THE AUXILIARY SPACE PRECONDITIONER FOR THE DE RHAM COMPLEX *

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Abstract. We generalize the construction and analysis of auxiliary space preconditioners to the n-dimensional finite element subcomplex of the de Rham complex. These preconditioners are based on a generalization of a decomposition of Sobolev space functions into a regular part and a potential. A discrete version is easily established using the tools of finite element exterior calculus. We then discuss the four-dimensional de Rham complex in detail. By identifying forms in four dimensions (4D) with simple proxies, form operations are written out in terms of familiar algebraic operations on matrices, vectors, and scalars. This provides the basis for our implementation of the preconditioners in 4D. Extensive numerical experiments illustrate their performance, practical scalability, and parameter robustness, all in accordance with the theory.

Key words. regular decomposition, HX preconditioner, 4D, skew-symmetric matrix fields, exterior derivative, proxies

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1. Introduction. The auxiliary space preconditioners for problems posed in $H(\mathrm{curl})$ and $H(\mathrm{div})$, initially studied by Hiptmair and Xu [24], are now well understood both theoretically and practically, in two and three space dimensions. These preconditioners have been used for accelerating a wide variety of solution techniques, thanks to their highly scalable parallel implementations, known as AMS and ADS preconditioners (see the software libraries HYPRE [25] and MFEM [33]). The goal of the present work is two-fold. First, we generalize the mathematical design and analysis of these preconditioners to n dimensions. Second, we provide an implementation of the preconditioners in 4D and detail the techniques we used to transform 4D exterior calculus into matrix and vector operations.

An important ingredient in the analysis of the auxiliary space preconditioners in two and three dimensions was the so-called regular decomposition, which splits a Sobolev space function into a component of higher regularity and a scalar or vector potential. Such decompositions were known early on [9]. But the key to the success of the auxiliary space preconditioners was a discrete version of this decomposition found in [24], now also known as the HX decomposition. Its practical use was elaborated in [29] and [30], where slightly stronger results were established (using [39]) to prove robustness of the solvers in a general setting involving a stiffness term and a mass term weighted with a parameter. Further related interesting studies on solvers in $H(\mathrm{curl})$ and $H(\mathrm{div})$ can be found in [10, 11, 48].

One of the motivations for this work, especially our 4D implementation, is the recent increased interest in spacetime discretizations. In three space dimensions, they

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yield large linear systems built on 4D meshes and discretizations. Starting as early as the eighties, literature on spacetime methods began to accumulate [5, 6, 20, 27, 28, 45]. As methods that parallelize only spatial degrees of freedom created increasingly larger computational bottlenecks in temporal simulations [17], the potential for higher scalability of the spacetime methods received more attention, resulting in a resurgence of interest in recent years [1, 6, 7, 31, 32, 35, 37, 38, 41, 42, 43, 44]. Further reasons for pursuing spacetime discretizations, such as limited regularity [15] and spacetime adaptivity [19] have also been noted. Among these reasons, perhaps the most relevant to this work is the above-mentioned potential of spacetime methods to break through temporal causality barriers when exploiting parallelism. However, this potential is unlikely to be realized without highly scalable solvers. In turn, spacetime solvers in 4D are unlikely to be developed without a complete understanding of preconditioners for the norm generated by each of the four canonical first order partial differential operators in 4D. Herein lies one of our contributions. By showing how to build scalable preconditioners for the norm of all the first order Sobolev spaces in 4D, we provide building blocks for designing spacetime solvers.

To describe a specific scenario illustrating the need for preconditioners in 4D, recall that conservation laws take the form $\operatorname{div} F = 0$ for some flux F depending on the unknown fields. Here, "div" is the 4D spacetime divergence when the conservation law in posed in three space dimensions. One can construct a spacetime discretization for this equation, following along the lines of [38] for scalar conservation laws. The resulting system of equations, as shown in [38], is of saddle-point form. Its leading blocks on the diagonal correspond to bilinear forms that are equivalent to the canonical norms arising from the 4D de Rham sequence. Therefore, a block-diagonal preconditioner for that saddle point system is obtained using diagonal blocks consisting of preconditioners for the relevant canonical 4D norms. This shows an immediate impact of our preconditioners in Section 3 on existing work. Our later discussions on 4D implementation are also of immediate relevance to this example. Indeed, one of the solvers considered in [38] utilizes iterations in a divergence-free space, which benefits from explicit knowledge of that subspace. Our considerations in Section 4 characterize this subspace as Div of certain skew-symmetric matrix-valued functions (where Div defined later –see (4.7)– is such that $\operatorname{div} \circ \operatorname{Div}$ applied to skew-symmetric matrix-valued functions vanishes). Beyond these comments, we shall not dwell on further details of applications in this paper.

The remainder of the paper is structured as follows. We begin in Section 2 with the necessary background on finite element exterior calculus and introduce the regular decomposition in n-dimensions. This section also reviews a few new tools available thanks to the recent intensive research on finite element exterior calculus, such as the bounded cochain projections and their commutativity and approximation properties. Section 3, introduces the auxiliary space preconditioner, which is the main object of this study. After its definition and complete analysis, we proceed to Section 4, which specializes the discussion to 4D exterior calculus and presents techniques and identities used for the implementation of the preconditioner and its 4D ingredients. Section 5 contains a large set of numerical results illustrating the scalable and robust performance of the method, all in accordance with the theory.

2. Preliminaries. We use finite element exterior calculus, for which standard references include [4, 22]. In this section, we establish the exterior calculus notations used in this paper and recall results pertinent for the analysis of preconditioners.

2.1. Sobolev spaces of exterior forms. First, we set notations for k-forms in n-dimensions ($0 \le k \le n$). The set of increasing multi-indices with k components is denoted by $\mathcal{I}_k = \{\alpha = (\alpha_1, \ldots, \alpha_k) : 1 \le \alpha_1 < \alpha_2 < \cdots \alpha_k \le n\}$. For $\alpha \in \mathcal{I}^k$ and $x = (x^1, x^2, \ldots, x^n)$, we abbreviate the elementary k-form $dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_k}$ to simply dx^α. The space of k-forms on $\mathbb{R}^n$ is denoted by $\mathbb{A}^k = \{\sum_{\alpha \in \mathcal{I}_k} c_\alpha dx^\alpha : c_\alpha \in \mathbb{R}\}$ and its dimension is $n_k \equiv \binom{n}{k}$. Let $H^s(\Omega)$ denote the standard Sobolev space on any open $\Omega \subset \mathbb{R}^n$. The Sobolev space of exterior k-forms is defined by

$$H^s(\Omega, \mathbb{A}^k) = \left\{ w = \sum_{\alpha \in \mathcal{I}_k} w_\alpha(x)\, dx^\alpha : \; w_\alpha \in H^s(\Omega) \right\}.$$

Its norm is given by

$$\|w\|^2_{H^s(\Omega, \mathbb{A}^k)} = \sum_{\alpha \in \mathcal{I}_k} \|w_\alpha\|^2_{H^s(\Omega)}. \tag{2.1}$$

The above notation scheme generalizes to analogously define other spaces of forms like $L^2(\Omega, \mathbb{A}^k)$, $C(\Omega, \mathbb{A}^k)$, etc. Thus $\mathcal{D}'(\Omega, \mathbb{A}^k)$ denotes the space of k-forms whose components $\varphi_{i_1 \cdots i_k}$ are distributions in $\mathcal{D}'(\Omega)$ (where $\mathcal{D}(\Omega)$ is the space of smooth compactly supported test functions). The inner product and norm of $L^2(\Omega, \mathbb{A}^k)$ is denoted simply by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively. In either case the form degree k will be understood from context.

Let $d \equiv d^{(k)}$ denote the kth exterior derivative, e.g., when applied to $w = w_\alpha dx^\alpha \in H^1(\Omega, \mathbb{A}^k)$, the exterior derivative dw is given by

$$dw = \sum_{i=1}^n \partial_i w_\alpha \, dx^i \wedge dx^\alpha, \tag{2.2}$$

where $\partial_i w_\alpha$ is the usual ith partial derivative $\partial w_\alpha / \partial x^i$ of the scalar multivariate function w_α. In three dimensions, d^0 generates the familiar gradient, d^1 generates curl, and d^2 generates the divergence operator. In four dimensions, the exterior derivative has analogous interpretations, which are worked out in detail later in §4.

We are interested in the Sobolev spaces

$$H(d, \Omega, \mathbb{A}^k) = \{w \in L^2(\Omega, \mathbb{A}^k) : \; dw \in L^2(\Omega, \mathbb{A}^{k+1})\}$$

normed by

$$\|w\|^2_{H(d,\Omega,\mathbb{A}^k)} = \|w\|^2 + \|dw\|^2.$$

Note that when $k = n$, this space coincides with $L^2(\Omega, \mathbb{A}^n)$ (since $d = 0$ then). When $n = 3$, these spaces coincide with the familiar three spaces $H^1(\Omega)$, $H(\mathrm{curl}, \Omega)$ and $H(\mathrm{div}, \Omega)$ for $k = 0, 1$, and 2, respectively. For $n = 4$, the corresponding four spaces are studied in detail in Section 4.

2.2. Regular decomposition. From now on, within this section, we tacitly assume that Ω is an open bounded domain that is starlike with respect to a ball B, by which we mean that for any $x \in \Omega$, the convex hull of x and B is contained in Ω. This assumption implies that topology of Ω is trivial, i.e., Ω is homotopy equivalent to a ball, and that the boundary of Ω is Lipschitz. Under this assumption, certain regularized versions of homotopy operators of Poincaré are constructed in [34] (where its called averaged Cartan-like operators) and [14] (where its called regularized Poincaré-type integral operators). In the proof below, we shall follow the notation of [14] and denote these by R_k. We use them to obtain a decomposition of $H(d,\Omega,\mathbb{A}^k)$ into a more regular part and a remaining potential, as stated next.

THEOREM 2.1 (Regular decomposition). *For each integer* $1 \le k \le n$, *there is a* $C_1 > 0$ *and continuous linear maps*

$$\mathcal{S} : H(d,\Omega,\mathbb{A}^k) \to H^1(\Omega,\mathbb{A}^k), \qquad \mathcal{P} : H(d,\Omega,\mathbb{A}^k) \to H^1(\Omega,\mathbb{A}^{k-1})$$

such that for all $w \in H(d,\Omega,\mathbb{A}^k)$,

$$w = \mathcal{S}w + d\mathcal{P}w$$

and

$$\|\mathcal{S}w\|_{H^1(\Omega,\mathbb{A}^k)} \le C_1\|dw\|, \qquad \|\mathcal{P}w\|_{H^1(\Omega,\mathbb{A}^{k-1})} \le C_1\|w\|_{H(d,\Omega,\mathbb{A}^k)}.$$

Proof. The regularized Poincaré-type integral operators of [14, Corollary 3.4] are continuous linear operators $R_k : L^2(\Omega,\mathbb{A}^k) \to H^1(\Omega,\mathbb{A}^{k-1})$ for all $k = 1,2,\ldots,n-1$ satisfying $dR_k u + R_{k+1} du = u$ for all $u \in H(d,\Omega,\mathbb{A}^k)$. Moreover, the results of [14] when $k = n$ also yield $dR_n u = u$. Therefore, setting $\mathcal{P} = R_k$ and $\mathcal{S} = R_{k+1}$ the result follows for all $k = 1,2,\ldots,n-1$. It also follows for $k = n$ once we set $\mathcal{S} = 0$ and $\mathcal{P} = R_n$. ◻

We note that regular decompositions were also given in [23, Theorem 5.2] and [16, Lemma 5], but their results do not state the first inequality of Theorem 2.1, which we need in the ensuing analysis.

2.3. Interpolation into finite element spaces. Recall the well-known finite element subspaces [4, 22] of $H(d,\Omega,\mathbb{A}^k)$. Let P_r denote the space of polynomials in n variables of degree at most r, $P_r\mathbb{A}^k = \{\sum_{\alpha\in\mathcal{I}_k} p_\alpha dx^\alpha : p_\alpha \in P_r\}$, and let $P_r^-\mathbb{A}^k \subseteq P_r\mathbb{A}^k$, for all integers $r \ge 1$, be as defined in [4, §5.1.3]. Let Ω_h denote a geometrically conforming shape-regular simplicial finite element mesh of Ω. Let h denote the maximal mesh diameter $h = \max_{K\in\Omega_h} \operatorname{diam}(K)$. To simplify technicalities, we assume that the mesh Ω_h is quasiuniform, so the diameter of every element is bounded above and below by some fixed constant multiples of h. The standard finite element subspaces of $H(d,\Omega,\mathbb{A}^k)$, indexed by maximal mesh element diameter h, are $V_h^{(k)} = \{v \in H(d,\Omega,\mathbb{A}^k) : v|_K \in P_r^-\mathbb{A}^k$ for all n-simplices K that are elements of the mesh $\Omega_h\}$. The Lagrange finite element space $V_h^{(0)}$ will play a special role in our discussions. We now introduce three operators that map various functions into $V_h^{(k)}$ that will be used in the sequel.

The first operator we need is the L^2 projection. Identifying the n_k-fold product of $V_h^{(0)}$ as a subspace of $H^1(\Omega,\mathbb{A}^k)$, we denote it by $V_h^{(0),k}$. Let $Q_h = Q_h^{(k)} : L^2(\Omega,\mathbb{A}^k) \to V_h^{(0),k}$ be defined by $(Q_h z, v_h) = (z, v_h)$ for all $v_h \in V_h^{(0),k}$. Then, it follows from [12] that for any $v \in H^1(\Omega,\mathbb{A}^k)$,

$$|Q_h v|_{H^1(\Omega,\mathbb{A}^k)} + h^{-1}\|Q_h v - v\| \prec |v|_{H^1(\Omega,\mathbb{A}^k)}. \tag{2.3}$$

Here and throughout, we write $A \prec B$ to indicate that the quantities A and B satisfy $A \le CB$ with a constant C that is independent of h (but may depend on the shape regularity of Ω_h).

The next operator is the finite element interpolant $\varPi_h \equiv \varPi_h^{(k)}$, often called the *canonical interpolant*. A standard set of degrees of freedom of $P_r^- \mathbb{A}^k$ is well known (see [4, Theorem 5.5] or [22]). It defines the canonical finite element interpolant $\varPi_h$ in the usual way. Although the domain of $\varPi_h$ is often viewed as contained in a general (sufficiently regular) Sobolev space, an important point of departure in this paper is to view $\varPi_h$ as a bounded linear operator on *discrete spaces*, namely

$$\varPi_h^{(k)} : V_h^{(0),k} \to V_h^{(k)}.$$

Lemma 3.3 below provide continuity and approximation estimates for $\varPi_h$ on the above domain.

Since $\varPi_h$ is, in general, unbounded on $H(d, \Omega, \mathbb{A}^k)$, ideas to construct bounded projectors into $V_h^{(k)}$ were proposed in [40] and its antecedents. Such projectors are now well known [4] by the name "bounded cochain projectors," Denoting them by $\mathcal{B}_h^{(k)}$, we recall the standard result [4, Theorem 5.9] that $\mathcal{B}_h^{(k)} : L^2(\Omega, \mathbb{A}^k) \to V_h^{(k)}$ is a bounded projection satisfying

$$\|w - \mathcal{B}_h^{(k)} w\| \prec h^s \|w\|_{H^s(\Omega, \mathbb{A}^k)} \tag{2.4a}$$

$$d\mathcal{B}_h^{(k)} = \mathcal{B}_h^{(k-1)} d \tag{2.4b}$$

for all $0 \le s \le r$.

As a final note on the notation, *we will omit the superscript* (k) *indicating the form degree from any notation* when no confusion can arise. For example, just as d abbreviates $d^{(k)}$, we shall use $\mathcal{B}_h$ for $\mathcal{B}_h^{(k)}$ when the form degree k can be understood from context.

3. The preconditioner.

3.1. Definition. Let $\tau > 0$ and let $A \equiv A^{(k)} : V_h^{(k)} \to V_h^{(k)}$ denote the operator defined by

$$(A^{(k)} u, v) = \tau(u, v) + (du, dv) \tag{3.1}$$

for all $u, v \in V_h^{(k)}$. Algebraic multigrid preconditioners for $A^{(k)}$, for any form degree k, can be built by generalizing the ideas in [24] and [29] as we shall see in this section.

The norm generated by A is defined by $\|u\|_A = (Au, u)^{1/2}$. Given two closed subspaces V, W of L^2 and a linear operator $R : V \to W$ we use $R^t : W \to V$ to denote its Hilbert adjoint defined by $(R^t w, v) = (w, Rv)$ for all $w \in W$ and $v \in V$. Let d_h denote the restriction of d on $V_h^{(k)}$, i.e., $d_h : V_h^{(k)} \to V_h^{(k+1)}$. Then its adjoint $d_h^t : V_h^{(k+1)} \to V_h^{(k)}$ is calculated by the above-mentioned definition.

We define the preconditioner $B \equiv B^{(k)} : V_h^{(k)} \to V_h^{(k)}$ for $k \ge 1$ by induction on k, supposing that for $k = 0$, we are given a good preconditioner $B^{(0)} : V_h^{(0)} \to V_h^{(0)}$, i.e., there exists a $\beta \ge 1$ such that

$$\beta^{-1}(B^{(0)} w, w) \le ((A^{(0)})^{-1} w, w) \le \beta(B^{(0)} w, w) \tag{3.2}$$

for all $w \in V_h^{(0)}$. Of course, we have in mind practically useful scenarios where β is completely independent of (or very mildly dependent on) τ and h. The supposition of (3.2) is justified since there are good algebraic preconditioners [21] for the Dirichlet operator (arising from $A^{(0)}$). Then the n_k-fold product of $B^{(0)}$, denoted by $B^{(0),k}$:

$V_h^{(0),k} \to V_h^{(0),k}$ preconditions $A^{(0),k} : V_h^{(0),k} \to V_h^{(0),k}$, the n_k-fold product of $A^{(0)}$. Our aim is to use this to precondition $B^{(k)}$ for $k > 0$.

We need one more ingredient, the operator $D_h \equiv D_h^{(k)} : V_h^{(k)} \to V_h^{(k)}$ defined by

$$(D_h u, v) = (h^{-2} + \tau)(u, v)$$

for all $u, v \in V_h^{(k)}$. Finally, we define the preconditioner by

$$B \equiv B^{(k)} = D_h^{-1} + \Pi_h B^{(0),k} \Pi_h^t + \tau^{-1} d_h B^{(k-1)} d_h^t \tag{3.3}$$

for all $1 \leq k \leq n$. Clearly, a practical implementation of this preconditioner would need implementations of Π_h, d_h, and $B^{(0),k}$. The latter has been amply clarified in the literature (see e.g. [21]). In Section 4, we will provide more details on the implementation of Π_h and d_h when $n = 4$.

Note that when the last term in (3.3) is recursively expanded, a simplification occurs, i.e., we have

$$\begin{aligned} d_h B^{(k-1)} d_h^t &= d_h^{(k-1)} \Big[(D_h^{(k-1)})^{-1} + \Pi_h^{(k-1)} B^{(0),k-1} (\Pi_h^{(k-1)})^t \\ &\qquad + \tau^{-1} d_h^{(k-2)} B^{(k-2)} (d_h^{(k-2)})^t \Big] (d_h^{(k-1)})^t \\ &= d_h^{(k-1)} \left[(D_h^{(k-1)})^{-1} + \Pi_h^{(k-1)} B^{(0),k-1} (\Pi_h^{(k-1)})^t \right] (d_h^{(k-1)})^t \end{aligned} \tag{3.4}$$

because $d_h^{(k-1)} d_h^{(k-2)} = 0$. Thus the cost of applying the preconditioner $B^{(k)}$ (ignoring the cost of inversion of D_h and the application of Π_h) is dominated by the cost of applying $B^{(0),k-1}$ and $B^{(0),k}$, i.e., the cost of applying $B^{(0)}$

$$n_k + n_{k-1} = \binom{n+1}{k}$$

times. This also shows that an implementation of B using only the above-mentioned nonzero terms would be more efficient than simply implementing (3.3) recursively.

3.2. Analysis. We now proceed to prove a discrete version of the regular decomposition (as stated in Lemma 3.4 below). To this end, in addition to Theorem 2.1, we need bounds on $\Pi_h^{(k)}$. By viewing $\Pi_h^{(k)}$ as an operator acting on discrete spaces (as already mentioned earlier), we are able to use conclusions from scaling and finite dimensionality arguments for k-forms, such as the next two lemmas. We shall briefly display a proof of one of them using Euclidean coordinates. Let $\hat{K}$ denote the unit n-simplex. There is an affine homeomorphism $\Phi_K : \hat{K} \to K$ for any n-simplex K. Let $h_K = \mathrm{diam}(K)$. Suppose v is a k-form in $L^2(K, \mathbb{A}^k)$. Its pullback under Φ_K is a k-form on $\hat{K}$ denoted by $\Phi_K^* v$.

Lemma 3.1 (Inverse inequality). *For all $v_h \in V_h^{(k)}$, we have $\|dv_h\| \prec h^{-1} \|v_h\|$.*

Lemma 3.2 (Scaling of pullback). *For all $v \in L^2(K, \mathbb{A}^k)$,*

$$\|\Phi_K^* v\|^2_{L^2(\hat{K}, \mathbb{A}^k)} \prec h_K^{2k-n} \|v\|^2_{L^2(K, \mathbb{A}^k)} \prec \|\Phi_K^* v\|^2_{L^2(\hat{K}, \mathbb{A}^k)} \tag{3.5}$$

and for all $v \in H^1(K, \mathbb{A}^k)$,

$$|\Phi_K^* v|^2_{H^1(\hat{K}, \mathbb{A}^k)} \prec h_K^{2+2k-n} |v|^2_{H^1(K, \mathbb{A}^k)} \prec |\Phi_K^* v|^2_{H^1(\hat{K}, \mathbb{A}^k)}. \tag{3.6}$$

Proof. Let $v = \sum_{\alpha\in\mathcal{I}_k} v_\alpha dx^\alpha$. Its pullback $\hat{v} = \Phi_K^* v$ when expanded in elementary form basis at any $\hat{x} \in \hat{K}$, takes the form

$$\hat{v}(\hat{x}) = \sum_{\alpha\in\mathcal{I}_k} \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} v_{\alpha_1,\dots,\alpha_k}(\Phi_K^{-1}\hat{x}) \frac{\partial x^{\alpha_1}}{\partial \hat{x}^{i_1}} \frac{\partial x^{\alpha_2}}{\partial \hat{x}^{i_2}} \cdots \frac{\partial x^{\alpha_k}}{\partial \hat{x}^{i_k}} \, d\hat{x}^{i_1} \wedge d\hat{x}^{i_2} \wedge \cdots \wedge d\hat{x}^{i_k}.$$

Note that $\|\partial x^{\alpha_l}/\partial \hat{x}^{i_l}\|_{L^\infty(\hat{K})} \prec h_K$. To prove (3.6), applying (2.1) but with norm replaced by seminorm,

$$\begin{aligned}
|\hat{v}|^2_{H^1(\hat{K},\mathbb{A}^k)} &= \sum_{\beta\in\mathcal{I}_k} |\hat{v}_\beta|^2_{H^1(\hat{K},\mathbb{A}^k)} \\
&\prec \sum_{\alpha\in\mathcal{I}_k} \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \sum_{l=1}^{n} \left\| \frac{\partial}{\partial \hat{x}^l} \left(v_{\alpha_1,\dots,\alpha_k}(\Phi_K^{-1}\hat{x}) \frac{\partial x^{\alpha_1}}{\partial \hat{x}^{i_1}} \frac{\partial x^{\alpha_2}}{\partial \hat{x}^{i_2}} \cdots \frac{\partial x^{\alpha_k}}{\partial \hat{x}^{i_k}} \right) \right\|^2_{L^2(\hat{K})} \\
&\prec \sum_{\alpha\in\mathcal{I}_k} \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \sum_{l=1}^{n} \sum_{m=1}^{n} \left\| \frac{\partial x^m}{\partial \hat{x}^l} \partial_m v_{\alpha_1,\dots,\alpha_k} \frac{\partial x^{\alpha_1}}{\partial \hat{x}^{i_1}} \frac{\partial x^{\alpha_2}}{\partial \hat{x}^{i_2}} \cdots \frac{\partial x^{\alpha_k}}{\partial \hat{x}^{i_k}} \right\|^2_{L^2(K)} \frac{|\hat{K}|}{|K|} \\
&\prec h_K^{2+2k} \frac{|\hat{K}|}{|K|} \sum_{\alpha\in\mathcal{I}_k} \sum_{m=1}^{n} \|\partial_m v_\alpha\|^2_{L^2(K)} \prec h_K^{2+2k-n} |v|^2_{H^1(\hat{K},\mathbb{A}^k)}.
\end{aligned}$$

The reverse inequality can be established by considering the inverse map Φ_K^{-1}. The inequalities of (3.5) are proved similarly. □

LEMMA 3.3. *For all $v_h \in V_h^{(0),k}$*

$$\|\Pi_h v_h\| \prec \|v_h\| \tag{3.7}$$
$$\|\Pi_h v_h - v_h\| \prec h\|v_h\|_{H^1(\Omega,\mathbb{A}^k)} \tag{3.8}$$
$$\|d\Pi_h v_h\| \prec \|v_h\|_{H^1(\Omega,\mathbb{A}^k)}. \tag{3.9}$$

Proof. Let $\Pi_K : P_r\mathbb{A}^k(K) \to P_r^-\mathbb{A}^k(K)$ be the canonical interpolant on K, i.e., $\Pi_K v = (\Pi_h v)|_K$ for any $v \in V_h^{(0),k}$. Recall that any v in $V_h^{(0),k} \subset H^1(\Omega,\mathbb{A}^k)$, when restricted to K, lies in $P_r\mathbb{A}^k$. It is easy to check that for any $v \in P_r\mathbb{A}^k$, $\Phi_K^*\Pi_K v = \Pi_{\hat{K}}\Phi_K^* v$. Since $\Pi_{\hat{K}} : P_r\mathbb{A}^k \to P_r^-\mathbb{A}^k(\hat{K})$ is a linear map between finite dimensional spaces, it is bounded. Using Lemma 3.2, we have

$$\begin{aligned}
\|\Pi_h v\|^2_{L^2(K,\mathbb{A}^k)} &\prec h^{n-2k}\|\Phi_K^*\Pi v\|^2_{L^2(\hat{K},\mathbb{A}^k)} = h^{n-2k}\|\Pi_{\hat{K}}\Phi_K^* v\|^2_{L^2(\hat{K},\mathbb{A}^k)} \\
&\prec h^{n-2k}\|\Phi_K^* v\|^2_{L^2(\hat{K},\mathbb{A}^k)} \prec h^{n-2k}h^{2k-n}\|v\|^2_{L^2(K,\mathbb{A}^k)}.
\end{aligned}$$

When summed over all $K \in \Omega_h$, this proves (3.7).

To prove (3.8), we note that $c - \Pi_{\hat{K}} c = 0$ for any constant function c. Hence choosing c to be the mean value of $\Phi_K^* v$ on $\hat{K}$,

$$\begin{aligned}
\|\Pi_h v - v\|^2_{L^2(K,\mathbb{A}^k)} &\prec h^{n-2k}\|(\Pi_{\hat{K}} - I)(\Phi_K^* v - c)\|^2_{L^2(\hat{K},\mathbb{A}^k)} \\
&\prec h^{n-2k}\|\Phi_K^* v - c\|^2_{L^2(\hat{K},\mathbb{A}^k)} \prec h^{n-2k}|\Phi_K^* v|^2_{H^1(\hat{K},\mathbb{A}^k)} \\
&\prec h^{n-2k}h^{2+2k-n}|v|^2_{H^1(K,\mathbb{A}^k)}
\end{aligned}$$

where we have again used Lemma 3.2. Summing over all elements, this proves (3.8).

Finally to prove (3.9), we note that the canonical interpolant commutes with the exterior derivative when applied to smooth functions. In particular, on any $v \in P_r \mathbb{A}^k(K)$, we have $d\Pi_h^{(k)} v|_K = \Pi_h^{(k+1)} dv|_K$. Hence, using the already established (3.7),

$$\|d\Pi_h v\|^2_{L^2(K,\mathbb{A}^k)} \prec \|dv\|^2_{L^2(K,\mathbb{A}^k)} \prec |v|^2_{H^1(K,\mathbb{A}^k)}.$$

Summing over all elements, this proves (3.9). □

LEMMA 3.4 (Stable decomposition). *For any $u_h \in V_h^{(k)}$, there are functions $s_h \in V_h^{(k)}$, $z_h \in V_h^{(0),k}$, and $p_h \in V_h^{(k-1)}$ such that*

$$u_h = s_h + \Pi_h z_h + dp_h \tag{3.10}$$

and

$$\begin{aligned} (h^{-2}+\tau)\|s_h\|^2 + \tau\|p_h\|^2 + \tau\|dp_h\|^2 \\ + \tau\|z_h\|^2 + |z_h|^2_{H^1(\Omega,\mathbb{A}^k)} \ \prec \ \tau\|u_h\|^2 + (1+\tau)\|du_h\|^2. \end{aligned} \tag{3.11}$$

Proof. We apply Theorem 2.1 to $u_h \in V_h^{(k)} \subset H(d,\Omega,\mathbb{A}^k)$ to obtain

$$u_h = z + dp, \tag{3.12}$$

$$\|z\|_{H^1(\Omega,\mathbb{A}^k)} \le C_1 \|du_h\|, \qquad \|p\|_{H^1(\Omega,\mathbb{A}^{k-1})} \le C_1 \|u_h\|_{H(d,\Omega,\mathbb{A}^k)}, \tag{3.13}$$

where $z = \mathcal{S}u$ and $p = \mathcal{P}u$. Now let $z_h = Q_h^{(k)} z \in V_h^{(0),k}$. Applying $\mathcal{B}_h^{(k)}$ to both sides of (3.12) and using (2.4),

$$u_h = \mathcal{B}_h z + d\mathcal{B}_h p.$$

Then (3.10) follows with

$$s_h = \mathcal{B}_h z - \Pi_h z_h, \quad p_h = \mathcal{B}_h p$$

and it only remains to prove the estimate (3.11).

Observe that

$$\begin{aligned}
\|z_h\|^2_{A^{(0)}} &= \tau\|z_h\|^2 + \|d^0 z_h\|^2 \prec \tau\|z\|^2 + |z|^2_{H^1(\Omega,\mathbb{A}^k)} && \text{by (2.3)} \\
&\prec (1+\tau)\,\|du_h\|^2 && \text{by (3.13).} \\
\tau\|p_h\|^2 &= \tau\|\mathcal{B}_h p\|^2 \le \tau\|p\|^2 && \text{by (2.4)} \\
&\prec \tau\|u_h\|^2 + \tau\|du_h\|^2 && \text{by (3.13).} \\
\|dp_h\|^2 &= \|d\mathcal{B}_h p\|^2 = \|\mathcal{B}_h dp\|^2 \prec \|dp\|^2 && \text{by (2.4)} \\
&\prec \|u_h\|^2 + \|z\|^2 && \text{by (3.12)} \\
&\prec \|u_h\|^2 + \|du_h\|^2 && \text{by (3.13).} \\
\|s_h\|^2 &\le (\|\mathcal{B}_h^{(k)} z - z\| + \|z - z_h\| + \|z_h - \Pi_h^{(k)} z_h\|)^2 \\
&\prec h^2\|z\|^2_{H^1(\Omega,\mathbb{A}^k)} \prec h^2\|du_h\|^2
\end{aligned}$$

by (2.4), (2.3) and (3.8). Inequality (3.11) follows by combining these estimates. □

With the above lemmas, we are ready to conclude the analysis. The basis for the analysis of auxiliary space preconditioners is the standard "fictitious space lemma" (see e.g., [24, 36, 47]) which we state without proof below in a form convenient for us. Suppose we want to precondition a self-adjoint positive definite operator Λ on a finite-dimensional Hilbert space V using

1. a selfadjointpositive definite operator $S : V \to V$ whose inverse is easy to apply,
2. two "auxiliary" Hilbert spaces $\tilde{V}_1$ and $\tilde{V}_2$ and linear operators $\tilde{R}_i : \tilde{V}_i \to V$, and
3. two further selfadjointpositive definite operators $\tilde{\Lambda}_i : \tilde{V}_i \to \tilde{V}_i$ on the auxiliary spaces whose inverses are easy to apply.

In this setting, the following result guides the preconditioner design. Here, we denote norms generated by selfadjoint positive definite operators in accordance with our prior notation scheme, e.g., $\|w\|_{\tilde{\Lambda}_i} = (\tilde{\Lambda}_i w, w)^{1/2}_{\tilde{V}_i}$.

Lemma 3.5 (Nepomnyaschikh lemma). *Suppose there are positive constants $c_1, c_2, c_s > 0$ such that for all $\tilde{v}_j \in \tilde{V}_j$, $j = 1, 2$, and $v \in V$,*

$$\|\tilde{R}_1 \tilde{v}_1\|_\Lambda \le c_1 \|\tilde{v}\|_{\tilde{\Lambda}_1}, \qquad \|\tilde{R}_2 \tilde{v}_2\|_\Lambda \le c_2 \|\tilde{v}\|_{\tilde{\Lambda}_2}, \qquad \|v\|_\Lambda \le c_s \|v\|_S. \tag{3.14}$$

Suppose also that given any $v \in V$ there are $s \in V$, $\tilde{v}_i \in \tilde{V}_i$ such that $s + \tilde{R}_1 \tilde{v}_1 + \tilde{R}_2 \tilde{v}_2 = v$ and

$$\|s\|_S^2 + \|\tilde{v}_1\|^2_{\tilde{\Lambda}_1} + \|\tilde{v}_2\|^2_{\tilde{\Lambda}_2} \le c_0^2 \|v\|^2_\Lambda. \tag{3.15}$$

Then $P = S^{-1} + \tilde{R}_1 \tilde{\Lambda}_1^{-1} \tilde{R}_1^t + \tilde{R}_2 \tilde{\Lambda}_2^{-1} \tilde{R}_2^t$ preconditions Λ and the spectrum of $P\Lambda$ is contained in the interval $[c_0^{-2}, c_1^2 + c_2^2 + c_s^2]$.

Theorem 3.6. *Let $0 < \tau \prec 1$ and let A and B be defined by* (3.1) *and* (3.3)*, respectively. Suppose* (3.2) *holds. Then for each $1 \le k \le n-1$, there is an $\alpha \ge 1$ independent of h and τ such that spectral condition number of BA satisfies*

$$\kappa(BA) \le \alpha^2 \beta^2.$$

Proof. First, we analyze the preconditioner

$$P^{(k)} = D_h^{-1} + \Pi_h (A^{(0),k})^{-1} \Pi_h^t + \tau^{-1} d_h (A^{(k-1)})^{-1} d_h^t. \tag{3.16}$$

For this, we apply Lemma 3.5 with

$$\begin{aligned} V &= V_h^{(k)}, & \Lambda &= A^{(k)}, & S &= D_h, \\ \tilde{V}_1 &= V_h^{(0),k}, & \tilde{\Lambda}_1 &= A^{(0),k}, & \tilde{R}_1 &= \Pi_h^{(k)}, \\ \tilde{V}_2 &= V_h^{(k-1)} & \tilde{\Lambda}_2 &= \tau A^{(k-1)} & \tilde{R}_2 &= d_h^{(k-1)}. \end{aligned}$$

Note that V and $\tilde{V}_i$ are endowed with L^2 inner products as before, so e.g., $\|w\|^2_{\tilde{\Lambda}_1} = (\tilde{\Lambda}_1 w, w) = (A^{(0),k} w, w) = \tau \|w\|^2 + (d^0 w, d^0 w)$. We must verify the conditions (3.14) and (3.15) of the lemma.

To verify (3.14), we use the following bounds, which hold for any $z_h \in \tilde{V}_1$, $p_h \in \tilde{V}_2$, and $v_h \in V$:

$$\begin{aligned} \|\tilde{R}_1 z_h\|_A^2 &= \|\Pi_h z_h\|_A^2 = \tau \|\Pi_h z_h\|^2 + \|d \Pi_h z_h\|^2 \\ &\prec \tau \|z_h\|^2 + |z_h|^2_{H^1(\Omega, \mathbb{A}^k)} = \|z_h\|^2_{\tilde{\Lambda}_1}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} \|\tilde{R}_2 p_h\|_A^2 &= \|dp_h\|_A^2 = \tau \|dp_h\|^2 \le \tau \left(\|dp_h\|^2 + \tau \|p_h\|^2 \right) \\ &= \tau \|p_h\|_A^2 = \|p_h\|^2_{\tilde{\Lambda}_2}, \\ \|v_h\|_A^2 &= \tau \|v_h\|^2 + \|dv_h\|^2 \\ &\prec (h^{-2} + \tau) \|v_h\|^2 = \|v_h\|^2_{D_h}. \end{aligned} \tag{3.18}$$

We have used the inverse inequality of Lemma 3.1 in the last bound. With the above bounds, we have verified (3.14).

Next, to verify (3.15), we use Lemma 3.4 to decompose any u_h in V into $u_h = s_h + \tilde{R}_1 z_h + \tilde{R}_2 p_h = s_h + \Pi_h z_h + dp_h$ and apply (3.11). Since $\tau \leq 1$, (3.11) implies

$$\begin{aligned} \|s_h\|^2_{D_h} + \|p_h\|^2_{\tilde{A}_2} + \|z_h\|^2_{\tilde{A}_1} &= \|s_h\|^2_{D_h} + \tau\|p_h\|^2_A + \|z_h\|^2_{\tilde{A}_1} \\ &\leq \|s_h\|^2_{D_h} + \tau(\tau\|p_h\|^2 + \|dp_h\|^2) + \|z_h\|^2_{\tilde{A}_1} \\ &\prec \tau\|u_h\|^2 + (1+\tau)\|du_h\|^2 \\ &\prec \|u_h\|^2_A. \end{aligned}$$

This verifies (3.15). Thus Lemma 3.5 yields the existence of an $\alpha_k \geq 1$ (after overestimating the constants if necessary) such that

$$\frac{1}{\alpha_k}(P^{(k)}v, v) \leq ((A^{(k)})^{-1}v, v) \leq \alpha_k(P^{(k)}v, v) \tag{3.19}$$

for all $v \in V$.

To complete the proof, we use the quadratic form of $P^{(k)}$ to estimate that of B. For any $v \in V$,

$$\begin{aligned} (B^{(k)}v, v) &= (D_h^{-1}v, v) + (B^{(0),k}\Pi_h^t v, \Pi_h^t v) + \tau^{-1}(B^{(k-1)}d_h^t v, d_h^t v) \\ &= ((D_h^{(k)})^{-1}v, v) + (B^{(0),k}\Pi_h^t v, \Pi_h^t v) \\ &\quad + \tau^{-1}\left[((D_h^{(k-1)})^{-1}d_h^t v, d_h^t v) + ((B^{(0),k-1})^{-1}\Pi_h^t d_h^t, \Pi_h^t d_h^t v)\right] \end{aligned}$$

where we have used (3.4). Now, using (3.2) and (3.16), and (3.19),

$$\begin{aligned} (B^{(k)}v, v) &\leq (D_h^{-1}v, v) + \beta((A^{(0),k})^{-1}\Pi_h^t v, \Pi_h^t v) \\ &\quad + \tau^{-1}\left[((D_h^{(k-1)})^{-1}d_h^t v, d_h^t v) + \beta((A^{(0),k-1})^{-1}\Pi_h^t d_h^t v, \Pi_h^t d_h^t v)\right] \\ &\leq \beta\left[(D_h^{-1}v, v) + ((A^{(0),k})^{-1}\Pi_h^t v, \Pi_h^t v) + \tau^{-1}(P^{(k-1)}d_h^t v, d_h^t v)\right] \\ &\leq \beta\left[(D_h^{-1}v, v) + ((A^{(0),k})^{-1}\Pi_h^t v, \Pi_h^t v) + \tau^{-1}\alpha_{k-1}((A^{(k-1)})^{-1}d_h^t v, d_h^t v)\right] \\ &\leq \beta\alpha_{k-1}(P^{(k)}v, v) \leq \beta\alpha_{k-1}\alpha_k((A^{(k)})^{-1}v, v). \end{aligned}$$

Combining with a similarly provable lower inequality, we have

$$\beta^{-1}\alpha_{k-1}^{-1}\alpha_k^{-1}(B^{(k)}v, v) \leq ((A^{(k)})^{-1}v, v) \leq \beta\alpha_{k-1}\alpha_k(B^{(k)}v, v)$$

for all $v \in V_h^{(k)}$. □

3.3. A variant. The preconditioner in (3.3) is a generalization of auxiliary space preconditioner in the form given in [24]. An auxiliary space preconditioner in a slightly different form was proposed in [29, 30]. It can also be extended to higher dimensions as we now show. To define this variant, let Q_0 denote the projection satisfying

$$Q_0 u \in dV_h^{(k-1)}: \qquad (Q_0 u, \kappa) = (u, \kappa) \quad \text{for all } \kappa \in dV_h^{(k-1)}.$$

Then $A_0 = Q_0 A|_{dV_h} : dV_h^{(k-1)} \to dV_h^{(k-1)}$ satisfies

$$(A_0\kappa_1, \kappa_2) = (A\kappa_1, \kappa_2) = \tau(\kappa_1, \kappa_2) \tag{3.20}$$

for all $\kappa_1, \kappa_2 \in dV_h^{(k-1)}$ because $d \circ d = 0$. Clearly A_0 is invertible for all $\tau > 0$. Let $B_0 : dV_h^{(k-1)} \to dV_h^{(k-1)}$ be a preconditioner for A_0, i.e., there is a $\beta_0 \geq 1$ such that

$$\beta_0^{-1}(B_0\kappa, \kappa) \leq (A_0^{-1}\kappa, \kappa) \leq \beta_0(B_0\kappa, \kappa) \tag{3.21}$$

for all κ in $dV_h^{(k-1)}$. Using B_0, we define our second auxiliary space preconditioner $C : V_h^{(k)} \to V_h^{(k)}$ by

$$C = D_h^{-1} + \varPi_h B^{(0),k} \varPi_h^t + B_0 Q_0. \tag{3.22}$$

Unlike (3.3), some care is needed to design and implement B_0. Consider, for instance, the case $B_0 = A_0^{-1}$. Although it appears from (3.20) that A_0^{-1} is simply the inverse of a mass matrix scaled by τ^{-1}, the difficulty is that we usually do not have a basis for $dV_h^{(k-1)}$ in a typical implementation. To compute the action of the last term in (3.22) on some $v \in V_h^{(k)}$, namely $\kappa_1 = A_0^{-1}Q_0 v$, we write $A_0\kappa_1 = Q_0 v$ and apply (3.20) to observe that κ_1 solves

$$\tau(\kappa_1, \kappa_2) = (v, \kappa_2) \qquad \text{for all } \kappa_2 \in dV_h^{(k-1)}. \tag{3.23}$$

Since we do not have a basis for $dV_h^{(k-1)}$, we use potentials p_1, p_2 in $V_h^{(k-1)}$ (for which we do have a basis) to express $\kappa_i = dp_i$. Then (3.23) implies that p_1 in $V_h^{(k-1)}$ solves

$$\tau(dp_1, dp_2) = (v, dp_2) \qquad \text{for all } p_2 \in V_h^{(k-1)}. \tag{3.24}$$

Even if these equations do not uniquely determine p_1, this approach does lead to a practical algorithm because we only need dp_1 to apply (3.22). Note that p_1 is determined only up to the kernel of d, but $\kappa_1 = dp_1$ is uniquely determined. One strategy to compute dp_1 is to apply d after computing a solution p_1 given by the pseudoinverse of the system in (3.24). Another is to use an iterative technique that converges to one solution of (3.24). One may also use a combination of such strategies, such as a multilevel iteration with smoothers that are convergent despite the singularity in (3.24), combined with a coarse-level solver obtained from a pseudoinverse. For more details, the reader may consult [29, 30] or the implementations in [25, 33]. Notwithstanding the complications in implementation, the analysis is a straightforward application of the previous results.

Theorem 3.7. *Let $0 < \tau \leq 1$ and let A and C be defined by* (3.1) *and* (3.22), *respectively. Suppose* (3.2) *and* (3.21) *holds and let $\beta_1 = \max(\beta, \beta_0)$. Then for each $1 \leq k \leq n-1$, there is an $\alpha \geq 1$ independent of h and τ such that spectral condition number of CA satisfies*

$$\kappa(CA) \leq \alpha^2 \beta_1^2.$$

Proof. As in the proof of Theorem 3.6, we consider an intermediate $P = D_h^{-1} + \varPi_h (A^{(0),k})^{-1} \varPi_h^t + A_0^{-1} Q_0$, which is the preconditioner of Lemma 3.5 with

$$\begin{aligned} &V = V_h^{(k)}, && \varLambda = A^{(k)}, && S = D_h, \\ &\tilde{V}_1 = V_h^{(0),k}, && \tilde{\varLambda}_1 = A^{(0),k}, && \tilde{R}_1 = \varPi_h^{(k)}, \\ &\tilde{V}_2 = dV_h^{(k-1)} && \tilde{\varLambda}_2 = A_0 = \tau I, && \tilde{R}_2 = I. \end{aligned}$$

We proceed to verify the conditions (3.14) and (3.15). The following estimates holds for all $z_h \in \tilde{V}_1$, $\kappa_h \in \tilde{V}_2$, and $v_h \in V$:

$$
\begin{aligned}
\|\tilde{R}_1 z_h\|_A^2 &\prec \|z_h\|_{\tilde{A}_1}^2, && \text{by (3.17)},\\
\|\tilde{R}_2 \kappa_h\|_A^2 &= \|\kappa_h\|_A^2 = \tau\|\kappa_h\|^2 = \|\kappa_h\|_{\tilde{A}_2}^2, &&\\
\|v_h\|_A^2 &\prec \|v_h\|_{D_h}^2, && \text{by (3.18)}.
\end{aligned}
$$

Hence we have verified (3.14). To verify (3.15), as before, we use Lemma 3.4 to decompose any u_h in V into $u_h = s_h + \tilde{R}_1 z_h + \tilde{R}_2 \kappa_h = s_h + \Pi_h z_h + \kappa_h$, where $\kappa_h = dp_h \in \tilde{V}_2$, and apply (3.11) to get

$$\|s_h\|_{D_h}^2 + \tau\|\kappa_h\|^2 + \|z_h\|_{\tilde{A}_1}^2 \prec \|u_h\|_A^2.$$

where we have used the assumption that τ is bounded. This verifies (3.15). Thus Lemma 3.5 yields the existence of an $\alpha_k \geq 1$ (after overestimating the constants if necessary) such that

$$\alpha_k^{-1}(Pv, v) \leq (A^{-1}v, v) \leq \alpha_k(Pv, v)$$

for all $v \in V$.

To complete the proof, observe that for any $v \in V$,

$$
\begin{aligned}
(Cv, v) &= (D_h^{-1}v, v) + (B^{(0),k}\Pi_h^t v, \Pi_h^t v) + (B_0 Q_0 v, Q_0 v)\\
&\leq (D_h^{-1}v, v) + \beta((A^{(0),k})^{-1}\Pi_h^t v, \Pi_h^t v) + \beta_0(A_0^{-1}Q_0 v, Q_0 v)\\
&\leq \beta_1(Pv, v) \leq \beta_1\alpha_k(A^{-1}v, v).
\end{aligned}
$$

Together with a similarly provable other-side bound, we have $\beta_1^{-1}\alpha_k^{-1}(Cv, v) \leq (A^{-1}v, v) \leq \beta_1\alpha_k(Cv, v)$ for all $v \in V$. □

4. Implementation in 4 dimensions. In this section, we detail the implementation of the building blocks of the preconditioner in four dimensions. These details form the basis for our publicly available implementation of the preconditioner in the MFEM package [33]. The vector space $\mathbb{A}^k$ in general dimensions is not usually implemented in finite element packages (yet). Therefore, our approach is to view forms using elements (called "proxies" below) of the more standard vector spaces like $\mathbb{R}$, $\mathbb{R}^4$, and the vector space of 4×4 skew symmetric matrices $\mathbb{K}$.

4.1. Proxies of forms. As already mentioned in §2.1, any $\varphi \in \mathbb{A}^k$ has the basis expansion

$$\varphi \;=\; \sum_{1 \leq i_1 < \cdots < i_k \leq 4} \varphi_{i_1 \cdots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \tag{4.1}$$

Here the sum runs over all indices in $\mathcal{I}_k$ with four components. The numbers $\varphi_{i_1 \cdots i_k}$, called the "components" or the "coefficients" of the form, are arranged into vectors or matrices that form "proxies" of k-forms, as defined below.

The proxy of a k-form φ is denoted by $[\varphi]^{(k)}$ and is defined as follows. In the case of a 0-form φ, we set $[\varphi]^{(0)} = \varphi$. In the case of higher form degrees, we use the components of φ in (4.1), namely φ_i for 1-form, φ_{ij} for 2-form, and φ_{ijk} for 3-form, to define proxies:

$$[\varphi]^{(1)} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix}, \quad [\varphi]^{(2)} = \begin{bmatrix} 0 & \varphi_{34} & -\varphi_{24} & \varphi_{23} \\ -\varphi_{34} & 0 & \varphi_{14} & -\varphi_{13} \\ \varphi_{24} & -\varphi_{14} & 0 & \varphi_{12} \\ -\varphi_{23} & \varphi_{13} & -\varphi_{12} & 0 \end{bmatrix}, \quad [\varphi]^{(3)} = \begin{bmatrix} \varphi_{234} \\ -\varphi_{134} \\ \varphi_{124} \\ -\varphi_{123} \end{bmatrix}.$$

Finally a $\varphi \in \mathbb{A}^4$ has only one component φ_{1234}, so we set $[\varphi]^{(4)} = \varphi_{1234}$. Thus $[\cdot]^{(k)}$ introduces a one-to-one onto correspondence from $\mathbb{A}^k$ to $\mathbb{R}, \mathbb{R}^4, \mathbb{K}, \mathbb{R}^4$, and $\mathbb{R}$, for $k = 0, 1, 2, 3$, and 4, respectively.

Some identities are expressed better using the permutation (or the Levi-Civita) symbol $\varepsilon_{i_1\, i_2 \dots\, i_n}$, whose definition we recall briefly. For any n and any indices i_k in $\{1, 2, \dots, n\}$, the value of $\varepsilon_{i_1\, i_2 \dots\, i_n}$ is zero when any two indices are equal. When the indices are distinct, $i_1\, i_2 \dots i_n$ is a permutation of $1, 2, \dots, n$ and the value of $\varepsilon_{i_1\, i_2 \dots i_n}$ is set to the sign of the permutation. It can be easily verified that the (i, j)th entry of the above-defined skew symmetric matrix proxy of a $\varphi \in \mathbb{A}^2$ and the ith component of the proxy vector of a $\varphi \in \mathbb{A}^3$ are given by

$$[\varphi]^{(2)}_{ij} = \sum_{1 \le k < l \le 4} \varepsilon_{ijkl} \varphi_{kl}, \qquad [\varphi]^{(3)}_{i} = \sum_{1 \le j < k < l \le 4} \varepsilon_{ijkl} \varphi_{jkl}, \tag{4.2}$$

where the sums run over increasing multi-indices in $\mathcal{I}_2$ and $\mathcal{I}_3$, respectively.

Next, we define *two cross products* (both denoted by $\times$) in the four-dimensional case. Recall that for any $u, v \in \mathbb{R}^3$, the ith component of the standard cross product $u \times v$ is given by $[u \times v]_i = \sum_{j,k=1}^3 \varepsilon_{ijk} u_j v_k$. Analogously, we define

$$[u \times v]_{ij} = \sum_{k,l=1}^{4} \varepsilon_{ijkl} u_k v_l, \qquad u, v \in \mathbb{R}^4,$$

i.e., this cross product of two 4-dimensional vectors yields a skew-symmetric matrix. We also define the cross product of two skew symmetric matrices κ, η in $\mathbb{K}$ by

$$\kappa \times \eta = \sum_{1 \le i < j \le 4} \sum_{1 \le k < l \le 4} \varepsilon_{ijkl} \kappa_{ij} \eta_{kl}, \qquad \kappa, \eta \in \mathbb{K},$$

i.e., result of the cross product of two matrices in $\mathbb{K}$ produces a real number by a formula analogous to the cross product of two-dimensional vectors (and the analogy is clear once we view the skew-symmetric matrices as vectors in $\mathbb{R}^6$).

These operations, together with other standard multiplication operations yield (after some elementary, albeit tedious calculations) formulas for the wedge product and form action in terms of proxies, as summarized in the next result. The standard products used below include the scalar multiplication, the inner product of two vectors in $\mathbb{R}^4$ (denoted by $\cdot$), the matrix-vector product of elements in $\mathbb{K}$ with $\mathbb{R}^4$, and the Frobenius inner product between matrices (denoted by :).

Proposition 4.1. *The following identities hold for the wedge product:*

$$[\eta \wedge \varphi]^{(k)} = [\varphi \wedge \eta]^{(k)} = [\varphi]^{(0)}\, [\eta]^{(k)}, \qquad \varphi \in \mathbb{A}^0,\ \eta \in \mathbb{A}^k,\ k = 0, \dots, 4, \tag{4.3a}$$

$$-[\eta \wedge \varphi]^{(2)} = [\varphi \wedge \eta]^{(2)} = [\varphi]^{(1)} \times [\eta]^{(1)}, \qquad \varphi \in \mathbb{A}^1,\ \eta \in \mathbb{A}^1, \tag{4.3b}$$

$$[\eta \wedge \varphi]^{(3)} = [\varphi \wedge \eta]^{(3)} = [\eta]^{(2)}\, [\varphi]^{(1)}, \qquad \varphi \in \mathbb{A}^1,\ \eta \in \mathbb{A}^2, \tag{4.3c}$$

$$-[\eta \wedge \varphi]^{(4)} = [\varphi \wedge \eta]^{(4)} = [\varphi]^{(1)} \cdot [\eta]^{(3)}, \qquad \varphi \in \mathbb{A}^1,\ \eta \in \mathbb{A}^3, \tag{4.3d}$$

$$[\eta \wedge \varphi]^{(4)} = [\varphi \wedge \eta]^{(4)} = [\varphi]^{(2)} \times [\eta]^{(2)} \qquad \varphi \in \mathbb{A}^2,\ \eta \in \mathbb{A}^2. \tag{4.3e}$$

The values of forms applied to vectors $u, v, w, z \in \mathbb{R}^4$ are given by

$$\varphi(v) = [\varphi]^{(1)} \cdot v, \qquad \varphi \in \Lambda^1, \tag{4.4a}$$

$$\varphi(u, v) = [\varphi]^{(2)} : (u \times v), \qquad \varphi \in \Lambda^2, \tag{4.4b}$$

$$\varphi(u, v, w) = \det\left[[\varphi]^{(3)}, u, v, w\right], \qquad \varphi \in \Lambda^3, \tag{4.4c}$$

$$\varphi(u, v, w, z) = [\varphi]^{(4)} \det[u, v, w, z], \qquad \varphi \in \Lambda^4. \tag{4.4d}$$

4.2. The four derivatives. Analogous to the three fundamental first order differential operators (grad, curl, div) in three dimensions, there are four first order differential operators in four dimensions, which we denote by

$$\operatorname{grad}, \ \operatorname{Curl}, \ \operatorname{Div}, \ \operatorname{div}. \tag{4.5}$$

The first and the last operators in (4.5) are standard: For any $u \in \mathcal{D}'(\Omega, \mathbb{R})$ define $\operatorname{grad} u \in \mathcal{D}'(\Omega, \mathbb{R}^4)$ as the vector whose ith component is $\partial_i u$. For any $v \in \mathcal{D}'(\Omega, \mathbb{R}^4)$, we set $\operatorname{div} v = \sum_{i=1}^4 \partial_i v_i$.

Next, we define the four-dimensional Curl in a way that brings out the analogies with the three-dimensional case. Recall that the ith component of the curl of a vector function w in three dimensions can be expressed as $[\operatorname{curl} w]_i = \sum_{j,k=1}^3 \varepsilon_{ijk} \partial_j w_k$. Analogously, for any $w \in \mathcal{D}'(\Omega, \mathbb{R}^4)$, we define $\operatorname{Curl} w$ as the matrix in $\mathbb{K}$ whose (i, j)th entry is defined by

$$[\operatorname{Curl} w]_{ij} = \sum_{k,l=1}^4 \varepsilon_{ijkl} \partial_k w_l \tag{4.6}$$

i.e.,

$$\operatorname{Curl} w = \begin{bmatrix} 0 & \partial_3 w_4 - \partial_4 w_3 & \partial_4 w_2 - \partial_2 w_4 & \partial_2 w_3 - \partial_3 w_2 \\ \partial_4 w_3 - \partial_3 v_4 & 0 & \partial_1 w_4 - \partial_4 w_1 & \partial_3 v_1 - \partial_1 w_3 \\ \partial_1 v_4 - \partial_4 v_2 & \partial_4 v_1 - \partial_1 v_4 & 0 & \partial_1 v_2 - \partial_2 v_1 \\ \partial_3 v_2 - \partial_2 v_3 & \partial_1 v_3 - \partial_3 v_1 & \partial_2 v_1 - \partial_1 v_2 & 0 \end{bmatrix}.$$

Finally, the remaining operation Div acts on $\kappa \in \mathcal{D}'(\Omega, \mathbb{K})$ and produces $\operatorname{Div} \kappa \in \mathbb{R}^4$ by taking divergence row-wise, i.e.,

$$[\operatorname{Div} \kappa]_i = \sum_{j=1}^4 \partial_j \kappa_{ij}. \tag{4.7}$$

Note that the identities $\operatorname{Curl}(\operatorname{grad} u) = 0$, $\operatorname{Div}(\operatorname{Curl} w) = 0$, and $\operatorname{div}(\operatorname{Div} \kappa) = 0$ follow immediately from the above definitions.

By connecting the inputs and outputs of the above-introduced four differential operators to proxies of forms, we may understand them as manifestations of exterior derivatives. In fact, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}'(\Omega, \Lambda^0) \arrow[r, "d^{(0)}"] \arrow[d, "{[\cdot]^{(0)}}"] & \mathcal{D}'(\Omega, \Lambda^1) \arrow[r, "d^{(1)}"] \arrow[d, "{[\cdot]^{(1)}}"] & \mathcal{D}'(\Omega, \Lambda^2) \arrow[r, "d^{(2)}"] \arrow[d, "{[\cdot]^{(2)}}"] & \mathcal{D}'(\Omega, \Lambda^3) \arrow[r, "d^{(3)}"] \arrow[d, "{[\cdot]^{(3)}}"] & \mathcal{D}'(\Omega, \Lambda^4) \arrow[d, "{[\cdot]^{(4)}}"] \\
\mathcal{D}'(\Omega, \mathbb{R}) \arrow[r, "\operatorname{grad}"] & \mathcal{D}'(\Omega, \mathbb{R}^4) \arrow[r, "\operatorname{Curl}"] & \mathcal{D}'(\Omega, \mathbb{K}) \arrow[r, "\operatorname{Div}"] & \mathcal{D}'(\Omega, \mathbb{R}^4) \arrow[r, "\operatorname{div}"] & \mathcal{D}'(\Omega, \mathbb{R}).
\end{tikzcd}$$

This follows from the identities collected next, which can again be proved by elementary calculations.

PROPOSITION 4.2. *The following identities hold:*

$$[d^{(0)}\varphi]^{(1)} = \operatorname{grad}([\varphi]^{(0)}), \qquad \varphi \in \mathcal{D}'(\Omega, \mathbb{A}^0), \tag{4.8a}$$
$$[d^{(1)}\varphi]^{(2)} = \operatorname{Curl}([\varphi]^{(1)}), \qquad \varphi \in \mathcal{D}'(\Omega, \mathbb{A}^1), \tag{4.8b}$$
$$[d^{(2)}\varphi]^{(3)} = \operatorname{Div}([\varphi]^{(2)}), \qquad \varphi \in \mathcal{D}'(\Omega, \mathbb{A}^2), \tag{4.8c}$$
$$[d^{(3)}\varphi]^{(4)} = \operatorname{div}([\varphi]^{(3)}), \qquad \varphi \in \mathcal{D}'(\Omega, \mathbb{A}^3). \tag{4.8d}$$

In addition to the operator Curl, another curl operator deserves mention because it fits in an alternate sequence of spaces in four dimensions. Let $\mathbb{M}$ denote the space of 4×4 matrices and let $\operatorname{skw} m = (m - m^t)/2$ for any $m \in \mathbb{M}$. Define the curl of a skew-symmetric matrix, namely $\operatorname{curl} : \mathcal{D}'(\Omega, \mathbb{K}) \to \mathcal{D}'(\Omega, \mathbb{R}^4)$, and an antisymmetrization operator $K : \mathbb{M} \to \mathbb{K}$, by

$$[\operatorname{curl} \omega]_i = \sum_{k,l=1}^{4} \varepsilon_{ijkl} \partial_j \omega_{kl}, \qquad [Km]_{ij} = \sum_{k,l=1}^{4} \varepsilon_{ijkl} m_{kl} \tag{4.9}$$

for any $\omega \in \mathcal{D}'(\Omega, \mathbb{K})$ and $m \in \mathbb{M}$. Also let the gradient of a vector field, $\operatorname{Grad} : \mathcal{D}'(\Omega, \mathbb{R}^4) \to \mathcal{D}'(\Omega, \mathbb{M})$, be defined by $[\operatorname{Grad} u]_{ij} = \partial_j u_i$. Now, analogous to the previously discussed sequence,

$$\mathcal{D}'(\Omega, \mathbb{R}) \xrightarrow{\operatorname{grad}} \mathcal{D}'(\Omega, \mathbb{R}^4) \xrightarrow{\operatorname{Curl}} \mathcal{D}'(\Omega, \mathbb{K}) \xrightarrow{\operatorname{Div}} \mathcal{D}'(\Omega, \mathbb{R}^4) \xrightarrow{\operatorname{div}} \mathcal{D}'(\Omega, \mathbb{R}),$$

we may study the following sequence with the newly defined curl:

$$\mathcal{D}'(\Omega, \mathbb{R}) \xrightarrow{\operatorname{grad}} \mathcal{D}'(\Omega, \mathbb{R}^4) \xrightarrow{\operatorname{skw}\operatorname{Grad}} \mathcal{D}'(\Omega, \mathbb{K}) \xrightarrow{\operatorname{curl}} \mathcal{D}'(\Omega, \mathbb{R}^4) \xrightarrow{\operatorname{div}} \mathcal{D}'(\Omega, \mathbb{R}).$$

The properties of the second sequence can be derived from that of the first using

$$K \operatorname{skw} \operatorname{Grad} u = \operatorname{Curl} u, \qquad \operatorname{curl} \omega = \operatorname{Div} K\omega \tag{4.10}$$

for all $u \in \mathcal{D}'(\Omega, \mathbb{R}^4)$ and $\omega \in \mathcal{D}'(\Omega, \mathbb{K})$. In particular, $\operatorname{skw} \operatorname{Grad} \operatorname{grad} = 0$, $\operatorname{curl} \operatorname{skw} \operatorname{Grad} = 0$ and $\operatorname{div} \operatorname{curl} = 0$.

4.3. Sobolev spaces. In view of the identities of Proposition 4.2, the spaces $H(d, \Omega, \mathbb{A}^k)$ in four dimensions are identified to be the same as

$$\begin{aligned}
H(\operatorname{grad}, \Omega, \mathbb{R}) &= \{u \in L^2(\Omega, \mathbb{R}) : \ \operatorname{grad} u \in L^2(\Omega, \mathbb{R}^4)\}, \\
H(\operatorname{Curl}, \Omega, \mathbb{R}^4) &= \{v \in L^2(\Omega, \mathbb{R}^4) : \ \operatorname{Curl} v \in L^2(\Omega, \mathbb{K})\}, \\
H(\operatorname{Div}, \Omega, \mathbb{K}) &= \{\kappa \in L^2(\Omega, \mathbb{K}) : \ \operatorname{Div} \kappa \in L^2(\Omega, \mathbb{R}^4)\}, \\
H(\operatorname{div}, \Omega, \mathbb{R}^4) &= \{q \in L^2(\Omega, \mathbb{R}^4) : \ \operatorname{div} q \in L^2(\Omega, \mathbb{R})\},
\end{aligned}$$

for $k = 0, 1, 2, 3$, respectively. Also setting

$$H(\operatorname{curl}, \Omega, \mathbb{K}) = \{\omega \in L^2(\Omega, \mathbb{K}) : \ \operatorname{curl} \omega \in L^2(\Omega, \mathbb{R}^4)\},$$

we note that the operator K defined in (4.9) yields a one-to-one onto homeomorphism $K : H(\operatorname{curl}, \Omega, \mathbb{K}) \to H(\operatorname{Div}, \Omega, \mathbb{K})$.

In addition to the usual Green's formula involving gradient and divergence, one can derive other integration by parts formulae, which also clarify the nature of traces in the new spaces. Let $\mathcal{D}(\bar{\Omega}, \mathbb{K})$ and $\mathcal{D}(\bar{\Omega}, \mathbb{R}^4)$ denote the sets of restrictions to Ω of functions in $\mathcal{D}(\mathbb{R}^4, \mathbb{K})$ and $\mathcal{D}(\mathbb{R}^4, \mathbb{R}^4)$, respectively. Let Ω have Lipschitz boundary so that the unit outward normal n on $\partial\Omega$ is well defined a.e. Then we can show that the traces $(n \times u)|_{\partial\Omega}$ and $(\omega n)|_{\partial\Omega}$ have meaning for $u \in H(\mathrm{Curl}, \Omega, \mathbb{R}^4)$ and $H(\mathrm{Div}, \Omega, \mathbb{K})$. More precisely, define

$$(\mathrm{tr}^{(1)} u)(\omega) = \int_{\partial\Omega} (n \times u) : \omega, \qquad (\mathrm{tr}^{(2)} \omega)(u) = \int_{\partial\Omega} \omega n \cdot u$$

for all $u \in \mathcal{D}(\Omega, \mathbb{R}^4)$ and $\omega \in \mathcal{D}(\Omega, \mathbb{K})$.

PROPOSITION 4.3. *Suppose Ω has Lipschitz boundary. Then* $\mathrm{tr}^{(1)}$ *and* $\mathrm{tr}^{(2)}$ *extend to continuous linear operators* $\mathrm{tr}^{(1)} : H(\mathrm{Curl}, \Omega, \mathbb{R}^4) \to H(\mathrm{curl}, \Omega, \mathbb{K})'$, *and* $\mathrm{tr}^{(2)} : H(\mathrm{Div}, \Omega, \mathbb{K}) \to H(\mathrm{Curl}, \Omega, \mathbb{R}^4)'$ *satisfying*

$$\begin{aligned}(\mathrm{tr}^{(1)} u)(\omega) &= \int_\Omega \mathrm{Curl}\, u : \omega - \int_\Omega u \cdot \mathrm{curl}\, \omega \\ (\mathrm{tr}^{(2)} \kappa)(u) &= \int_\Omega \mathrm{Div}\, \kappa \cdot u \, dx - \int_\Omega \kappa \times \mathrm{Curl}\, u\end{aligned}$$

for all $u \in H(\mathrm{Curl}, \Omega, \mathbb{R}^4)$, $\kappa \in H(\mathrm{Div}, \Omega, \mathbb{K})$, *and* $\omega \in H(\mathrm{curl}, \Omega, \mathbb{K})$.

Proof. For $u \in \mathcal{D}(\Omega, \mathbb{R}^4)$ and $\omega \in \mathcal{D}(\Omega, \mathbb{K})$, integrating by parts each term that makes up the products below and using the properties of ε to simplify the result, we derive

$$\int_\Omega \mathrm{Curl}\, u : \omega - \int_\Omega u \cdot \mathrm{curl}\, \omega = \int_{\partial\Omega} (n \times u) : \omega. \tag{4.11}$$

Similarly, we also derive

$$\int_\Omega \mathrm{Div}\, \omega \cdot u - \int_\Omega \omega \times \mathrm{Curl}\, u = \int_{\partial\Omega} \omega n \cdot u. \tag{4.12}$$

Now viewing $H(\mathrm{Curl}, \Omega, \mathbb{R}^4)$, $H(\mathrm{Div}, \Omega, \mathbb{K})$, and $H(\mathrm{curl}, \Omega, \mathbb{K})$ as graph spaces of Curl, Div, and curl, we apply the well-known extensions of classical density proofs to graph spaces (see e.g., [26]) to conclude that $\mathcal{D}(\bar{\Omega}, \mathbb{R}^4)$ is dense in $H(\mathrm{Curl}, \Omega, \mathbb{R}^4)$ and that $\mathcal{D}(\bar{\Omega}, \mathbb{K})$ is dense in $H(\mathrm{curl}, \Omega, \mathbb{K})$ as well as $H(\mathrm{Div}, \Omega, \mathbb{K})$. Hence the result follows from (4.11) and (4.12). □

The Sobolev spaces we have introduced above have their analogues with essential boundary conditions:

$$\begin{aligned}H_0(\mathrm{Curl}, \Omega, \mathbb{R}^4) &= \{v \in H(\mathrm{Curl}, \Omega, \mathbb{R}^4) : \ \mathrm{tr}^{(1)} v = 0\}, \\ H_0(\mathrm{Div}, \Omega, \mathbb{K}) &= \{\omega \in H(\mathrm{Div}, \Omega, \mathbb{K}) : \ \mathrm{tr}^{(2)} \omega = 0\}.\end{aligned}$$

The construction of the HX preconditioner for these spaces follows along the same lines as before, now using standard preconditioners in $H_0^1(\Omega)$. To highlight the changes required in the analysis, first, instead of the regularized Poincaré operator R_k (appearing in the proof of Theorem 2.1) , we must now use the generalized Bogovskiĭ operator (see [14] or [34, Theorem 1.5]) to get the appropriate regular decomposition with boundary conditions. We then continue along the previous lines after replacing $Q_h^{(k)}$ by the L^2 projection into $V_h^{(k)} \cap H_0^1(\Omega, \mathbb{A}^k)$. Note that the H_0^1-stability of this projection holds as remarked in [12, p. 153]. Bounded cochain projectors preserving homogeneous boundary conditions are also known [13], so all the ingredients are available to generalize our analysis to the case of homogeneous essential boundary conditions.

4.4. Finite element spaces. Let T be a 4-simplex with vertices a_i, $i = 1, \ldots, 5$. Let λ_i denote its ith barycentric coordinate, i.e., $\lambda_i(x)$ is the unique affine function (of the Euclidean coordinate x of points in T) that equals 1 at a_i and equals 0 at all the remaining vertices of T. Let

$$g_i = \operatorname{grad} \lambda_i \in \mathbb{R}^4, \qquad g_{ij} = g_i \times g_j \in \mathbb{K}, \qquad g_{ijk} = g_{ij} g_k \in \mathbb{R}^4.$$

Let $f_{i_1,\ldots,i_k}$ denote the subsimplex of T formed by the convex hull of $a_{i_1}, \ldots, a_{i_k}$ for any $k = 1, \ldots, 5$ and let $\triangle(k, T)$ denote the set of all k-subsimplices of T. To a 0-subsimplex $f_i = a_i$ we associate the function λ_i and to other subsimplices f_{ij}, f_{ijk} and f_{ijkl}, we associate, respectively, the following functions.

$$\lambda_{ij} = \lambda_i g_j - \lambda_j g_i, \tag{4.13a}$$

$$\lambda_{ijk} = \lambda_i g_{jk} - \lambda_j g_{ik} + \lambda_k g_{ij}, \tag{4.13b}$$

$$\lambda_{ijkl} = \lambda_i g_{jkl} - \lambda_j g_{ikl} + \lambda_k g_{ijl} - \lambda_l g_{ijk}. \tag{4.13c}$$

Note that these expressions depend on the ordering of the vertices and on T. When such dependence is to be made explicit, we write the function associated to any $f_{i_1,\ldots,i_k} \in \triangle(k, T)$, namely $\lambda_{i_1,\ldots,i_k}$, as $\lambda^T_{a_{i_1},\ldots,a_{i_k}}$ or $\lambda^T_{a(f)}$ where $a(f) = (a_{i_1}, \ldots, a_{i_k})$.

We implemented the lowest order polynomial space $P_1^{-,(k)}(T) = \operatorname{span}\{\lambda_{a(f)} : f \in \triangle(k, T)\}$ for $k = 0, 1, 2$ and 3. Using Propositions 4.1 and 4.2, these spaces may be immediately recognized as the space of proxies of the *Whitney basis* [2, 3, 46] for $P_1^- \mathbb{A}^k$. To construct the global finite element spaces, we consider the set of all k-subsimplices of the simplicial mesh Ω_h, denoted by $\triangle(k, \Omega_h)$. An element f of $\triangle(k, \Omega_h)$ is in the set $\triangle(k, T_j)$ for one or more mesh elements $T_1, \ldots, T_{n_f}$ in Ω_h. To each $f \in \triangle(k, \Omega_h)$, we associate an *ordered* set of its vertices $a(f)$. The ordering fixes a global orientation of f independently of T_j. Let λ_f, for each $f \in \triangle(k, \Omega_h)$, be the function that vanishes on all elements of the mesh except $T_1, \ldots, T_{n_f}$ where its values are given by $\lambda_f|_{T_j} = \lambda^{T_j}_{a(f)}$. These functions define the global finite space by

$$\mathcal{V}_h^{(k)} = \operatorname{span}\{\lambda_f : f \in \triangle(k, \Omega_h)\}$$

for each $k = 0, 1, 2, 3$. One can easily show that $\mathcal{V}_h^{(1)} \subseteq H(\mathrm{Curl}, \Omega, \mathbb{R}^4)$, and $\mathcal{V}_h^{(2)} \subseteq H(\mathrm{Div}, \Omega, \mathbb{K})$, either directly integrating by parts using Proposition 4.3 on each mesh element, or by observing that $\mathcal{V}_h^{(k)}$ consists of all proxies of $V_h^{(k)}$ (when $r = 1$) and recalling [2] that $V_h^{(k)} \subseteq H(d, \Omega, \mathbb{A}^k)$. Of course, we also have $\mathcal{V}_h^{(0)} \subseteq H(\mathrm{grad}, \Omega, \mathbb{R}^4)$ and $\mathcal{V}_h^{(3)} \subseteq H(\mathrm{div}, \Omega, \mathbb{R}^4)$.

Our actual implementation uses an alternate, but equivalent technique that proceeds by implementing the expressions in (4.13) only on the unit 4-simplex and then mapping the basis functions to each mesh simplex appropriately (see the code in [33] for more details).

4.5. Finite element interpolant. The implementation of the HX preconditioner for $\mathcal{V}_h^{(1)} \subseteq H(\mathrm{Curl}, \Omega, \mathcal{R}^4)$, $\mathcal{V}_h^{(2)} \subseteq H(\mathrm{Div}, \Omega, \mathbb{K})$, and $\mathcal{V}_h^{(3)} \subseteq H(\mathrm{div}, \Omega, \mathbb{R}^4)$ requires us to implement canonical finite element interpolants $\Pi_h^{\mathrm{Curl}}, \Pi_h^{\mathrm{Div}}$, and Π_h^{div}, into these spaces, respectively. Since the last one is standard, we only describe the first two.

Let T be a 4-simplex with vertices a_i and let $u : T \to \mathbb{R}^4$ be a smooth vector function. Let e_{ij} denote the segment connecting a_i and a_j. Then $\Pi_h^{\mathrm{Curl}} u|_T$ is the unique function in $P_1^{-,(1)}(T)$ satisfying $\sigma_{ij}(u - \Pi_h^{\mathrm{Curl}} u) = 0$ for every edge e_{ij} of T, where

$$\sigma_{ij}(u) = \frac{1}{|e_{ij}|} \int_{e_{ij}} u \cdot (a_i - a_j)$$

and $|e_{ij}|$ denotes the length of the edge e_{ij}.

Next, let $\omega : T \to \mathbb{K}$ be a smooth function and let f_{ijk} denote the triangle formed by the convex hull of a_i, a_j and a_k. Then $\Pi_h^{\mathrm{Div}} \omega|_T$ is the unique function in $P_1^{-,(2)}(T)$ satisfying $\sigma_{ijk}(\omega - \Pi_h^{\mathrm{Div}} \omega) = 0$ for all 2-subsimplices f_{ijk} of T, where

$$\sigma_{ijk}(\omega) = \frac{1}{|f_{ijk}|} \iint_{f_{ijk}} \omega : (a_j - a_i) \times (a_k - a_i)$$

and $|f_{ijk}|$ denotes the area of the triangle f_{ijk}.

To compute the preconditioner action, we need to apply Π_h^{Curl} to functions $u : \Omega \to \mathbb{R}^4$ whose components are in the lowest order Lagrange finite element space. Then defining $\Pi_h^{\mathrm{Curl}} u|_T$ for each T in Ω_h as above, the continuity of components of u imply that the resulting global function $\Pi_h^{\mathrm{Curl}} u$ in $\mathcal{V}_h^{(1)}$. Similarly, when ω has components in the Lagrange finite element space, $\Pi_h^{\mathrm{Div}} \omega$ is in $\mathcal{V}_h^{(2)}$.

The unisolvency of these degrees of freedom follow from [4, Theorem 5.5] after identifying the degrees of freedom given there (for $k = 1, 2$) in terms of our proxies using Proposition 4.1. In particular, Π_h^{Curl} and Π_h^{Div} can be viewed as proxies of $\Pi_h^{(k)}$ for $k = 1$ and 2 in four dimensions.

5. Numerical results. In this section, we report the results of numerical experiments obtained using our implementation of the preconditioners in 4D. We implemented the lowest order finite element subspaces of $H^1(\Omega)$, $H(\mathrm{Curl}, \Omega)$, $H(\mathrm{Div}, \Omega)$ and $H(\mathrm{div}, \Omega)$ on general unstructured (conforming) meshes of 4-simplices. The preconditioners were built atop this discretization. Below we will perform verification of the discretization as well as report on the performance of the preconditioners.

5.1. Convergence studies. In the first series of examples, we fix $\Omega = (0, 1)^4$ and solve the linear systems arising from the lowest order finite element discretization of the following problem: Find $u \in H(d, \Omega, \mathbb{A}^k)$, such that

$$(u, v) + (du, dv) = \langle F, v \rangle \qquad \text{for all } v \in H(d, \Omega, \mathbb{A}^k), \tag{5.1}$$

where $F \in H(d, \Omega, \mathbb{A}^k)'$ is a bounded linear functional given below for each $k = 0, 1, 2, 3$. The domain Ω was initially subdivided into a mesh Ω_h of 96 4-simplices of uniform size (see also [37]). Afterwards we apply successive refinement based on the algorithm of Freudenthal (see [8, 18, 37] for more details). The arising linear systems are solved using preconditioned conjugate gradient iterations, where the preconditioner is set to the ones given in §3.1 for each k. In all the presented experiments set the smoother D_h by three steps of a Chebyshev smoother with respect to the operator A. We iterate until a relative residual error reduction of 10^{-6} is obtained.

Under these numerical settings, we study two types of convergence, namely the convergence rates of the lowest order 4D discretizations, and the iterative convergence of the preconditioned conjugate gradient iterations.

To establish a baseline, we start with 0-forms, i.e. $d = \text{grad}$. The F in (5.1) is set so that the exact solution is

$$u(x) = \cos(\pi x_1)\cos(\pi x_2)\cos(\pi x_3)\cos(\pi x_4).$$

The $L^2(\Omega)$ distance between this u and the computed solution u_h in the 4D lowest order Lagrange finite element space is reported in one of the columns of Table 5.1. Clearly the observed convergence rate is close to two, the best possible rate for this approximation space. For solving the linear systems we set the preconditioner to the algebraic multigrid preconditioner BoomerAMG of the hypre package [21]. The iteration counts reported in the last column of the same table show small iteration numbers with small growth. Recall that one of the basic assumptions in the auxiliary space preconditioner construction is that we have a good preconditioner for the Laplacian. Therefore this report of the performance of BoomerAMG in 4D gives us a measure of how well this baseline assumption is verified in practice.

For 1-forms, i.e. $d = \text{Curl}$, we set an F in (5.1) that yields the exact solution

$$u(x) = [s_1c_2c_3c_4, -c_1s_2c_3c_4, c_1c_2s_3c_4, -c_1c_2c_3s_4]^\top,$$

where $c_i = \cos(\pi x_i)$ and $s_i = \sin(\pi x_i)$ for $i = 1, \ldots, 4$. In Table 5.2 we summarize the convergence results for the lowest order finite elements, i.e., edge-elements in 4D. Here, we again observe a convergence rate close to the theoretically expected rate of one. For solving the linear system we use the proposed preconditioner given in Subsection 3.1. Here we obtain small iteration counts. But observe that they are slightly increasing. We believe this is due to the fact that the BoomerAMG's performance (reported in the previous table) is not strictly uniform.

When considering 2-forms, i.e. $d = \text{Div}$ we use the manufactured solution

$$u(x) = \begin{bmatrix} 0 & c_1c_2s_3s_4 & -c_1s_2c_3s_4 & c_1s_2s_3c_4 \\ -c_1c_2s_3s_4 & 0 & s_1c_2c_3s_4 & -s_1c_2s_3c_4 \\ c_1s_2c_3s_4 & -s_1c_2c_3s_4 & 0 & s_1s_2c_3c_4 \\ -c_1s_2s_3c_4 & s_1c_2s_3c_4 & -s_1s_2c_3c_4 & 0 \end{bmatrix},$$

where c_i, s_i are as above. Using the lowest order finite elements for 2-forms in 4D, we observe in Table 5.3 the optimal convergence rate of one. Moreover the preconditioner given in Subsection 3.1 leads to similar iteration numbers as in the previous example.

For 3-forms, i.e. $d = \text{div}$ we consider the exact solution

$$u(x) = [c_1s_2s_3s_4, s_1c_2s_3s_4, s_1s_2c_3s_4, s_1s_2s_3c_4]^\top.$$

For these lowest order Raviart-Thomas finite elements in 4D, we again observe the correct convergence rate of one from Table 5.4. The iteration numbers for the auxiliary space preconditioner again exhibit a small growth.

5.2. Parameter robustness. In the following experiments we will study the preconditioners when a parameter $\tau > 0$ is involved, namely instead of (5.1), we consider the following problem: Find $u \in H(d, \Omega, \mathbb{A}^k)$, such that

$$\tau(u, v) + (du, dv) = \langle F, v\rangle \qquad \text{for all } v \in H(d, \Omega, \mathbb{A}^k),$$

level	cores	elements	dof	$\|\|u-u_h\|\|_{L^2(\Omega)}$	eoc	iter
0	16	96	25	2.21816E-1	-	6
1	16	1 536	169	1.64804E-1	0.43	11
2	128	24 576	1 681	7.38929E-2	1.16	15
3	16 384	393 216	21 025	2.63863E-2	1.49	18
4	32 768	6 291 456	297 025	7.77695E-3	1.76	20
5	32 768	100 663 296	4 464 769	2.06687E-3	1.91	22
6	32 768	1 610 612 736	69 239 041	5.26637E-4	1.97	25

TABLE 5.1
Convergence result and iteration numbers for $H(\mathrm{grad}, \Omega, \mathbb{R})$.

level	cores	elements	dof	$\|\|u-u_h\|\|_{L^2(\Omega)}$	eoc	iter
0	16	96	144	3.77018E-1	-	10
1	16	1 536	1 512	3.05056E-1	0.31	15
2	128	24 576	19 344	1.92098E-1	0.67	18
3	16 384	393 216	276 000	1.10859E-1	0.79	22
4	32 768	6 291 456	4 167 744	5.94185E-2	0.90	25
5	32 768	100 663 296	64 774 272	3.05043E-2	0.96	28
6	32 768	1 610 612 736	1 021 411 584	1.53829E-2	0.99	35

TABLE 5.2
Convergence result and iteration numbers for $H(\mathrm{Curl}, \Omega, \mathbb{R}^4)$.

where F is set for each k as described previously. All other parameters, including the domain and stopping criterion, are set as in the previous experiments. The results summarized in Tables 5.5-5.8 show iteration numbers for the preconditioned conjugate gradient method. For weights τ ranging from 10^{-6} to 10^{6} we observe quite small iteration numbers which vary only slightly with τ. For small values of τ, these observations are consistent with the analytical conclusions of Theorem 3.6.

Conclusion. We presented the auxiliary space preconditioning technique in arbitrary dimensions. The presentation extends previous results of Hiptmair and Xu [24] using recent estimates on regularized homotopy operators [14, 34] and recent developments in finite element exterior calculus [4]. Although, we only analyzed the additive version of the auxiliary space preconditioners, their multiplicative versions can be similarly derived and analyzed.

This work also provides an implementation of the 4D auxiliary space preconditioners (currently available as one of the public domain development branches of [33]). During this work, we also implemented the finite element subspaces (currently the lowest order ones) of the 4D de Rham complex, their canonical interpolants using proxy identities, and attendant 4D mesh operations. Use of this technology for space-time applications and the addition of geometric multigrid to the tool set are subjects of ongoing research.

REFERENCES

[1] R. ANDREEV, *Stability of sparse space-time finite element discretizations of linear parabolic evolution equations*, IMA J. Numer. Anal., 33 (2013), pp. 242–260.

[2] D. N. ARNOLD, R. S. FALK, AND R. WINTHER, *Finite element exterior calculus, homological techniques, and applications*, Acta Numer., 15 (2006), pp. 1–155.

level	cores	elements	dof	$\|\|u-u_h\|\|_{L^2(\Omega)}$	eoc	iter
0	16	96	312	4.40957E-1	-	16
1	16	1 536	4 032	4.58250E-1	-	26
2	128	24 576	57 600	2.83070E-1	0.69	27
3	16 384	393 216	869 376	1.53587E-1	0.88	28
4	32 768	6 291 456	13 504 512	7.95018E-2	0.95	28
5	32 768	100 663 296	212 877 312	4.02155E-2	0.98	29
6	32 768	1 610 612 736	3 380 674 560	2.01713E-2	1.00	33

TABLE 5.3

Convergence result and iteration numbers for $H(\mathrm{Div}, \Omega, \mathbb{K})$.

level	cores	elements	dof	$\|\|u-u_h\|\|_{L^2(\Omega)}$	eoc	iter
0	16	96	288	3.74239E-1	-	9
1	16	1 536	4 224	1.94337E-1	0.95	18
2	128	24 576	64 512	9.76605E-2	0.99	19
3	16 384	393 216	1 007 616	4.88885E-2	1.00	19
4	32 768	6 291 456	15 925 248	2.43047E-2	1.01	19
5	32 768	100 663 296	253 231 104	1.20968E-2	1.01	20
6	32 768	1 610 612 736	4 039 114 752	6.03196E-3	1.00	24

TABLE 5.4

Convergence result and iteration numbers for $H(\mathrm{div}, \Omega, \mathbb{R}^4)$.

	weight τ												
L	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1	10	10^{2}	10^{3}	10^{4}	10^{5}	10^{6}
0	5	5	6	6	7	7	6	5	9	11	12	12	12
1	10	11	12	12	12	12	11	9	5	10	12	12	12
2	15	15	15	15	15	15	15	13	8	9	12	12	12
3	19	19	19	19	19	19	18	17	13	7	10	12	12
4	20	20	20	20	20	20	20	19	15	9	9	11	12
5	23	23	23	23	23	23	22	21	16	11	6	8	11
6	27	27	27	27	27	26	25	23	17	12	8	5	9

TABLE 5.5

Iteration numbers for different number of weights τ and refinements for the space $H(\mathrm{grad}, \Omega, \mathbb{R})$.

	weight τ												
L	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1	10	10^{2}	10^{3}	10^{4}	10^{5}	10^{6}
0	11	11	12	12	11	11	10	11	16	18	19	19	19
1	14	15	15	15	14	15	15	14	19	24	27	27	27
2	21	19	19	18	17	19	18	15	16	21	28	29	29
3	23	21	20	19	21	23	22	20	15	18	22	26	26
4	22	21	20	21	23	24	25	22	16	14	20	25	28
5	22	23	25	24	27	29	28	25	18	14	16	22	29
6	28	29	29	31	32	33	35	29	22	14	13	19	27

TABLE 5.6

Iteration numbers for different number of weights τ and refinements for the space $H(\mathrm{Curl}, \Omega, \mathbb{R}^4)$.

L	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1	10	10^{2}	10^{3}	10^{4}	10^{5}	10^{6}
	weight τ												
0	12	13	14	14	15	15	16	17	21	23	23	23	23
1	20	21	22	23	24	26	26	26	25	31	34	34	34
2	20	21	23	24	26	27	27	27	26	27	34	35	35
3	20	21	23	24	26	27	28	27	26	24	27	30	31
4	20	21	22	24	26	27	28	27	26	25	24	28	30
5	22	22	24	25	27	28	29	28	25	25	24	25	31
6	28	28	30	30	31	32	33	30	26	23	24	21	28

TABLE 5.7

Iteration numbers for different number of weights τ and refinements for the space $H(\mathrm{Div}, \Omega, \mathbb{K})$.

L	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1	10	10^{2}	10^{3}	10^{4}	10^{5}	10^{6}
	weight τ												
0	8	8	9	9	9	9	9	10	13	14	14	14	14
1	20	20	19	19	18	17	18	18	17	16	17	17	17
2	22	21	20	19	18	18	19	20	20	16	17	17	17
3	21	20	18	17	17	18	19	20	20	18	15	16	16
4	19	18	17	16	17	18	19	20	19	19	16	15	16
5	18	20	20	18	18	19	20	20	19	18	18	14	16
6	22	22	22	22	23	23	24	23	20	16	17	16	15

TABLE 5.8

Iteration numbers for different number of weights τ and refinements for the space $H(\mathrm{div}, \Omega, \mathbb{R}^4)$.

[3] ———, *Geometric decompositions and local bases for spaces of finite element differential forms*, Comput. Methods Appl. Mech. Engrg., 198 (2009), pp. 1660–1672.

[4] ———, *Finite element exterior calculus: from Hodge theory to numerical stability*, Bull. Amer. Math. Soc. (N.S.), 47 (2010), pp. 281–353.

[5] I. Babuška and T. Janik, *The h-p version of the finite element method for parabolic equations. Part I. the p-version in time*, Numerical Methods for Partial Differential Equations, 5 (1989), pp. 363–399.

[6] ———, *The h-p version of the finite element method for parabolic equations. II. the h-p version in time*, Numerical Methods for Partial Differential Equations, 6 (1990), pp. 343–369.

[7] R. E. Bank, P. S. Vassilevski, and L. T. Zikatanov, *Arbitrary dimension convection–diffusion schemes for space–time discretizations*, Journal of Computational and Applied Mathematics, 310 (2017), pp. 19–31.

[8] J. Bey, *Simplicial grid refinement: on Freudenthal's algorithm and the optimal number of congruence classes*, Numer. Math., 85 (2000), pp. 1–29.

[9] M. Birman and M. Solomyak, *Construction in a piecewise smooth domain of a function of the class H^2 from the value of the conormal derivative*, J. Math. Sov., 49 (1990), pp. 1128—1136.

[10] P. Bochev, J. Hu, C. Siefert, and R. Tuminaro, *An algebraic multigrid approach based on a compatible gauge reformulation of Maxwell's equations*, SIAM J. Sci. Comput., 31 (2008), pp. 557–583.

[11] P. Bochev, C. Siefert, R. Tuminaro, J. Xu, and Y. Zhu, *Compatible gauge approaches for $H(\mathrm{div})$ equations*, in CSRI Summer Proceedings, 2007.

[12] J. H. Bramble, J. E. Pasciak, and O. Steinbach, *On the stability of the L^2 projection in $H^1(\Omega)$*, Math. Comp., 71 (2002), pp. 147–156.

[13] S. H. Christiansen and R. Winther, *Smoothed projections in finite element exterior calculus*, Math. Comp., 77 (2008), pp. 813–829.

[14] M. Costabel and A. McIntosh, *On Bogovskiĭ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains*, Mathematische Zeitschrift, (2010), pp. DOI:10.1007/s00209–009–0517–8.

[15] L. Demkowicz, J. Gopalakrishnan, S. Nagaraj, and P. Sepulveda, *A spacetime DPG*

method for the Schrödinger equation, SIAM J. Numer. Anal., 55 (2017), pp. 1740–1759.

[16] A. Demlow and A. N. Hirani, *A posteriori error estimates for finite element exterior calculus: the de Rham complex*, Found. Comput. Math., 14 (2014), pp. 1337–1371.

[17] R. D. Falgout, S. Friedhoff, T. V. Kolev, S. P. MacLachlan, and J. B. Schroder, *Parallel time integration with multigrid*, SIAM Journal on Scientific Computing, 36 (2014), pp. C635–C661.

[18] H. Freudenthal, *Simplizialzerlegungen von beschränkter Flachheit*, Ann. of Math. (2), 43 (1942), pp. 580–582.

[19] J. Gopalakrishnan and P. Sepúlveda, *A spacetime DPG method for acoustic waves*, Preprint arXiv:1709.08268, (2017).

[20] P. Hansbo, *Space-time oriented streamline diffusion methods for nonlinear conservation laws in one dimension*, Comm. Numer. Methods Engrg., 10 (1994), pp. 203–215.

[21] V. Henson and U. Yang, *BoomerAMG: a parallel algebraic multigrid solver and preconditioner*, Applied Numerical Mathematics, 41 (2002), pp. 155–177.

[22] R. Hiptmair, *Canonical construction of finite elements*, Math. Comp., 68 (1999), pp. 1325–1346.

[23] R. Hiptmair, J. Li, and J. Zhou, *Universal extension for Sobolev spaces of differential forms and applications*, Journal of Functional Analysis, 263 (2012), pp. 364–382.

[24] R. Hiptmair and J. Xu, *Nodal auxiliary space preconditioning in* $\mathbf{H}(\mathbf{curl})$ *and* $\mathbf{H}(\mathrm{div})$ *spaces*, SIAM J. Numer. Anal., 45 (2007), pp. 2483–2509.

[25] hypre*: High performance preconditioners.* http://www.llnl.gov/CASC/hypre/.

[26] M. Jensen, *Discontinuous Galerkin Methods for Friedrichs Systems with Irregular Solutions*, PhD thesis, University of Oxford, 2004.

[27] C. Johnson, U. Nävert, and J. Pitkäranta, *Finite element methods for linear hyperbolic problems*, Comput. Methods Appl. Mech. Engrg., 45 (1984), pp. 285–312.

[28] C. Johnson and J. Saranen, *Streamline diffusion methods for the incompressible Euler and Navier-Stokes equations*, Math. Comp., 47 (1986), pp. 1–18.

[29] T. V. Kolev and P. S. Vassilevski, *Parallel auxiliary space AMG for* $H(\mathrm{curl})$ *problems*, J. Comput. Math., 27 (2009), pp. 604–623.

[30] ———, *Parallel auxiliary space AMG solver for* $H(\mathrm{div})$ *problems*, SIAM J. Sci. Comput., 34 (2012), pp. A3079–A3098.

[31] U. Langer, S. Moore, and M. Neumüller, *Space-time isogeometric analysis of parabolic evolution equations*, Comput. Methods Appl. Mech. Engrg., 306 (2016), pp. 342–363.

[32] S. Larsson and M. Molten, *Numerical solution of parabolic problems based on a weak space-time formulation*, Comput. Methods Appl. Math., 17 (2017), pp. 65–84.

[33] *MFEM: Modular finite element methods library.* `http://mfem.org`.

[34] D. Mitrea, M. Mitrea, and S. Monniaux, *The Poisson problem for the exterior derivative operator with Dirichlet boundary condition in nonsmooth domains*, Commun. Pure Appl. Anal., 7 (2008), pp. 1295–1333.

[35] C. Mollet, *Stability of Petrov-Galerkin discretizations: Application to the space-time weak formulation for parabolic evolution problems*, Computational Methods in Applied Mathematics, 14 (2014), pp. 231–255.

[36] S. Nepomnyaschikh, *Domain decomposition methods*, in Lectures on advanced computational methods in mechanics, vol. 1 of Radon Ser. Comput. Appl. Math., Walter de Gruyter, Berlin, 2007, pp. 89–159.

[37] M. Neumüller and O. Steinbach, *Refinement of flexible space–time finite element meshes and discontinuous Galerkin methods*, Comput. Visual. Sci., 14 (2011), pp. 189–205.

[38] M. Neumüller, P. S. Vassilevski, and U. E. Villa, *Space-time CFOSLS methods with AMGe upscaling*, in Domain Decomposition Methods in Science and Engineering XXIII, C.-O. Lee, X.-C. Cai, D. E. Keyes, H. H. Kim, A. Klawonn, E.-J. Park, and O. B. Widlund, eds., Cham, 2017, Springer International Publishing, pp. 253–260.

[39] J. E. Pasciak and J. Zhao, *Overlapping Schwarz Methods in* $H(curl)$ *on Nonconvex Domains*, East West J. Num. Anal., 10 (2002), pp. 221–234.

[40] J. Schöberl, *A posteriori error estimates for Maxwell equations*, Math. Comp., 77 (2008), pp. 633–649.

[41] C. Schwab and R. Stevenson, *Space–time adaptive wavelet methods for parabolic evolution problems*, Math. Comput., 78 (2009), pp. 1293–1318.

[42] I. Smears, *Robust and efficient preconditioners for the discontinuous galerkin time-stepping method*, IMA J. Numer. Anal., 00 (2016), pp. 1–25.

[43] O. Steinbach, *Space-time finite element methods for parabolic problems*, Comput. Meth. Appl. Math., 15 (2015), pp. 551–566.

[44] K. Urban and A. T. Patera, *An improved error bound for reduced basis approximation of*

linear parabolic problems, Math. Comput., 83 (2014), pp. 1599–1615.

[45] J. J. W. van der Vegt and H. van der Ven, *Space-time discontinuous Galerkin finite element method with dynamic grid motion for inviscid compressible flows. I. General formulation*, J. Comput. Phys., 182 (2002), pp. 546–585.

[46] H. Whitney, *Geometric integration theory*, Princeton University Press, Princeton, N. J., 1957.

[47] J. Xu, *The auxiliary space method and optimal multigrid preconditioning techniques for unstructured grids*, Computing, 56 (1996), pp. 215–235.

[48] J. Xu, L. Chen, and R. H. Nochetto, *Optimal multilevel methods for* $H(\mathrm{grad})$, $H(\mathrm{curl})$, *and* $H(\mathrm{div})$ *systems on graded and unstructured grids*, in Multiscale, nonlinear and adaptive approximation, Springer, Berlin, 2009, pp. 599–659.