} K$, respectively. These maps are obviously well defined for smooth functions. Below we prove that they extend to W. Let $L^2_w(S)$ denote the set of all measurable functions s on S with finite $\int_S ws^2$.

Lemma 4.1. *For the W in this example, the following maps are continuous:*

$$\tau_{\mathrm{i}} : W \to L^2_x(0,1), \quad \tau_{\mathrm{o}} : W \to L^2_x(0,1), \quad \textit{and} \quad \tau_{\mathrm{i}} - \tau_{\mathrm{o}} : W \to L^2_{1/x}(0,1),$$

i.e., there is a constant $C_0 > 0$ such that

$$\int_0^1 x|\tau_{\mathrm{i}} w|^2\,dx + \int_0^1 x|\tau_{\mathrm{o}} w|^2\,dx + \int_0^1 \frac{|\tau_{\mathrm{i}} w - \tau_{\mathrm{o}} w|^2}{x}\,dx \le C_0 \|w\|_W^2 \tag{4.2}$$

for all $w \in W$.

Proof. A general density result in [13, Theorem 4] implies that $C^1(\bar{K})$ is dense in W, so it suffices to prove (4.2) for all $w \in C^1(\bar{K})$. Beginning with the fundamental theorem of calculus,

$$\tau_{\mathrm{i}} w(x) = w(x,r) - \int_0^r \partial_t w(x,s)\,ds,$$

squaring, integrating over r, and overestimating,

$$x|\tau_{\mathrm{i}} w(x)|^2 = \int_0^x |\tau_{\mathrm{i}} w(x)|^2\,dr \le 2\int_0^x |w(x,r)|^2\,dr + 2\int_0^x r\int_0^r |\partial_t w(x,s)|^2\,ds\,dr.$$

Now integrating over x and overestimating again,

$$\begin{aligned}\frac{1}{2}\int_0^1 x|\tau_{\mathrm{i}} w(x)|^2\,dx &\le \int_0^1\!\!\int_0^x |w(x,r)|^2\,dr\,dx + \int_0^1\!\!\int_0^1 1\int_0^x |\partial_t w(x,s)|^2\,ds\,dr\,dx \\ &= \|w\|_W^2.\end{aligned}$$

A similar argument shows that the same inequality holds with τ_{i} replaced by τ_{o}.

To complete the proof, we therefore only need to show that

$$\int_0^1 \frac{|\tau_{\mathrm{i}} w - \tau_{\mathrm{o}} w|^2}{x}\,dx \le \|w\|_W^2. \tag{4.3}$$

But this follows from

$$|\tau_{\rm i} w(x) - \tau_{\rm o} w(x)|^2 = \left| \int_0^x \partial_t w(x,s)\, ds \right|^2 \le x \int_0^x |\partial_t w(x,s)|^2\, ds,$$

dividing through by x and integrating over x. $\square$

Lemma 4.2. *Assumption* (2.6) *holds for this example after setting*

$$V = \{w \in W : \tau_{\rm i} w = 0\}, \tag{4.4a}$$

$$V^* = \{w \in W : \tau_{\rm o} w = 0\}. \tag{4.4b}$$

Proof. For $v, w \in C^1(\bar{K})$, the definition of D implies that

$$\begin{aligned} \langle Dw, v\rangle_W &= \int_0^1\int_0^x (\partial_t w)v + w(\partial_t v)\, dt\, dx \\ &= \int_0^1 (\tau_{\rm o} w)(\tau_{\rm o} v)\, dx - \int_0^1 (\tau_{\rm i} w)(\tau_{\rm i} v)\, dx. \end{aligned} \tag{4.5}$$

In order to apply the density argument, we rewrite this expression:

$$\langle Dw, v\rangle_W = \int_0^1 (x^{1/2}\tau_{\rm o} w) \left(\frac{\tau_{\rm o} v - \tau_{\rm i} v}{x^{1/2}}\right) dx + \int_0^1 \left(\frac{\tau_{\rm o} w - \tau_{\rm i} w}{x^{1/2}}\right)(x^{1/2}\tau_{\rm i} v)\, dx. \tag{4.6}$$

Now, one can immediately verify using Cauchy-Schwarz inequality and Lemma 4.1, that both the integrals extend continuously to W. Hence (4.6) holds for all v and w in W. Similarly, the expression

$$\langle Dw, v\rangle_W = \int_0^1 \left(\frac{\tau_{\rm o} w - \tau_{\rm i} w}{x^{1/2}}\right)(x^{1/2}\tau_{\rm o} v)\, dx + \int_0^1 (x^{1/2}\tau_{\rm i} w) \left(\frac{\tau_{\rm o} v - \tau_{\rm i} v}{x^{1/2}}\right) dx, \tag{4.7}$$

also holds for all v and w in W.

Let us verify (2.6a). For any $v \in V$, since $\tau_{\rm i} v = 0$, we have from (4.7) that

$$\langle Dv, v\rangle_W = \int_0^1 \left(\frac{\tau_{\rm o} v - 0}{x^{1/2}}\right)(x^{1/2}\tau_{\rm o} v)\, dx \ge 0.$$

Hence $V \subseteq C^+$. Similarly, $V^* \subseteq C^-$.

To prove (2.6b), let $v \in V$. Then using (4.6) and putting $\tau_{\rm i} v = 0$, we have,

$$\langle Dv^*, v\rangle_W = \int_0^1 (x^{1/2}\tau_{\rm o} v^*) \left(\frac{\tau_{\rm o} v - 0}{x^{1/2}}\right) dx$$

which vanishes for any $v^* \in V^*$. Hence $V \subseteq {}^\perp D(V^*)$. For the reverse inclusion, let $v^\perp \in {}^\perp D(V^*)$. Then, since $\tau_{\rm o} v^* = 0$ for all $v^* \in V^*$, we have from (4.6) that

$$\langle Dv^*, v^\perp\rangle_W = \int_0^1 \left(\frac{\tau_{\rm o} v^* - \tau_{\rm i} v^*}{x^{1/2}}\right)(x^{1/2}\tau_{\rm i} v^\perp)\, dx = -\int_0^1 (\tau_{\rm i} v^*)(\tau_{\rm i} v^\perp)\, dx.$$

Since all functions in $\mathcal{D}(0,1)$ can be written as $\tau_{\rm i} v^*$ for some $v^* \in V^*$, this implies that $\tau_{\rm i} v^\perp = 0$ a.e. in $(0,1)$, so $v^\perp \in V$. Thus, $V = {}^\perp D(V^*)$. A similar argument shows that $V^* = {}^\perp D(V)$. $\square$

Remark 4.3. Note that although the two integrals in (4.5) need not generally exist for all $w, v \in W$, those in the identities (4.6) and (4.7) exist for all $w, v \in W$.

Remark 4.4. It is proved in [9, Lemma 4.4] that if $V + V^*$ is closed, then an M that satisfies (2.4) can be constructed. They then write, "it is not yet clear to us whether properties (2.6a)–(2.6b) actually imply that $V + V^*$ is closed in W." This issue was settled in [1] where they showed by a counterexample that (2.6a)–(2.6b) does not in general imply $V+V^*$ is closed. Our study above provides another simpler counterexample: Specifically, for $n \ge 2$, let χ_n denote the indicator function of the interval $[1/n, 1]$. Then $v_n(x,t) = \chi_n(x)t/x$ is in V and $v_n^* = \chi_n(x)(x-t)/x$ is in V^*. Clearly, as $n \to \infty$, the sequence $v_n + v_n^* = \chi_n \in V + V^*$ converges in W. But its limit, the function 1, is not in $V+V^*$. Indeed, if 1 were to equal $v+v^*$ for some $v \in V$ and $v^* \in V^*$, then by Lemma 4.1,

$$\int_0^1 \frac{1}{x}\, dx = \int_0^1 \frac{|\tau_{\mathrm{i}} v + \tau_{\mathrm{o}} v^*|^2}{x}\, dx \le 2\int_0^1 \frac{|\tau_{\mathrm{i}} v|^2}{x} + \frac{|\tau_{\mathrm{o}} v^*|^2}{x}\, dx \le 2C\left(\|v\|_W^2 + \|v^*\|_W^2\right)$$

which is impossible.

Lemma 4.5. *The inequalities of* (2.11) *hold for this example.*

Proof. Given any $v \in V$, by the density of $C^1(\bar K)$ in W, there is a sequence $w_n \in C^1(\bar K)$ converging to v in W. Let $v_n(x,t) = w_n(x,t) - (\tau_{\mathrm{i}} w_n)(x)$. Clearly $v_n \in V \cap C^1(\bar K)$. Moreover,

$$\begin{aligned}
\|v_n - v\|_W &\le \|w_n - v\|_W + \|\tau_{\mathrm{i}} w_n\|_{L^2(K)} \\
&= \|w_n - v\|_W + \left(\int_0^1\int_0^x |w_n(x,0)|^2\, dt\, dx\right)^{1/2} \\
&= \|w_n - v\|_W + \|\tau_{\mathrm{i}} w_n\|_{L^2_x(0,1)} \\
&= \|w_n - v\|_W + \|\tau_{\mathrm{i}}(w_n - v)\|_{L^2_x(0,1)}.
\end{aligned}$$

This, together with Lemma 4.1, and the convergence of w_n to v in W, imply the convergence of v_n to v in W. Thus $V \cap C^1(\bar K)$ is dense in V. Similarly, $V^* \cap C^1(\bar K)$ is dense in V^*. Hence, it suffices to prove the inequalities of (2.11) for the dense subsets.

For any $C^1(\bar K)$ function v in V, we have

$$v(x,t)^2 = \int_0^t \frac{\partial}{\partial s}\left(v(x,s)^2\right)\, ds = \int_0^t 2v(x,s)\frac{\partial}{\partial s} v(x,s)\, ds,$$

which implies

$$\begin{aligned}
\int_0^1\int_0^x v(x,t)^2\, dt\, dx &\le 2\int_0^1\int_0^x \left(\int_0^t v(x,s)^2\, ds\right)^{1/2}\left(\int_0^t \left|\frac{\partial}{\partial s} v(x,s)\right|^2\, ds\right)^{1/2} dt\, dx, \\
&\le 2\int_0^1 \left(\int_0^x\int_0^t v(x,s)^2\, ds\, dt\right)^{1/2}\left(\int_0^x\int_0^t \left|\frac{\partial}{\partial s} v(x,s)\right|^2\, ds\, dt\right)^{1/2} dx.
\end{aligned}$$

Since $x \le 1$, this shows that $\|v\|_L \le 2\|Av\|_L$ for all $v \in V \cap C^1(\bar K)$ and hence for all $v \in V$. The proof of (2.11b) is similar. $\square$

Theorem 4.6. *Formulations* (3.4) *and* (3.6) *are well posed for this example.*

Proof. By Lemmas 4.2 and 4.5, assumptions (2.6) and (2.11) hold, so Theorem 2.2 implies that (2.10) holds. Therefore, Theorem 3.1 gives the required result. □

4.2. **Unidirectional advection.** The above calculations have a straightforward generalization to multidimensional tent-shaped domains. We say that K_0 is a vertex patch around a point p if it is an open polyhedron in $\mathbb{R}^d$ ($d \geq 1$) that can be partitioned into a finite number of d-simplices with a common vertex $p \in \mathbb{R}^d$.

We first consider domains K built on (spatial) vertex patches of the form

$$K = \{(x,t) : \; x \in K_0, \; g_{\mathrm{i}}(x) < t < g_{\mathrm{o}}(x)\} \tag{4.8}$$

(and later, after Definition 4.8 below, specialize to tent-shaped domains). Above, $g_{\mathrm{o}}(x)$ and $g_{\mathrm{i}}(x)$ are Lipschitz functions on K_0 such that K is a nonempty open set in $\mathbb{R}^{d+1}$. Then the unit outward normal vector $n = (n_x, n_t)$ exists a.e. on ∂K. Continuing to consider the same operator as in (4.1), namely $A = \partial_t$, but on the new domain K, the following defines inflow, outflow, and characteristic parts of the boundary:

$$\partial_{\mathrm{i}} K = \{(x,t) \in \partial K : n_t < 0\}, \qquad \partial_{\mathrm{o}} K = \{(x,t) \in \partial K : n_t > 0\}, \tag{4.9a}$$

$$\partial_{\mathrm{c}} K = \{(x,t) \in \partial K : n_t = 0\}. \tag{4.9b}$$

We can immediately prove the following by extending the arguments of §4.1.

Theorem 4.7. *Let K be as in* (4.8) *and let $A = \partial_t$. Then the inflow and outflow trace maps, $\tau_{\mathrm{i}} : v(x,t) \mapsto v(x, g_{\mathrm{i}}(x))$ and $\tau_{\mathrm{o}} : v(x,t) \mapsto v(x, g_{\mathrm{o}}(x))$, extend to continuous linear operators*

$$\tau_{\mathrm{i}} : W \to L^2_{g_{\mathrm{o}}-g_{\mathrm{i}}}(K_0), \quad \tau_{\mathrm{o}} : W \to L^2_{g_{\mathrm{o}}-g_{\mathrm{i}}}(K_0), \quad \text{and} \quad \tau_{\mathrm{i}} - \tau_{\mathrm{o}} : W \to L^2_{1/(g_{\mathrm{o}}-g_{\mathrm{i}})}(K_0).$$

As in (4.4), *set $V = \ker(\tau_{\mathrm{i}})$ and $V^* = \ker(\tau_{\mathrm{o}})$. Then, the assumptions of* (2.6) *and the inequalities of* (2.11) *hold. Hence the formulations* (3.4) *and* (3.6) *are well-posed.*

Identities similar to (4.6) and (4.7) prove the continuity properties of the trace maps stated above. To prove the stated wellposedness, we need to verify the assumptions in (2.6) and (2.11), which can be done by simple generalizations of the arguments in the proofs of Lemmas 4.2 and 4.5. Next, we proceed to consider a convection operator on tent-shaped domains.

Definition 4.8. Suppose K and K_0 are as in (4.8). If, in addition, K can be divided into finitely many $(d+1)$-simplices with a common edge $\{(p,t) : g_{\mathrm{i}}(p) < t < g_{\mathrm{o}}(p)\}$, then we call K a space-time *tent*. We refer to the common edge as its *tent pole*. Clearly, in this case, g_{o} and g_{i} are linear on each simplex of K_0. We split the tent's boundary into the these parts:

$$\partial_{\mathrm{i}} K = \{(x, g_{\mathrm{i}}(x)) : x \in K_0\}, \qquad \partial_{\mathrm{o}} K = \{(x, g_{\mathrm{o}}(x)) : x \in K_0\}, \tag{4.10a}$$

$$\partial_{\mathrm{b}} K = \partial K \setminus (\partial_{\mathrm{i}} K \cup \partial_{\mathrm{o}} K). \tag{4.10b}$$

We refer to the two parts in (4.10a) as the tent's *inflow* and *outflow* boundaries, respectively. (Using such terms without regard to an underlying flow operator is an abuse of terminology that we overlook for expediency.)

The equation modeling advection along a fixed direction $\alpha \equiv (\alpha_i) \in \mathbb{R}^d$ is of the form $Au = f$ with

$$Au = \frac{\partial u}{\partial t} + \sum_{i=1}^{d} \alpha_i \frac{\partial u}{\partial x_i}. \tag{4.11}$$

Setting $L = L^2(K)$, $\mathcal{D} = \mathcal{D}(K)$, and noting that $\tilde{A} = -A$, we can put this into the Friedrichs framework since the prerequisite (2.1) holds.

Let $n \in \mathbb{R}^{d+1}$ denote the outward unit normal on ∂K. We often write it separating its space and time components as $n = (n_x, n_t)$ with $n_x \in \mathbb{R}^d$. We now assume that the tent boundaries are such that

$$\partial_{\mathrm{i}} K \subseteq \{(x,t) \in \partial K : n \text{ at } (x,t) \text{ satisfies } n_t + \alpha \cdot n_x < 0\} \tag{4.12a}$$

$$\partial_{\mathrm{o}} K \subseteq \{(x,t) \in \partial K : n \text{ at } (x,t) \text{ satisfies } n_t + \alpha \cdot n_x > 0\}. \tag{4.12b}$$

The vertical part of the boundary, namely $\partial_{\mathrm{b}} K$, is further split into three parts $\partial_{\mathrm{b}}^{+} K, \partial_{\mathrm{b}}^{-} K$, and $\partial_{\mathrm{b}}^{0} K$ where $n_t + \alpha \cdot n_x = \alpha \cdot n_x$ is positive, negative, and zero, respectively (see Figure 2). Let $\mathit{\Gamma}_{\mathrm{i}}$ and $\mathit{\Gamma}_{\mathrm{o}}$ denote the closures of $\partial_{\mathrm{i}} K \cup \partial_{\mathrm{b}}^{-} K$ and $\partial_{\mathrm{o}} K \cup \partial_{\mathrm{b}}^{+} K$, respectively, and let

$$\mathit{\Gamma}_{\mathrm{io}} = \mathit{\Gamma}_{\mathrm{i}} \cap \mathit{\Gamma}_{\mathrm{o}}.$$

Define $\delta(z) = \operatorname{dist}(z, \mathit{\Gamma}_{\mathrm{io}})$. We will use the restriction of this function to $\mathit{\Gamma}_{\mathrm{i}}$ and $\mathit{\Gamma}_{\mathrm{o}}$ as weight functions while describing the norm continuity of traces below. For smooth functions w on K, let

$$\tau_{\mathrm{i}} w = w|_{\mathit{\Gamma}_{\mathrm{i}}}, \qquad \tau_{\mathrm{o}} w = w|_{\mathit{\Gamma}_{\mathrm{o}}}.$$

Theorem 4.9. *Let K be a tent and A be given by* (4.11). *Suppose* (4.12) *holds. Then the above-defined maps τ_{i} and τ_{o} extend to continuous linear operators*

$$\tau_{\mathrm{i}} : W \to L^2_\delta(\mathit{\Gamma}_{\mathrm{i}}) \quad \text{and} \quad \tau_{\mathrm{o}} : W \to L^2_\delta(\mathit{\Gamma}_{\mathrm{o}}).$$

Hence $V = \ker(\tau_{\mathrm{i}})$ and $V^* = \ker(\tau_{\mathrm{o}})$ are closed subspaces of W. When restricted to these subspaces, the traces have an additional continuity property, namely

$$\tau_{\mathrm{i}} : V^* \to L^2_{1/\delta}(\mathit{\Gamma}_{\mathrm{i}}) \quad \text{and} \quad \tau_{\mathrm{o}} : V \to L^2_{1/\delta}(\mathit{\Gamma}_{\mathrm{o}}) \tag{4.13}$$

are continuous. Finally, with this V and V^*, the weak formulations (3.4) *and* (3.6) *are well-posed.*

Proof. The idea is to use a change of variable that brings the operator to the previously analyzed operator ∂_t. The new variables are $\hat{x} = x - \alpha t$ and $\hat{t} = t$, i.e.,

$$\begin{bmatrix} x \\ t \end{bmatrix} = H \begin{bmatrix} \hat{x} \\ \hat{t} \end{bmatrix} \quad \text{where} \quad H = \begin{bmatrix} I & \alpha \\ 0 & 1 \end{bmatrix}.$$

Let $\hat{K} = H^{-1}K$. (Note that $\hat{K}$ is not a tent, in general.) Pulling back functions w on K to functions $\hat{w} = w \circ H$ on $\hat{K}$, the chain rule gives

$$\hat{A}\hat{w} = (Aw) \circ H, \qquad \text{where} \qquad \hat{A} = \frac{\partial}{\partial \hat{t}}. \tag{4.14}$$

Thus $w \in W$ if and only if $\hat{w} \in \hat{W} = \{\hat{z} \in L^2(\hat{K}) : \hat{A}\hat{z} \in L^2(\hat{K})\}$.

Next, let $\hat{n} = (\hat{n}_{\hat{x}}, \hat{n}_{\hat{t}})$ denote the unit outward normal on $\partial\hat{K}$. Then $\hat{n} = (\hat{n}_{\hat{x}}, \hat{n}_{\hat{t}}) = H^t n/\|H^t n\|_2$. Defining $\partial_{\mathrm{i}}\hat{K}, \partial_{\mathrm{o}}\hat{K}$, and $\partial_{\mathrm{c}}\hat{K}$ as in (4.9), we claim that

$$\partial_{\mathrm{i}}\hat{K} \equiv \{(\hat{x},\hat{t}) \in \partial\hat{K} : \hat{n}_{\hat{t}} < 0\} = H^{-1}(\partial_{\mathrm{i}}K \cup \partial_{\mathrm{b}}^{-}K) \tag{4.15a}$$

$$\partial_{\mathrm{o}}\hat{K} \equiv \{(\hat{x},\hat{t}) \in \partial\hat{K} : \hat{n}_{\hat{t}} > 0\} = H^{-1}(\partial_{\mathrm{o}}K \cup \partial_{\mathrm{b}}^{+}K), \tag{4.15b}$$

$$\partial_{\mathrm{c}}\hat{K} \equiv \{(\hat{x},\hat{t}) \in \partial\hat{K} : \hat{n}_{\hat{t}} = 0\} = H^{-1}(\partial_{\mathrm{b}}^{0}K). \tag{4.15c}$$

For example, to sketch a proof of the first identity, note that n at $(x, g_{\mathrm{i}}(x))$ is in the direction of $(\nabla_x g_{\mathrm{i}}, -1)$ where ∇_x denotes the gradient with respect to x. Hence, because of (4.12), we have $\alpha \cdot \nabla_x g_{\mathrm{i}} - 1 < 0$ on $\partial_{\mathrm{i}}\hat{K}$. Since the mapped normal $\hat{n}$ is in the direction of

$$H^t n = \begin{bmatrix} I & 0 \\ \alpha^t & 1 \end{bmatrix} \begin{bmatrix} \nabla_x g_{\mathrm{i}} \\ -1 \end{bmatrix} = \begin{bmatrix} \nabla_x g_{\mathrm{i}} \\ \alpha \cdot \nabla_x g_{\mathrm{i}} - 1 \end{bmatrix}$$

we conclude that $\hat{n}_{\hat{t}} < 0$. Applying similar arguments on the remaining parts of the boundary, the claim (4.15) is proved.

Let $\hat{K}_0$ be the projection of $\hat{K}$ on the $\hat{t} = 0$ plane. There are (continuous piecewise linear) functions $\hat{g}_{\mathrm{o}}$ and $\hat{g}_{\mathrm{i}}$ such that $\partial_{\mathrm{o}}\hat{K}$ and $\partial_{\mathrm{i}}\hat{K}$ are graphs of $\hat{g}_{\mathrm{o}}$ and $\hat{g}_{\mathrm{i}}$, respectively, over $\hat{K}_0$. On $\hat{K}$, since $\hat{A} = \partial_{\hat{t}}$, we apply Theorem 4.7 to conclude that $\hat{\tau}_{\mathrm{i}}\hat{w} = \hat{w}|_{\partial_{\mathrm{i}}\hat{K}}$ and $\hat{\tau}_{\mathrm{o}}\hat{w} = \hat{w}|_{\partial_{\mathrm{o}}\hat{K}}$ extend to continous linear operators $\hat{\tau}_{\mathrm{i}} : \hat{W} \to L^2_{\hat{g}_{\mathrm{o}}-\hat{g}_{\mathrm{i}}}(\hat{K}_0)$ and $\hat{\tau}_{\mathrm{o}} : \hat{W} \to L^2_{\hat{g}_{\mathrm{o}}-\hat{g}_{\mathrm{i}}}(\hat{K}_0)$. Hence $\hat{V} = \ker(\hat{\tau}_{\mathrm{i}})$ and $\hat{V}^* = \ker(\hat{\tau}_{\mathrm{o}})$ are closed subspace of $\hat{W}$. By the additional continuity of $\hat{\tau}_{\mathrm{i}} - \hat{\tau}_{\mathrm{o}} : \hat{W} \to L^2_{1/(\hat{g}_{\mathrm{o}}-\hat{g}_{\mathrm{i}})}(\hat{K}_0)$ (also given by Theorem 4.7), we conclude that

$$\hat{\tau}_{\mathrm{i}} : \hat{V}^* \to L^2_{1/(\hat{g}_{\mathrm{o}}-\hat{g}_{\mathrm{i}})}(\hat{K}_0), \qquad \hat{\tau}_{\mathrm{o}} : \hat{V} \to L^2_{1/(\hat{g}_{\mathrm{o}}-\hat{g}_{\mathrm{i}})}(\hat{K}_0),$$

are also continuous.

These continuity results are more conveniently mapped to K by using $\hat{\delta}(z) = \mathrm{dist}(z, \hat{\varGamma}_{\mathrm{io}})$. Note that $\hat{g}_{\mathrm{o}} - \hat{g}_{\mathrm{i}}$ vanishes at $\hat{\varGamma}_{\mathrm{io}} = H^{-1}\varGamma_{\mathrm{io}}$. To restate the continuity properties of τ_{i} in terms of $\hat{\delta}$, we prove that there are $c_1, c_2 > 0$ such that

$$c_1\,\hat{\delta}(\hat{x}, \hat{g}_{\mathrm{i}}(\hat{x})) \le \hat{g}_{\mathrm{o}}(\hat{x}) - \hat{g}_{\mathrm{i}}(\hat{x}) \le c_2\,\hat{\delta}(\hat{x}, \hat{g}_{\mathrm{i}}(\hat{x})), \qquad \forall \hat{x} \in \hat{K}_0, \tag{4.16}$$

(and similarly for τ_{o}). When a point $N = (\hat{x}, \hat{g}_{\mathrm{i}}(\hat{x}))$ on $\partial_{\mathrm{i}}\hat{K}$ is sufficiently near to $\hat{\varGamma}_{\mathrm{io}}$, the point P nearest to it on $\hat{\varGamma}_{\mathrm{io}}$, together with $O = (\hat{x}, \hat{g}_{\mathrm{o}}(\hat{x}))$ form a triangle (as shown in Figure 2). Now we may restrict ourselves to the two-dimensional plane containing this triangle.

Consider the case when the segment PO lies on or below the plane of constant $\hat{t}$ passing through P, so that PO makes an angle $\theta_{\mathrm{o}} \ge 0$ with that plane. Let θ be the angle made

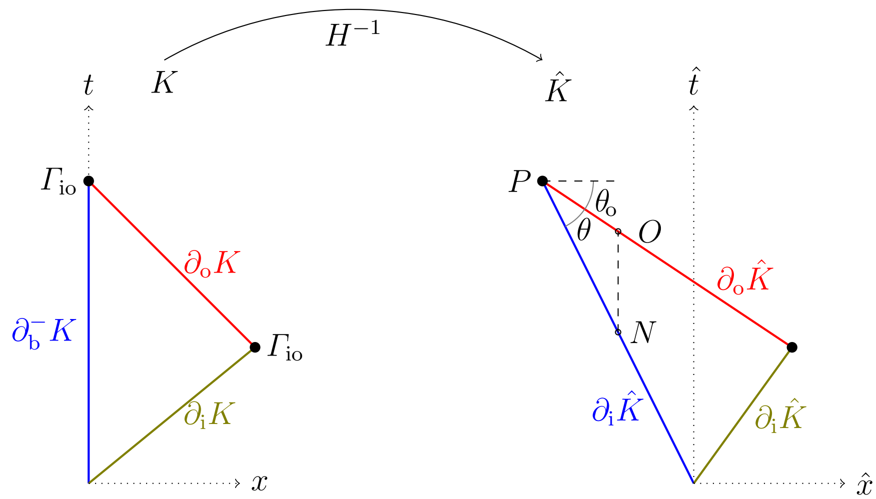


FIGURE 2. On the left is a tent K with $A = \partial_t + 0.5\partial_x$ that satisfies (4.12). On the right is $\hat{K}$ obtained after applying the map in the proof of Theorem 4.9 with mapped over operator $\hat{A} = \partial_{\hat{t}}$.

by PN and PO at P. Then, by elementary geometry,

$$\|P - N\|_2 = \frac{\cos\theta_{\rm o}}{\sin\theta}\|O - N\|_2. \tag{4.17}$$

Note that $\theta > 0$ and $0 \le \theta_{\rm o} < \pi/2$. Therefore, observing that $\|P - N\|_2 = \hat{\delta}(N)$ and $\|O - N\|_2 = \hat{g}_{\rm o} - \hat{g}_{\rm i}$, (4.17) proves (4.16). For the remaining geometrical configurations, identities similar to (4.17) can be derived to prove (4.16). Having established (4.16), we find that after mapping back to K, the stated continuity properties of $\tau_{\rm i} w = (\hat{\tau}_{\rm i}\hat{w}) \circ H^{-1}$ and $\tau_{\rm o} w = (\hat{\tau}_{\rm o}\hat{w}) \circ H^{-1}$ are proved.

It now only remains to prove the stated wellposedness of the weak formulations. By Theorem 4.7, for any $\hat{F} \in \hat{W}'$, there is a unique $\hat{u} \in L^2(\hat{K})$ and $\hat{q} = \hat{D}\hat{z} \in (\hat{V}^*)^\perp$ satisfying

$$-(\hat{u}, \hat{A}\hat{v})_{L^2(\hat{K})} + \langle \hat{D}\hat{z}, \hat{v}\rangle_{\hat{W}} = \hat{F}(\hat{v}), \qquad \forall\, \hat{v} \in \hat{W}, \tag{4.18}$$

where $\hat{D} \in \mathcal{L}(\hat{W}, \hat{W}')$ is defined as before by $\langle \hat{D}\hat{v}, \hat{w}\rangle_{\hat{W}} = (\hat{A}\hat{v}, \hat{w})_{L^2(\hat{K})} + (\hat{v}, \hat{A}\hat{v})_{L^2(\hat{K})}$. Here we have used (2.8) to find a $\hat{z} \in \hat{V}$ such that $\hat{q} = \hat{D}\hat{z}$. (While $\hat{q}$ is unique, $\hat{z}$ need not be unique.) It now follows from the properties of the mapping that $\hat{u}$ and $\hat{q} = \hat{D}\hat{z}$ satisfies (4.18) if and only if $u = \hat{u} \circ H^{-1}$ and $z = \hat{z} \circ H^{-1}$ satisfies

$$-(u, Av)_{L^2(K)} + \langle Dz, v\rangle_W = \hat{F}(v \circ H), \qquad \forall\, v \in W. \tag{4.19}$$

Here we have used the fact that (4.14) implies $(\hat{u}, \hat{A}\hat{v})_{L^2(\hat{K})} = (u, Av)_{L^2(K)}$ and consequently, $\langle \hat{D}\hat{z}, \hat{v}\rangle_{\hat{W}} = \langle Dz, v\rangle_W$. This shows that the weak formulations on $\hat{K}$ and K are equivalent, so the wellposedness of the latter, namely (3.4), follows from the former. The wellposedness of (3.6) is proved similarly. $\square$

Remark 4.10. Under additional assumptions, including $\operatorname{dist}(\partial_{\rm i}K, \partial_{\rm o}K) > 0$, a stronger trace result is proved in [9, Lemma 5.1]. However, on tents, $\operatorname{dist}(\partial_{\rm i}K, \partial_{\rm o}K)$ is always zero, so we are unable to use their result.

4.3. **A linear symmetric hyperbolic system in one space dimension.** Let $C \in \mathbb{R}^{m\times m}$ be an $m \times m$ real symmetric matrix and let K be a tent as in §4.2. Set $L = L^2(K)^m, \mathcal{D} = \mathcal{D}(K)^m$ and

$$Au = \frac{\partial u}{\partial t} + C\frac{\partial u}{\partial x} \tag{4.20}$$

where $\partial_t u$ and $\partial_x u$ are vectors in $\mathbb{R}^m$ with their ℓth component equal to $\partial_t u_\ell$ and $\partial_x u_\ell$, respectively. Since C is symmetric, $\tilde{A} = -A$, so assumption (2.1) is obviously satisfied.

Let Q be an orthogonal matrix and $\varLambda = \operatorname{diag}(\lambda_\ell)$ be a diagonal matrix such that $C = Q\varLambda Q^t$. Let $\partial_{\rm i}K, \partial_{\rm o}K$ and $\partial_{\rm b}K$ be as defined in (4.10). In this subsection, we assume – instead of (4.12) – that

$$\partial_{\rm i}K \subseteq \{x \in \partial K : n_t I + n_x C \text{ is negative definite}\}, \tag{4.21a}$$

$$\partial_{\rm o}K \subseteq \{x \in \partial K : n_t I + n_x C \text{ is positive definite}\}. \tag{4.21b}$$

For each $\ell = 1, \ldots, m$, we decompose $\partial_{\rm b}K$ into $\partial_{\rm b}^{+,\ell}K, \partial_{\rm b}^{-,\ell}K$, and $\partial_{\rm b}^{0,\ell}K$ where $\lambda_\ell n_x$ is positive, negative, and zero, respectively. Then we have the following theorem, which is proved using the diagonalization of C to separate each component and then appealing to the analysis in § 4.2. We now opt for a brief statement of the theorem, leaving the tacitly used properties of the traces to the proof.

Theorem 4.11. *Suppose* (4.21) *holds for the tent K and the operator A in* (4.20)*. Then, the formulations* (3.4) *and* (3.6) *with*

$$V = \{z \in W : [Q^t z]_\ell\big|_{\partial_{\rm i}K\cup\partial_{\rm b}^{-,\ell}K} = 0, \text{ for all } \ell = 1, \ldots, m\}, \tag{4.22a}$$

$$V^* = \{z \in W : [Q^t z]_\ell\big|_{\partial_{\rm o}K\cup\partial_{\rm b}^{+,\ell}K} = 0, \text{ for all } \ell = 1, \ldots, m\}, \tag{4.22b}$$

are well-posed.

Proof. Let $\breve{A} = Q^t A Q$, $\breve{W} = \{\breve{v} \in L^2(K)^m : \breve{A}\breve{v} \in L^2(K)^m\}$, and $\breve{D}$ be the corresponding boundary operator on $\breve{W}$. Then clearly, $v \in W$ if and only if $\breve{v} = Q^t v$ is in $\breve{W}$. Moreover, $\breve{A}\breve{w} = \partial_t\breve{w} + \varLambda\partial_x\breve{w}$, i.e., its ℓth component equals

$$\breve{A}_\ell\breve{w}_\ell \equiv \partial_t\breve{w}_\ell + \lambda_\ell\partial_x\breve{w}_\ell.$$

Note that $\breve{A}_\ell$ is a Friedrichs operator on K of the form (4.11) and has its associated graph space $\breve{W}_\ell$ and boundary operator $\breve{D}_\ell$.

Now, the assumptions of (4.21) imply that (4.12) holds for each $\breve{A}_\ell$ (with $\alpha = \lambda_\ell$) so Theorem 4.9 yields the continuity of the maps $\breve{\tau}_{\rm i}^\ell : \breve{w} \mapsto \breve{w}_\ell|_{\varGamma_{\rm i}^\ell}$ and $\breve{\tau}_{\rm o}^\ell : \breve{w} \mapsto \breve{w}_\ell|_{\varGamma_{\rm o}^\ell}$ on $\breve{W}$, where $\varGamma_{\rm i}^\ell = \partial_{\rm i}K \cup \partial_{\rm b}^{-,\ell}K$ and $\varGamma_{\rm o}^\ell = \partial_{\rm o}K \cup \partial_{\rm b}^{+,\ell}K$. Therefore, the full trace maps $\breve{\tau}_{\rm i} = (\breve{\tau}_{\rm i}^1, \ldots, \breve{\tau}_{\rm i}^m)$ and $\breve{\tau}_{\rm o} = (\breve{\tau}_{\rm o}^1, \ldots, \breve{\tau}_{\rm o}^m)$ are continuous on $\breve{W}$. Set $\breve{V} = \ker(\breve{\tau}_{\rm i})$ and

$\breve{V}^* = \ker(\breve{\tau}_{\mathrm{o}})$. Then the following variational equation for $\breve{u} \in L^2(K)^m$ and $\breve{q} = \breve{D}\breve{z}$ with $\breve{z} \in \breve{V}$,

$$-(\breve{u}, \breve{A}\breve{v})_{L^2(K)^m} + \langle \breve{q}, \breve{v} \rangle_{\breve{W}} = F(\breve{v}), \qquad \forall \breve{v} \in \breve{W}, \tag{4.23}$$

splits into m decoupled equations, namely

$$-(\breve{u}_\ell, \breve{A}_\ell \breve{v}_\ell)_{L^2(K)} + \langle \breve{D}_\ell \breve{z}_\ell, \breve{v}_\ell \rangle_{\breve{W}_\ell} = F(\breve{v}_\ell), \qquad \forall \breve{v}_\ell \in \breve{W}_\ell, \quad \forall \ell = 1, \ldots, m. \tag{4.24}$$

Here $\breve{u}_\ell \in L^2(K)$ and $\breve{z}_\ell \in \breve{V}_\ell \equiv \ker(\breve{\tau}_{\mathrm{i}}^\ell)$ are the ℓth components of $\breve{u}$ and $\breve{z}$, respectively. By Theorem 4.9, there is a unique $\breve{u}_\ell \in L^2(K)$ and $\breve{q}_\ell = \breve{D}_\ell \breve{z}_\ell \in (\breve{V}_\ell^*)^\perp$ solving (4.24) for each ℓ. This in turn proves the wellposedness of (4.23).

To transfer these results for $\breve{A}$ to A, we define

$$\tau_{\mathrm{i}} w = \breve{\tau}_{\mathrm{i}}(Q^t w), \qquad \tau_{\mathrm{o}} w = \breve{\tau}_{\mathrm{o}}(Q^t w).$$

Then (4.22) is the same as $V = \ker(\tau_{\mathrm{i}})$ and $V^* = \ker(\tau_{\mathrm{o}})$. Note that $z \in V$ if and only if $\breve{z} = Q^t z \in \breve{V}$. Also note that a $\breve{u} \in L^2(K)^m$ and $\breve{z} \in \breve{V}$ solves (4.23) if and only if $u = Q\breve{u}$ and $z = Q\breve{z}$ satisfies

$$-(u, Av)_{L^2(K)^m} + \langle Dz, v \rangle_W = F(Q^t v), \qquad \forall v \in W.$$

Here we have used $(\breve{u}, \breve{A}\breve{v})_{L^2(K)^m} = (u, Av)_{L^2(K)^m}$ and consequent identities for the corresponding boundary operators. Thus the stated wellposedness of (3.4) follows from the established wellposedness of (4.23). The proof of wellposedness of (3.6) is similar. □

Remark 4.12. Consider a tent K with empty $\partial_{\mathrm{b}} K$. Then, under the assumptions of Theorem 4.11, a function in V has all its m components equal to zero on the inflow boundary $\partial_{\mathrm{i}} K$. Moreover, if $v \in V \cap C(\bar{K})$, then applying the additional continuity property (4.13) to the operators $\breve{\tau}_{\mathrm{o}}^\ell$ in the above proof, we find that the outflow trace of each component of v must approach zero as we approach Γ_{io} where the inflow and outflow boundary parts meet.

5. The wave equation

We now apply the previous ideas to the important example of the wave equation and work out the resulting weak formulation in detail. Our model problem is to find a real-valued function ϕ on the space-time domain $\Omega = (0, S) \times (0, T)$, satisfying

$$c^{-2}\partial_{tt}\phi - \partial_{xx}\phi = g, \qquad 0 < x < S,\ 0 < t < T, \tag{5.1a}$$

$$\partial_t \phi = \phi = 0, \qquad t = 0,\ 0 < x < S, \tag{5.1b}$$

$$\partial_t \phi - c\,\partial_x \phi = 0, \qquad x = 0,\ 0 < t < T, \tag{5.1c}$$

$$\partial_t \phi + c\,\partial_x \phi = 0, \qquad x = S,\ 0 < t < T, \tag{5.1d}$$

where $c > 0$ is the wave speed. Here, we have imposed the outgoing impedance boundary conditions (but other boundary conditions can also be considered – see Section 7).

The above second order system for ϕ arises from a system of first order physical principles, which also matches the form of the problems we have been studying, namely (2.9). Set

$$u = \begin{bmatrix} c\,\partial_x \phi \\ \partial_t \phi \end{bmatrix}$$

and observe that $\partial_t u_1 = c\,\partial_{xt}\phi = c\,\partial_x u_2$ and $\partial_t u_2 = \partial_{tt}\phi = c\,\partial_x u_1 + c^2 g$. These two equations give the first order system $Au = f$ where

$$Au = \partial_t u - \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} \partial_x u, \qquad f = \begin{bmatrix} 0 \\ c^2 g \end{bmatrix}. \tag{5.2}$$

It fits into the framework of §4.3 after the diagonalization

$$C \equiv -\begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} = Q\Lambda Q^t, \quad Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

where $\lambda_1 = -c$ and $\lambda_2 = c$.

Analogous to (4.9), we define $\partial_{\mathrm{i}}\varOmega = (0,S) \times \{0\}$, $\partial_{\mathrm{o}}\varOmega = (0,S) \times \{T\}$ and $\partial_{\mathrm{b}}\varOmega = \partial\varOmega \setminus (\partial_{\mathrm{i}}\varOmega \cup \partial_{\mathrm{o}}\varOmega)$. The vertical parts $\partial_{\mathrm{b}}\varOmega$ are further split into

$$\partial_{\mathrm{b}}^{+,1}\varOmega = \partial_{\mathrm{b}}^{-,2}\varOmega = \{0\} \times [0,T], \qquad \partial_{\mathrm{b}}^{-,1}\varOmega = \partial_{\mathrm{b}}^{+,2}\varOmega = \{S\} \times [0,T].$$

Set $\varGamma_{\mathrm{i}}^{\ell}$ and $\varGamma_{\mathrm{o}}^{\ell}$ to the closures of $\partial_{\mathrm{i}}\varOmega \cup \partial_{\mathrm{b}}^{-,\ell}\varOmega$ and $\partial_{\mathrm{o}}\varOmega \cup \partial_{\mathrm{b}}^{+,\ell}\varOmega$, respectively, $\varGamma_{\mathrm{io}}^{\ell} = \varGamma_{\mathrm{i}}^{\ell} \cap \varGamma_{\mathrm{o}}^{\ell}$, and $\delta_\ell(x,t) = \operatorname{dist}((x,t), \varGamma_{\mathrm{io}}^{\ell})$ for $\ell = 1,2$. By a minor modification of the arguments in Section 4, one can prove that the global trace maps

$$\tau_{\mathrm{i}} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} (z_1 + z_2)\big|_{\varGamma_{\mathrm{i}}^1} \\ (z_1 - z_2)\big|_{\varGamma_{\mathrm{i}}^2} \end{bmatrix}, \qquad \tau_{\mathrm{i}} : W \to L^2_{\delta_1}(\varGamma_{\mathrm{i}}^1) \times L^2_{\delta_2}(\varGamma_{\mathrm{i}}^2)$$

$$\tau_{\mathrm{o}} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} (z_1 + z_2)\big|_{\varGamma_{\mathrm{o}}^1} \\ (z_1 - z_2)\big|_{\varGamma_{\mathrm{o}}^2} \end{bmatrix}, \qquad \tau_{\mathrm{o}} : W \to L^2_{\delta_1}(\varGamma_{\mathrm{o}}^1) \times L^2_{\delta_2}(\varGamma_{\mathrm{o}}^2)$$

are continuous. Set

$$V(\varOmega) = \ker(\tau_{\mathrm{i}}), \qquad V^*(\varOmega) = \ker(\tau_{\mathrm{o}}).$$

These spaces can be used to give a global weak formulation on $\varOmega$, but our focus in on local solvers.

In space-time tent pitching methods, we are required to numerically solve the wave equation on space-time tents, ordered so that inflow data on a tent can be provided by the outflow solution on previously handled tents or through given data. Hence we now focus on the formulation and discretization on one tent K.

5.1. **Weak formulation on a tent.** Consider the analogue of (5.1) on one tent K, with zero initial data on the inflow boundaries and with boundary conditions inherited from the global boundary conditions (5.1c)–(5.1d).

Define, as before, the boundary parts of a tent K, by

$$\partial_{\rm i} K = \{(x,t) \in \partial K : n_t < 0\}, \quad \partial_{\rm o} K = \{(x,t) \in \partial K : n_t > 0\}, \quad \partial_{\rm b} K = \partial K \setminus (\partial_{\rm i} K \cup \partial_{\rm o} K),$$
$$\partial_{\rm b}^{+,1} K = \partial_{\rm b}^{-,2} K = \{(x,t) \in \partial_{\rm b} K : c n_x < 0\},$$
$$\partial_{\rm b}^{+,2} K = \partial_{\rm b}^{-,1} K = \{(x,t) \in \partial_{\rm b} K : c n_x > 0\}.$$

Note that the boundary part $\partial_{\rm b} K$ may be empty in some tents. We consider the tent problem of solving for u satisfying

$$\begin{aligned} Au &= f && \text{on } K, && u_1 - u_2 = 0 && \text{on } \partial_{\rm b}^{+,1} K,\\ u &= 0 && \text{on } \partial_{\rm i} K, && u_1 + u_2 = 0 && \text{on } \partial_{\rm b}^{+,2} K. \end{aligned}$$

To obtain a well-posed weak formulation on one tent, we proceed to use Theorem 4.11.

To this end, we must assume that the tent satisfies Assumption (4.21), which now reads

$$\partial_{\rm i} K \subseteq \{x \in \partial K : n_t \pm n_x c < 0\}, \tag{5.3a}$$
$$\partial_{\rm o} K \subseteq \{x \in \partial K : n_t \pm n_x c > 0\}. \tag{5.3b}$$

Since $\tilde{A} = -A$ in this example, the weak formulation (3.4) reads

$$u \in L, \ q \in (V^*)^\perp : \qquad -(u, Av)_L + \langle q, v \rangle_W = F(v), \qquad \forall v \in W, \tag{5.4}$$

where the spaces are set following (4.22), namely

$$V = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in W : \begin{bmatrix} z_1 + z_2 \\ z_1 - z_2 \end{bmatrix}_\ell \bigg|_{\partial_{\rm i} K \cup \partial_{\rm b}^{-,\ell} K} = 0, \quad \text{for } \ell = 1, 2 \right\},$$
$$V^* = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in W : \begin{bmatrix} z_1 + z_2 \\ z_1 - z_2 \end{bmatrix}_\ell \bigg|_{\partial_{\rm o} K \cup \partial_{\rm b}^{+,\ell} K} = 0, \quad \text{for } \ell = 1, 2 \right\}.$$

Theorem 4.11 shows that (5.4) is a well-posed weak formulation on K provided the tent K satisfies (5.3). Note that the above spaces change from tent to tent and may arguably be better denoted by $V(K), V^*(K)$, etc., but to avoid notational bulk we will suppress the K-dependence.

5.2. **CFL condition.** Let us take a closer look at (5.3). First note that each tent, in this application, consists of either two triangles (on either side of the tent pole), or just one triangle. The tents are thus divided into three types, as shown in Figure 3.

The length of the tent pole is k, the numbers p_l and p_r are such that $p_l k$ and $p_r k$ give the heights of the outflow boundaries on the left and right side of the tent pole, respectively, and the spatial mesh size are $h_r, h_l \geq 0$. Writing down the normal vector on the tent boundaries, we immediately find that condition (5.3) on a tent is equivalent to

$$\left| c \frac{k p_r}{h_r} \right| < 1 \quad \text{and} \quad \left| c \frac{k p_l}{h_l} \right| < 1. \tag{5.5}$$

Clearly, by controlling the size of the tent pole we can satisfy these inequalities.

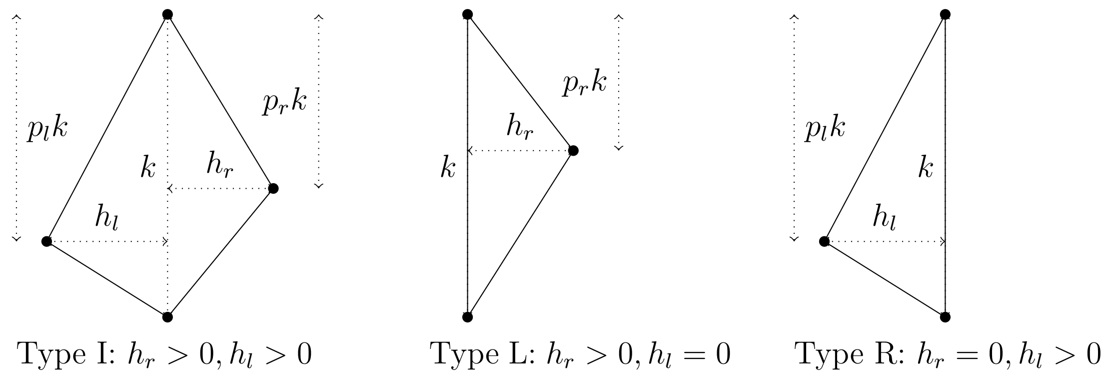


FIGURE 3. Three types of tents

The well-known Courant-Friedrichs-Levy (CFL) condition [5] identifies stability conditions as constraints on the time step size in terms of space mesh size in numerical discretizations. In our case, this condition manifests itself as geometrical constraints (5.5) on the tent. For this reason, we will refer to (5.3) – or (5.5) – as the CFL condition of our method.

6. The numerical scheme

In this section, continuing to work with the wave operator A defined by (5.2), we give an explicit numerical scheme for approximating $u(x, t)$ satisfying

(6.1a) $$Au = f, \qquad 0 < x < S,\ 0 < t < T,$$

(6.1b) $$u_i(x, 0) = u_i^0(x), \qquad 0 < x < S,\ i = 1, 2,$$

(6.1c) $$u_1(0, t) - u_2(0, t) = 0, \qquad 0 < t < T,$$

(6.1d) $$u_1(1, t) + u_2(1, t) = 0, \qquad 0 < t < T.$$

The scheme will allow varying spatial and temporal mesh sizes. Here f and u^0 are assumed to given smooth functions. We begin by describing the calculations within each tent, followed by the tent pitching technique to advance in time.

6.1. **Conforming discretization on a tent.** As seen above, a tent is comprised of one or two triangles. Let the space of continuous functions on a tent K whose restrictions to these triangles are linear be denoted by $P_1^h(K)$. We construct a conforming discretization of (5.4) within K using the discrete space

(6.2) $$V_1 = V \cap (P_1^h(K))^2.$$

By definition, $V_1 \subseteq V$, and consequently, functions in V_1 must satisfy the essential boundary conditions of V. Depending on the tent geometry, different boundary conditions must be imposed on different tents.

To examine what this entails for the nodal coefficients on mesh vertices, let $\zeta \in P_1^h(K)$ be the continuous scalar function (unique Lagrange basis function) that equals one at the "apex" of the tent K, equals zero at all its other vertices. The *apex* of a tent, irrespective of whether it consists of one or two triangles, is the vertex in $\partial_{\rm o}K$ that is away from $\partial_{\rm i}K$. Now, suppose $\mu \equiv \left[\begin{smallmatrix}\mu_1\\ \mu_2\end{smallmatrix}\right]$ in $\mathbb{R}^2$ is such that

$$\mu_1 - \mu_2 = 0 \qquad \text{if } \partial_{\rm b}^{+,1}K \text{ is nonempty,} \tag{6.3a}$$

$$\mu_1 + \mu_2 = 0 \qquad \text{if } \partial_{\rm b}^{-,1}K \text{ is nonempty.} \tag{6.3b}$$

(Note that if $\partial_{\rm b}K$ is empty, then μ is an arbitrary vector in $\mathbb{R}^2$.) Then, it is easy to see that

$$V_1 = \{\mu\zeta : \ \mu \text{ satisfies (6.3)}\} \tag{6.4}$$

provides an alternate characterization of (6.2).

A computable conforming discretization of (5.4) additionally requires finite-dimensional subspaces of L and $(V^*)^\perp$. For the latter, observe that (2.8) implies that

$$D(V_1) \subset D(V) = (V^*)^\perp.$$

Hence we choose an approximation q_1 of the solution component q in (5.4) to have the form

$$q_1 = Dz_1, \qquad z_1 \in V_1.$$

Then q_1 is clearly in $(V^*)^\perp$. Next, set $L_1 \subset L$ to be the space of vector functions whose components are constant functions on K. Finally, set

$$W_1 = \left\{w : \ w = \kappa + \mu\zeta, \quad \kappa, \mu \in \mathbb{R}^2, \quad \kappa\zeta \in V_1\right\}. \tag{6.5}$$

Our discretization of (5.4) now takes the following form: Find u_1 in L_1 and $q_1 \in D(V_1)$ satisfying $-(u_1, Aw) + \langle q_1, w\rangle = F(w)$, for all $w \in W_1$. Clearly, $\dim(W_1)$ is four or three, depending on whether $\partial_{\rm b}K$ is empty or not. This equation gives rise to an invertible discrete system, as a consequence of the unisolvency of the following slightly modified problem:

$$\begin{aligned} &\textit{Find } u_1 \in L_1 \textit{ and } z_1 \in V_1 \textit{ such that} \\ &\qquad -(u_1, Aw) + \langle Dz_1, w\rangle = F(w), \qquad \forall w \in W_1. \end{aligned} \tag{6.6}$$

Proposition 6.1. *There is a unique solution for Problem* (6.6).

Proof. Note the $\dim(L_1) + \dim(V_1) = \dim(W_1)$, so (6.6) gives a square (Petrov-Galerkin) system. Hence it suffices to set $F = 0$ and prove that $u_1 = z_1 = 0$. With $F = 0$, writing $z_1 = \alpha\zeta$ for some $\alpha \in \mathbb{R}^2$, we have $\langle D(\alpha\zeta), w\rangle = 0$ for all constant $w \in W$. Since $\alpha\zeta \in V_1$, by the definition of W_1, we may set $w = \alpha$ in (6.6), to get

$$\langle D(\alpha\zeta), \alpha\rangle = \int_{\partial_{\rm o}K \cup \partial_{\rm b}K} \mathscr{D}\alpha \cdot \alpha\zeta = 0$$

where

$$\mathscr{D} = n_t I + C n_x = \begin{bmatrix} n_t & -cn_x \\ -cn_x & n_t \end{bmatrix}. \tag{6.7}$$

If $\partial_{\rm b}K$ is empty, then since $\mathscr{D}\alpha \cdot \alpha = (\alpha_1 + \alpha_2)^2(n_t - cn_x)/2 + (\alpha_1 - \alpha_2)^2(n_t + cn_x)/2$, the CFL condition (5.3) gives $\alpha = 0$. If $\partial_{\rm b}K$ is nonempty, then whenever $\alpha\zeta \in V$ we have either $\alpha_1 - \alpha_2 = 0$ or $\alpha_1 + \alpha_2 = 0$, so we can continue to conclude that $\alpha = 0$. Of course $\alpha = 0$ implies $z_1 = 0$.

To prove that $u_1 = 0$, we use (2.3) after substituting $z_1 = 0$, to get

$$\langle Dw, u_1 \rangle = 0, \qquad \forall w \in W_1.$$

Since u_1 is a constant function, $u_1\zeta \in W_1$, so we may choose $w = u_1\zeta$ and conclude that $u_1 = 0$ by an argument analogous to what we used above. □

Remark 6.2. One can view $z_1|_{\partial_{\mathrm{o}} K}$ as an interface trace variable and $q_1 = Dz_1$ as an interface flux variable. By the trace theory we developed previously, outflow trace $z_1|_{\partial_{\mathrm{o}} K}$ must vanish at the points where outflow and inflow edges meet in order for z_1 to be in V. This motivates our choice (6.4) of V_1 to obtain a conforming method. Other non-conforming avenues to design approximations within a tent can be found in [10] and [19].

6.2. **Advancing in time by tent pitching.** We now show how the above ideas yield an explicit time marching algorithm for solving (6.1). First, we mesh the space-time domain $\Omega = (0, S) \times (0, T)$ by a collection Ω_h of tents K with these properties: The first property is that either $\partial_{\mathrm{b}} K$ is empty or

$$\partial_{\mathrm{b}} K \subseteq \partial_{\mathrm{b}} \Omega, \tag{6.8a}$$

for all $K \in \Omega_h$. Second, there exists an enumeration of all tents, $K_1, K_2, \ldots, K_J$, with the property that for each $j \in \{1, \ldots J\}$,

$$\partial_{\mathrm{i}} K_j \subseteq \bigcup_{k=1}^{j-1} \partial_{\mathrm{o}} K_k \cup \partial_{\mathrm{i}} \Omega. \tag{6.8b}$$

Finally, for all $j \in \{1, \ldots J\}$,

$$K_j \text{ satisfies the CFL condition (5.5).} \tag{6.8c}$$

It is well-known how to construct an algorithm (not only in one space dimension, but also in higher dimensions [6, 25]) that produces meshes satisfying (6.8), so we shall not dwell further on the meshing process.

The discrete space-time approximation on the mesh Ω_h is developed using

$$V_h = \big\{ z \in H^1(\Omega)^2 \cap V(\Omega) : \ z|_K \in P_1^h(K)^2, \ \forall K \in \Omega_h, \ \text{and } z(x, 0) = I_h u^0(x) \big\} \tag{6.9a}$$

$$L_h = \big\{ \alpha : \ \alpha|_K \in \mathbb{R}^2 \text{ is constant on each } K \in \Omega_h \big\} \tag{6.9b}$$

$$W_h = \{ w : \ w|_K \in W_1 \text{ on each } K \in \Omega_h \}, \tag{6.9c}$$

where I_h denote the linear nodal interpolant on the spatial mesh. The method finds approximations $u_h \in L_h$ and $z_h \in V_h$ satisfying

$$\sum_{K \in \Omega_h} \left(- \int_K u_h \cdot Aw + \int_{\partial K} \mathscr{D} z_h \cdot w \right) = \sum_{K \in \Omega_h} \int_K f \cdot w, \qquad \forall w \in W_h, \tag{6.10}$$

where $\mathscr{D}$ is as in (6.7).

Because of (6.8), we are able to use a time-marching algorithm to solve (6.10): Proceed in the ordering of (6.8b), and for each tent K, solve for $u_h|_K$ and $z_h|_K$. Specifically, if α

is the nodal (vector) value of z_h at the apex of K, then defining $z_{\mathrm{o}}^K = \alpha\zeta$, the problem on one tent is to find $u_h|_K \in L_1$ and $z_{\mathrm{o}}^K \in V_1$ satisfying (6.10), namely

$$-\int_K u_h \cdot Aw + \int_{\partial K} \mathscr{D} z_{\mathrm{o}}^K \cdot w = \int_K f \cdot w - \int_{\partial K} \mathscr{D} z_{\mathrm{i}}^K \cdot w, \qquad \forall w \in W_1. \tag{6.11}$$

where $z_{\mathrm{i}}^K = z_h - z_{\mathrm{o}}^K$. Note that z_{i}^K on right hand side will be a known quantity if (6.8b) holds and if we have already solved on every K' appearing before K in the ordering of tents in (6.8). Indeed, z_{i}^K is completely determined by its nodal values at (the three or two) vertices on $\partial_{\mathrm{i}} K$, which either lie at $t = 0$ or were apex vertices of previous tents. Problem (6.11) is exactly of the same type we discussed in § 6.1.

6.3. **Propagation formula.** Since the system (6.11) is small, we can explicitly calculate its solution. To see how information is propagated from inflow to outflow on a mesh of tents, we consider the case where the volume source f is zero. Write $z_h = \left[\begin{smallmatrix} z_{h,1} \\ z_{h,2} \end{smallmatrix}\right]$ in (6.10) and let the nodal values of the scalar Lagrange finite element functions $z_{h,1}$ and $z_{h,2}$ be $\left[\begin{smallmatrix} U^t \\ V^t \end{smallmatrix}\right], \left[\begin{smallmatrix} U^b \\ V^b \end{smallmatrix}\right], \left[\begin{smallmatrix} U^l \\ V^l \end{smallmatrix}\right], \left[\begin{smallmatrix} U^r \\ V^r \end{smallmatrix}\right]$, at the top, bottom, left and right vertices, respectively, of a tent of Type I, as in Figure 3. For the other two tent types, we omit the nodal values at the missing vertex.

Equation (6.11) finds $\left[\begin{smallmatrix} U^t \\ V^t \end{smallmatrix}\right]$ as a function of the remaining nodal values. After tedious simplifications (not displayed), this relationship is found to be as follows:

$$\begin{aligned}
\begin{bmatrix} U^t \\ V^t \end{bmatrix} &= \begin{bmatrix} U^b \\ V^b \end{bmatrix} + w_1 \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} \begin{bmatrix} U^r - U^l \\ V^r - V^l \end{bmatrix} + w_2 c \begin{bmatrix} U^r - U^l \\ V^r - V^l \end{bmatrix}, && \text{for Type I,} \\
\begin{bmatrix} U^t \\ V^t \end{bmatrix} &= \begin{bmatrix} U^b \\ V^b \end{bmatrix} + w_1 \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} \begin{bmatrix} U^r - U^b \\ V^r - V^b \end{bmatrix} + w_2 c \begin{bmatrix} U^r - U^b \\ V^r - V^b \end{bmatrix}, && \text{for Type L,} \\
\begin{bmatrix} U^t \\ V^t \end{bmatrix} &= \begin{bmatrix} U^b \\ V^b \end{bmatrix} + w_1 \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} \begin{bmatrix} U^b - U^l \\ V^b - V^l \end{bmatrix} + w_2 c \begin{bmatrix} U^b - U^l \\ V^b - V^l \end{bmatrix}, && \text{for Type R,}
\end{aligned} \tag{6.12}$$

where

$$\begin{aligned}
w_1 &= \frac{(h_r + h_l)k}{(h_r + h_l)^2 - c^2 k^2 (p_r - p_l)^2}, & w_2 &= \frac{c\, k^2 (p_r - p_l)}{(h_r + h_l)^2 - c^2 k^2 (p_r - p_l)^2}, && \text{for Type I,} \\
w_1 &= \frac{k}{2(ck(1 - p_r) + h_r)}, & w_2 &= w_1, && \text{for Type L,} \\
w_1 &= \frac{k}{2(ck(1 - p_l) + h_l)}, & w_2 &= -w_1, && \text{for Type R.}
\end{aligned}$$

6.4. **Error analysis on uniform grids.** We now work out the stencil given by the method on a uniform grid where all tents are shaped the same (see Figure 4). The stencil translates (6.12) into an equation that gives the nodal values of the outflow apex vertex, given the nodal values at the inflow vertices. Let $h > 0$ be the uniform spatial mesh size, $k > 0$ be the time step size measured, as before, by the height of the tent pole. At a point $(jh/2, kn/2)$ in the lattice $(h/2)\mathbb{Z} \times (k/2)\mathbb{Z}$, let (U_j^n, V_j^n) denote the nodal value of the approximation to z_h there. As shown in Figure 4, the scheme uses only a subset of

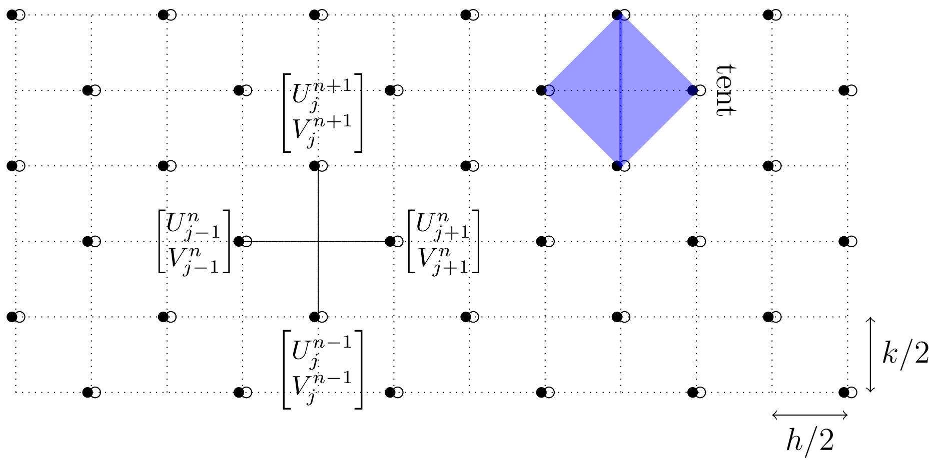


FIGURE 4. The stencil

lattice points in $h\mathbb{Z} \times k\mathbb{Z}$. Each grid point involved in the scheme has an associated U value (indicated in the figure by "$\bullet$") and a V value (indicated by "$\circ$").

Equation (6.12) now simplifies to

$$U_j^{n+1} = U_j^{n-1} + ac(V_{j+1}^n - V_{j-1}^n) \tag{6.13a}$$

$$V_j^{n+1} = V_j^{n-1} + ac(U_{j+1}^n - U_{j-1}^n) \tag{6.13b}$$

where $a = k/h$. This is simply the non-staggered leapfrog scheme (studied extensively for scalar equations) applied to the first order system. By a simple Taylor expansion about the stencil center, we see that the scheme is consistent and that the local truncation error is of second order (see [23, 24] for definitions of these and related terminology).

To examine stability, introduce a new vector variable X_j^n and rewrite the scheme (6.13) as follows:

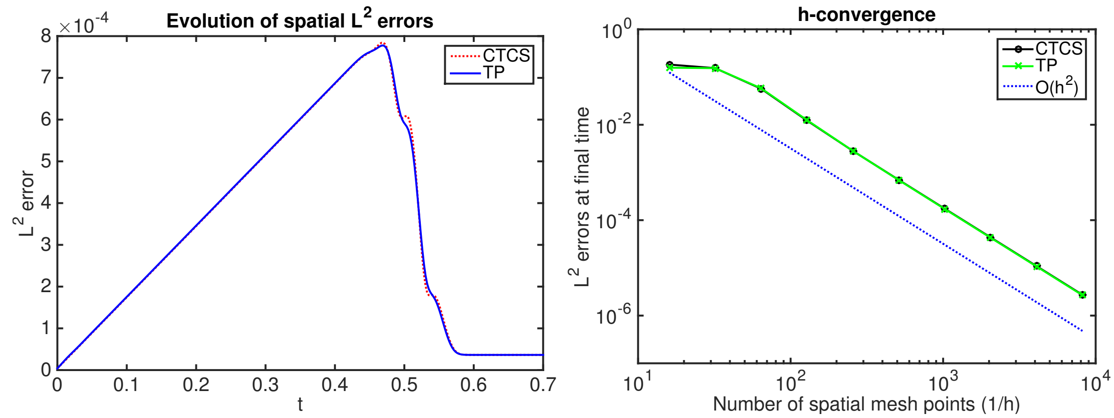
$$X_j^{n+1} = \begin{bmatrix} [X_j^n]_3 + ac[X_{j+1}^n - X_{j-1}^n]_2 \\ [X_j^n]_4 + ac[X_{j+1}^n - X_{j-1}^n]_1 \\ [X_j^n]_1 \\ [X_j^n]_2 \end{bmatrix}, \qquad \text{where } X_j^n = \begin{bmatrix} U_j^n \\ V_j^n \\ U_j^{n-1} \\ V_j^{n-1} \end{bmatrix}.$$

To this one-step scheme, we now apply von Neumann analysis [23, 24]. The amplification matrix G, connecting X_j^{n+1} to X_j^n can be readily calculated:

$$G = \begin{bmatrix} 0 & 2\hat{\imath}s & 1 & 0 \\ 2\hat{\imath}s & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = R\Lambda R^{-1}, \quad \text{where} \quad R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ g_1^{-1} & g_2^{-1} & g_3^{-1} & g_4^{-1} \\ g_1^{-1} & g_2^{-1} & -g_3^{-1} & -g_4^{-1} \end{bmatrix},$$

$\hat{\imath}$ denotes the imaginary unit, the eigenvalues of G are $g_1 = \hat{\imath}s - \sqrt{1-s^2}$, $g_2 = \hat{\imath}s + \sqrt{1-s^2}$, $g_3 = -\hat{\imath}s - \sqrt{1-s^2}$, $g_4 = -\hat{\imath}s + \sqrt{1-s^2}$, $\Lambda = \operatorname{diag}(g_i)$, $s = ac\sin(\theta)$, and $\theta \in [-\pi, \pi]$ gives the frequency in von Neumann analysis. If

$$|ac| < 1 \tag{6.14}$$



(A) Errors as a function of time

(B) Second order convergence

FIGURE 5. Comparison with CTCS scheme

then all eigenvalues satisfy $|g_i| = 1$. Furthermore, since $\det R = 4(g_1^{-1} - g_2^{-1})(g_4^{-1} - g_3^{-1})$ remains away from zero whenever (6.14) holds, the powers $G^n = R\Lambda^n R^{-1}$ are uniformly bounded for all n and all θ. Hence (6.14) implies that the scheme is stable.

We thus conclude, by the Lax-Richtmyer theorem, that the scheme is convergent and is of second order. Note that the CFL condition we previously found on general meshes, namely (5.5), when restricted to uniform meshes, gives exactly the same CFL condition (6.14) obtained above from von Neumann analysis.

7. Numerical results

7.1. **Convergence study.** First, we report numerical results from our tent pitching (TP) scheme and compare it with the well-known "central-time central-space" (CTCS) finite difference scheme (see [5, 23], sometimes also known as the Yee scheme [12, 27]). The only difference between the two is that while the TP scheme sets the U and V nodes on the same location (exactly as indicated in Figure 4), the CTCS scheme sets them on staggered locations on the same grid. Both schemes are applied to the model problem (6.1) on uniform grids with $S = 1$. We use a grid like that in Figure 4 for both methods.

To impose the outgoing impedance boundary conditions within the CTCS scheme, we use the standard finite difference technique of introducing ghost points to the left and right of the finite grid and eliminating the unknown values at those points using the boundary condition. In contrast, in the TP scheme, the impedance boundary conditions are essentially imposed within the finite element spaces, as we have already seen previously. We set $c = 1$ and impose the initial condition so that the exact solution is

$$u_1(x,t) = u_2(x,t) = e^{-1000((x+t)-1/2)^2},$$

i.e., the solution is a smooth pulse moving to the left at unit speed, eventually clearing out of the simulation domain. At every other time step (in the uniform space-time grid) we compute the $L^2(0,1)$-norm of the difference between the computed and exact solution.

The evolution of these errors in time on a grid of spatial mesh size $h = 0.0025$ and $k = 0.9h$ is shown in Figure 5a.

We observe from Figure 5a that the errors of both methods are comparable and remain low throughout the simulated time. Note also that after the pulse clears the simulation domain reflectionlessly (and the solution within $[0, 1]$ vanishes), the errors for both methods decrease markedly. In Figure 5, we display a log-log plot of the $L^2(0, 1)$-norm of the errors at $t = 0.5$ for $h = 1/2^3, \ldots, 1/2^{13}$, and $k = 0.9h$. The rate of decrease of this error is clearly seen to be of the order $O(h^2)$. This is in accordance with our von Neumann analysis of § 6.4 (although we did not take into account boundary conditions in that analysis).

Thus we conclude from Figure 5 that there is negligible difference between the performance of the two methods on uniform grids.

7.2. **Material interfaces and other boundary conditions.** Next, we consider a generalization of (6.1) given by

$$\partial_t \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} \partial_x \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = f, \qquad 0 < x < 1, 0 < t < T, \tag{7.1a}$$

$$u_1(x, 0) = u_1^0(x), \qquad 0 < x < 1, \tag{7.1b}$$

$$u_2(x, 0) = u_2^0(x), \qquad 0 < x < 1, \tag{7.1c}$$

$$z_0 u_1 - u_2 = 0, \qquad x = 0,\ 0 < t < T, \tag{7.1d}$$

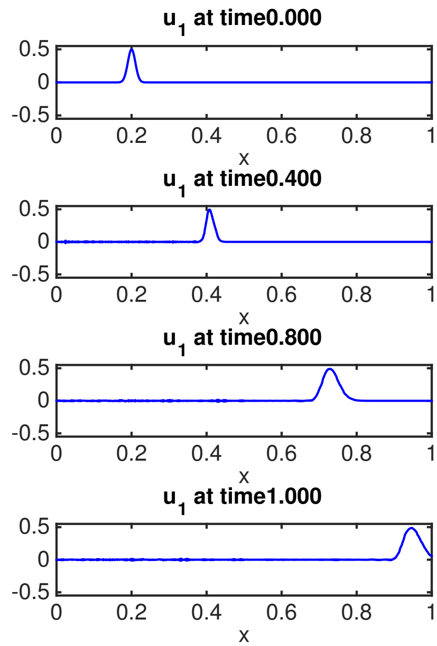
$$z_1 u_1 + u_2 = 0, \qquad x = 1,\ 0 < t < T. \tag{7.1e}$$

where $\kappa_1(x)$ and $\kappa_2(x)$ are time-independent material parameters and c, z_0 and z_1 are constants. Such systems arise from electromagnetics or acoustics [12, 16] on layered media and the differential equation is often written in the following equivalent, but non-symmetric form

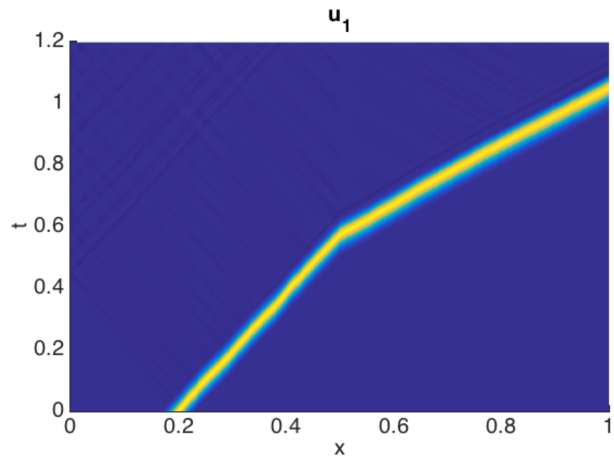
$$\partial_t \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} 0 & \beta_1 \\ \beta_2 & 0 \end{bmatrix} \partial_x \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \tilde{f}$$

where $\beta_i(x) = c/\kappa_i(x)$ and $\tilde{f} = \mathrm{diag}(\kappa_1^{-1}, \kappa_2^{-1}) f$ obtained by scaling the equations of (7.1a) by κ_1^{-1} and κ_2^{-1}. When $\kappa_1(x) \equiv \kappa_2(x) \equiv 1$ and $z_0 = z_1 = 1$, we obtain the model formulation we discussed previously in detail. Dirichlet boundary conditions can be imposed by putting $z_0 = z_1 = 0$, while exact outgoing impedance conditions can be imposed using $z_0 = \sqrt{\kappa_1/\kappa_2}$ and $z_1 = \sqrt{\kappa_1/\kappa_2}$. Intermediate values of z_i give damped impedance boundary conditions.

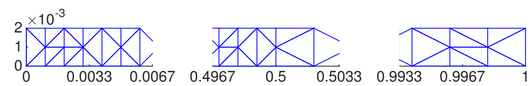
Whenever κ_i is a constant on each spatial mesh interval, a tent pitching scheme is suggested by a simple generalization of the previous algorithm for homogeneous media. We define the discrete spaces exactly as in (6.9), but noting that $V(\Omega)$ now has different essential boundary conditions – stemming from (7.1d)–(7.1e) – which are inherited by the spaces on tents with its tent pole on the boundary. The generalization of the scheme is



(A) Time snapshots of u_1 (cf. whole space-time plot of u_1 in Figure 6b)



(B) Entire space-time plot (over all time slabs) of the u_1-component of the solution



(C) Parts of the tent pitched mesh of the initial time slab, with varying spatial mesh sizes in the regions $x < 0.5$ and $x > 0.5$.

FIGURE 6. Wave propagation through an impedance-matched interface

derived by merely setting the A and $\mathscr{D}$ in (6.11) by

$$A = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix} \partial_t - \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} \partial_x, \qquad \mathscr{D} = \begin{bmatrix} n_t\kappa_1 & -cn_x \\ -cn_x & n_t\kappa_2 \end{bmatrix}.$$

Note that this A, appearing on the left hand side of (7.1a), satisfies (2.1). By solving this general version of (6.11) one can obtain propagation formulas similar to (6.12), but we omit these details and report only the numerical results.

First we consider the case

$$\kappa_1 = \begin{cases} 2, & 0 < x < 1/2, \\ 1, & 1/2 < x < 1, \end{cases} \qquad \kappa_2 = \begin{cases} 2, & 0 < x < 1/2, \\ 1, & 1/2 < x < 1, \end{cases}$$

and $c = 1$. The wave speed (equalling $c/\sqrt{\kappa_1\kappa_2}$), jumps from 0.5 in the left half to 1 in the right half. However, the impedance (equalling κ_1/κ_2 – see [16]) is one in both regions. Thus $x = 0.5$ is an impedance-matched interface about which we do not expect to see any reflection.

We use the tent pitching method to simulate a wave propagating to the right starting near $x = 0.2$. To this end, define a smooth pulse $g(x) = e^{-5000(x-0.2)^2}$ and set the data in (7.1) by

$$f = 0, \quad u_1^0(x) = (c/\kappa_1)g(x), \quad u_2^0(x) = -(c/\sqrt{\kappa_1\kappa_2})g(x), \tag{7.2}$$

and $z_0 = \sqrt{\kappa_1/\kappa_2}$ and $z_1 = \sqrt{\kappa_1/\kappa_2}$. We use a spatial mesh of mesh size $h = 10^{-3}$ in the left half and $h = 2 \times 10^{-3}$ in the right half. A simple tent meshing algorithm then produces a mesh of space-time tents based on this non-uniform spatial mesh that satisfies the CFL condition (5.5). The meshing algorithm proceeds as illustrated as in Figure 1 by simply picking a point with the lowest time coordinate to pitch a tent. When multiple locations have the minimal time coordinate, the algorithm picks a tent pitching location among them randomly, thus giving an unstructured mesh. To minimize the overhead in constructing the mesh of tents, instead of meshing the entire space-time domain at once, we first mesh a thin time slab $\{(x,t) : 0 < t < 0.002,\ 0 < x < 1\}$ and then repeatedly stack this mesh in time to cover the entire region of time simulation. The mesh of the initial slab is shown in Figure 6c.

One of the two components of the computed solution is shown in the remaining two plots of Figure 6. Clearly, the simulated wave packet travels left across the $x = 0.5$ interface without any reflected wave and expands as it enters the region of higher wave speed. In further (unreported) numerical experiments, we have noticed changes in the discrete wave speed depending on the space-time mesh. For example, the wave speed differs if one uses uniform space time meshes with positively sloped diagonals only or negatively sloped diagonals only. Such wave speed differences appear to approach to zero slowly as h is made smaller. High order methods may be needed to reduce these dispersive errors.

Our next and final example involves an interface where we expect both reflection and transmission. We set $c = 1$ and

$$\kappa_1 = \begin{cases} 4, & 0 < x < 1/2, \\ 1/2, & 1/2 < x < 1, \end{cases} \qquad \kappa_2 = \begin{cases} 1, & 0 < x < 1/2, \\ 1/2, & 1/2 < x < 1, \end{cases}$$

Both the wave speed and the impedance jumps from the left region to the right region (from 0.5 and 4 to 2 and 1, respectively). We set f and initial data as in the last simulation by (7.2), but in order to impose Dirichlet boundary condition, we set $z_0 = z_1 = 0$. This time, instead of using a non-uniform mesh, we use a spatially uniform mesh of $h = 10^{-3}$ and let the tent pitching algorithm adjust k to satisfy the CFL condition (5.5) in each tent. We found that the mesh obtained, displayed in Figure 7a, while not ideal due to the thin triangles, is adequate for the simulation. (Better tent pitched meshes can be obtained using non-uniform spatial mesh spacing, as we saw in the previous example and Figure 6c.) The solution components u_1 and u_2 obtained from the simulation are displayed in Figure 7b. The computed waves are transmitted as well as reflected both from the interface and the Dirichlet boundaries. The expected features of the solution are therefore recovered by the method.

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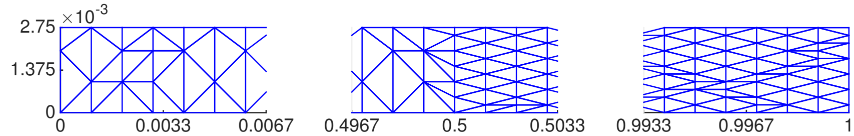
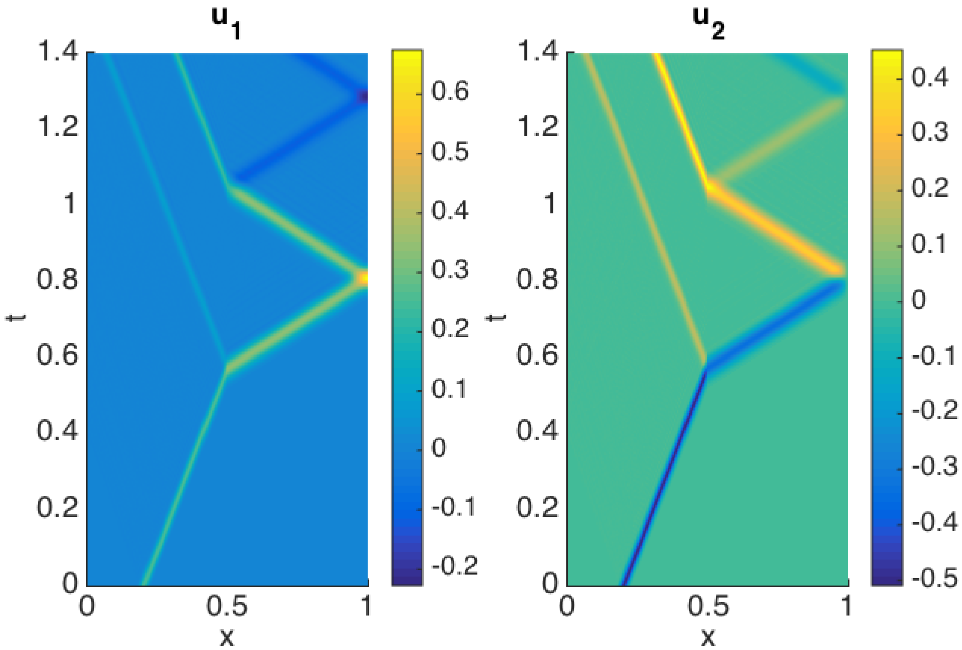


(A) Left, middle and right parts of the tent pitched mesh on one time slab



(B) Space-time plot of the two solution components computed by explicit tent pitching

FIGURE 7. Case of reflected and transmitted waves

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