New hybridization techniques

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In this paper we present an overview of some new hybridization techniques for linear second-order elliptic problems. We begin by introducing the hybridization technique through a simple one dimensional example. We then introduce a new point of view with which previously unsuspected applications of hybridization have become possible. Presentation of these applications is the main objective of this review. One such application is in comparing and establishing connections between mixed methods. Next we show how hybridization makes possible the construction of high order variable degree mixed methods. We develop a new error analysis for mixed methods making essential use of hybridization resulting in new error estimates for our new as well as old methods. Finally, we show that via hybridization one can solve the long standing research problem of computing numerical approximations to Stokes flow that are exactly divergence free. We show this for a discontinuous Galerkin method and then for a classical mixed method.

1 Introduction

Hybridization of finite element methods was born in [21]. In its inception, it was a clever *implementation* technique that allowed computation of the solution of an elasticity problem efficiently and conveniently. Twenty years later, it was realized that hybridization is much more than *just* an implementation technique: The solution of the hybridized form of the method was found to contain more information [2] than the solution of the non-hybridized form of the method. The papers [2, 8, 17] show how one can obtain solutions of enhanced accuracy by local post processing using this extra information. After another twenty years, a new perspective on hybridization was recently developed, thanks to which it is possible to achieve the following:

1. Uncover previously unsuspected relationships between some existing methods: Here we develop hybridization as a theoretical tool to compare different mixed methods (Sections 2 and 3). We will review old facts on the extra information contained in the solution obtained after hybridization in a new way.

2. Construct variable degree mixed methods (Sections 4 and 7): Many practical problems require the use of finite elements of varying degrees of approximation in different regions of the computational domain, hence the importance of variable degree mixed methods.

3. Develop a new technique of error analysis: Making essential use of hybridization, we shall obtain new error estimates for new as well as old methods (Section 5).

4. Design methods that give exactly divergence free numerical solutions for Stokes flow: The construction of such methods has been a long standing research problem, but one that has never before received the benefit of hybridization ideas. It is possible to solve this problem either by hybridizing

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a discontinuous Galerkin method (Section 6) or by developing new hybridization techniques for a classical mixed method (Section 7).

In this paper, we review all the new results mentioned above.

To introduce the reader to the subject, we begin by very simple considerations in the next section. We compare three standard methods for a one dimensional example and explain in detail why one of them, a hybridized method, is the superior choice. Although most of the results in this section are well known, the methodology for analysis is new even for this simple example. The new approach is based on a result that characterizes the Lagrange multiplier component alone as *the solution of a variational equation*. We illustrate this approach by applying it to the one dimensional example in Section 2 and later to the two dimensional Dirichlet problem in Section 3. In Section 3, we will also compare a few existing methods and reveal some old and some new connections between them. In particular, we show that the Lagrange multipliers of two popular independently developed mixed methods, namely the Raviart-Thomas (RT) method [33] and the Brezzi-Douglas-Marini (BDM) method [8], coincide for a large class of practically important data.

In Section 4, we develop a variable degree version of the RT method on simplices. Variable degree methods are essential for *hp*-adaptivity which in turn is needed for practical problems with singular solutions. In the early paper of [9], one finds a natural variable degree extension of the original BDM method, but results were achieved only under the assumption that the maximum variation of the polynomial degrees of two adjacent elements be one. This method was obtained by introducing suitably defined "transition elements". Such elements were thought to be necessary to maintain the continuity of the normal component of the flux along element interfaces when joining two elements of different degree. Indeed, all the popular *hp*-mixed method codes use the "minimum degree rule" and corresponding transitional shape functions. We will show that this is not necessary once hybridization is used. Moreover, the early approach was not general enough to develop variable degree versions of the RT method on simplices, while our approach yields variable degree versions of both the BDM and RT methods in a natural fashion without any restriction on the variation of the polynomial degrees. In Section 5 we develop a new technique for error analysis based on hybridization. We apply this to the variable degree RT method we constructed in the previous section. We prove error estimates that do not reduce to previously known estimates even in the case of the standard uniform degree method.

In Sections 6 and 7, we introduce two new and efficient ways to compute exactly divergence free approximations for Stokes flow. The construction of exactly divergence free finite element basis has been a long standing research question [26]. Basis functions for finite dimensional spaces of weakly incompressible functions were constructed in [25, 27, 39]. However, such constructions proved to be extremely difficult to extend to spaces of polynomials of higher degree. Exactly divergence free finite element spaces have been studied, but known results require the use of polynomials of degree four or higher for the two dimensional case [30, 36, 37] and no similar result exists for the three-dimensional case.

One way to circumvent this problem is described in Section 6. There we consider an optimally convergent method obtained by using a discontinuous Galerkin method to discretize a vorticity-velocity formulation of the Stokes equations. We then show that by applying a new *hybridization* to the resulting discretization, a globally divergence-free approximate velocity is obtained without having to construct globally divergence-free finite dimensional spaces – only element-wise divergence-free basis functions are used. Another important feature is that it has significantly less degrees of freedom than all other similar discontinuous Galerkin methods. In particular, the approximation to the pressure is only defined on the faces of the elements.

In Section 7, we again consider a velocity-vorticity formulation of the Stokes problem, but this time we pose it as a classical mixed problem. This mixed formulation has previously appeared in many works [1, 3, 23, 32]. But all the previous works resort to the introduction of a stream function variable to obtain exactly divergence free numerical velocities. This approach is beset with significant difficulties such as the increase in degrees of freedom (especially in three dimensions, when the stream function is a vector function), the introduction of a nontrivial "gauge condition", and finding "cuts" to simplify complicated topology [1]. All these difficulties disappear in our approach via hybridization. Because we do not introduce the stream function, our method requires nothing special to be done when the computational domain has nontrivial topology. We also never encounter the fourth order operators

that often appear in formulations with the streamfunction – our matrices represent discretizations of operators of second order only. Moreover, while the introduction of the stream function results in an increase in degrees of freedom, our approach using hybridization and elimination actually results in a decrease in degrees of freedom.

Finally, let us emphasize that the hybridization of this classical mixed method is more complicated than that of the above mentioned discontinuous Galerkin method. Indeed, although in both cases, we must apply a hybridization technique to deal with the divergence-free condition, in the case of the classical mixed we need to carry out another new hybridization. The purpose of this additional hybridization is to avoid having to work with $H(\mathbf{curl})$-conforming finite element spaces for the vorticity, thereby making it possible to eventually eliminate all the original unknowns and getting one system for the tangential velocity and the pressure approximations on the *boundary of the elements*. Once these variables are computed, the velocity, the vorticity, and the pressure in the whole domain can be easily obtained in an element-by-element fashion.

The organization of the paper is so that we start from simple hybridization techniques in Section 2 and proceed to more complicated ones in later sections. To elaborate, consider the following sequence of Sobolev spaces:

$$H^1(\Omega)/\mathbb{R} \xrightarrow{\mathbf{grad}} H(\mathbf{curl}\,,\Omega) \xrightarrow{\mathbf{curl}} H(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega). \tag{1}$$

Moving from right to left, the first space with nontrivial continuity requirements is $H(\mathrm{div},\Omega)$. Finite element subspaces of $H(\mathrm{div},\Omega)$ consist of functions whose normal component is continuous across element interfaces. Hybridization techniques to relax such continuity are well known and they form the foundation upon which our considerations in Sections 3 are based. Sections 4 and 5 build upon them further. Such hybridizations relaxed the continuity of finite element subspaces across interior mesh faces using traces from (just) two elements sharing an interior mesh face. In Section 6, we present a surprising application of this procedure to a discontinuous Galerkin method which allows to obtain exactly divergence-free velocities. Finally, in Section 7, we encounter a mixed formulation that uses the space $H(\mathbf{curl}\,,\Omega)$ further to the left in (1), whose continuity constraints are more complicated. It has finite element subspaces with edge degrees of freedom connected to more than two elements. This necessitates development of new hybridization techniques. Indeed, none of the old hybridization techniques could handle edge or vertex degrees of freedom. Since all previously known hybridization techniques relaxed continuity across mesh faces, we find a widespread belief that methods using edge and vertex degrees of freedom are not amenable to hybridization. We shall dispel this belief. We conclude in Section 8.

2 A one dimensional example

This section is aimed at readers not conversant with hybridization techniques. We will compare three standard numerical methods, namely the H^1-conforming method, the mixed method, and the hybridized method, for a simple one dimensional example. Our objective is to quickly show how and why hybridization results in a superior method.

Consider the one dimensional boundary value problem of finding $u(x)$ satisfying

$$-\frac{d^2u}{dx^2} = f(x), \qquad \text{for all } x \in (0,1), \tag{2}$$

$$u(0) = u(1) = 0, \tag{3}$$

for some f in $L^2(0,1)$. Perhaps the most natural and elegant numerical method for this problem is the H^1-conforming finite element method obtained from the variational formulation of the problem: Find u in the Sobolev space $H_0^1(0,1)$ satisfying

$$\int_0^1 u'v'\,\mathrm{d}x = \int_0^1 fv\,\mathrm{d}x, \quad \text{for all } v \in H_0^1(0,1),$$

where the primes denote differentiation. This formulation is obtained by multiplying (2) by a test function v and integrating by parts. A finite element approximation $\mathring{u}_h \in \mathring{S}_h$ is defined simply by

requiring that the same equation holds on a finite dimensional subspace $\mathring{S}_h$ of the infinite dimensional space $H^1_0(0,1)$:

$$\int_0^1 \mathring{u}_h' v_h' \, \mathrm{d}x = \int_0^1 f v_h \, \mathrm{d}x, \quad \text{for all } v_h \in \mathring{S}_h. \tag{4}$$

One choice of $\mathring{S}_h$ that gives the well known H^1-conforming finite element method is the space of linear splines based on knots $0 = x_0 < x_1 < \ldots x_{N-1} < x_N = 1$ that vanish at the end points 0 and 1. Many error estimates are known for this method, but perhaps the most spectacular one is

$$\mathring{u}_h(x_i) - u(x_i) = 0, \quad \text{for all } i = 1, 2, \ldots, N-1. \tag{5}$$

In other words, there is superconvergence (of infinite order) at the nodes [40].

In various physical applications, one needs approximations to u as well as the "flux" $q := -du/dx$. While this can be obtained from the H^1-conforming finite element method by numerical differentiation of the computed solution $\mathring{u}_h$, a better alternative is given by the *mixed method*, where fluxes are directly approximated by the method. The mixed method is obtained as follows: First rewrite (2) as the first order system

$$u' + q = 0, \qquad q' = f.$$

Next reformulate it (by multiplying by test functions and integrating by parts) into the variational problem of finding $u \in L^2(0,1)$ and $q \in H^1(0,1)$ satisfying

$$\int_0^1 q\, r \, \mathrm{d}x - \int_0^1 u\, r' \, \mathrm{d}x = 0, \qquad \text{for all } r \in H^1(0,1),$$
$$\int_0^1 q' v \, \mathrm{d}x = \int_0^1 f v \, \mathrm{d}x, \qquad \text{for all } v \in L^2(0,1).$$

Then approximate this formulation by imposing the equations on finite dimensional subspaces of $H^1(0,1)$ and $L^2(0,1)$. Let $S_h \subset H^1_0(0,1)$ be the space of linear splines based on knots $0 = x_0 < x_1 < \ldots x_{N-1} < x_N = 1$ as before, but now without any boundary conditions, and let W_h be the space of functions that are constant in each of the intervals $E_i := (x_i, x_{i+1})$. Then the mixed finite element approximations $q_h \in S_h$ and $u_h \in W_h$ are defined by

$$\int_0^1 q_h\, r \, \mathrm{d}x - \int_0^1 u_h\, r' \, \mathrm{d}x = 0, \qquad \text{for all } r \in S_h, \tag{6}$$
$$\int_0^1 q_h' v \, \mathrm{d}x = \int_0^1 f v \, \mathrm{d}x, \qquad \text{for all } v \in W_h. \tag{7}$$

Clearly the mixed method is not equivalent to the H^1-conforming method since the approximations to u provided by both the methods lie in different spaces. The main attraction of the mixed method is its flux approximation: They are usually more pleasing since they are continuous and they possess the physically important property of being conservative. A typical error estimate for this mixed method is

$$\|u - u_h\|_{L^2(0,1)} + \|q - q_h\|_{L^2(0,1)} \le \mathrm{C}h\|f\|_{L^2(0,1)}, \tag{8}$$

where $h = \max_i |x_{i+1} - x_i|$ and C is a mesh independent constant.

The *hybridized* version of this mixed method is obtained by relaxing the continuity constraints of the space S_h and reimposing them via Lagrange multipliers. Removing the continuity constraints of S_h, we obtain the space $\hat{S}_h$ of functions that are linear on each (x_i, x_{i+1}). We now seek a flux approximation $\hat{q}_h \in \hat{S}_h$ which satisfies the analogue of (7), namely

$$\sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \hat{q}_h' \hat{v} \, \mathrm{d}x = \int_0^1 f \hat{v} \, \mathrm{d}x, \quad \text{for all } \hat{v} \in W_h, \tag{9}$$

but additionally require that the method only select flux solutions with continuity, i.e., the hybridized method has the additional equation

$$[\hat{q}_h] = \mathbf{0}, \tag{10}$$

where $[\hat{q}_h]$ denotes the vector in $\mathbb{R}^{N-1}$ whose i-th component $[\hat{q}_h]_i$ is the jump in the value of $\hat{q}_h$ as we cross the interior node x_i, i.e., $[\hat{q}_h]_i = \lim_{x \downarrow x_i} \hat{q}_h(x) - \lim_{x \uparrow x_i} \hat{q}_h(x)$. Because we have relaxed the continuity constraint, a Lagrange multiplier $\boldsymbol{\lambda}_h \in \mathbb{R}^{N-1}$ appears in (6), which must now be rewritten as follows:

$$\left(\sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \hat{q}_h \, r \, \mathrm{d}x \right) - \left(\sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \hat{u}_h \, r' \, \mathrm{d}x \right) + \boldsymbol{\lambda}_h \cdot [r] = 0 \qquad \text{for all } r \in \hat{S}_h, \tag{11}$$

where "$\cdot$" denotes the Euclidean innerproduct in $\mathbb{R}^{N-1}$. Equations (9), (10), and (11) form the hybridized mixed method.

The last equation and the particular way in which $\boldsymbol{\lambda}_h$ is introduced can be understood in the framework of the well known process of converting constrained minimization problems to unconstrained ones via the introduction of a Lagrange multiplier (see e.g. [10, 20]). To describe this briefly, we first observe that by standard results in convex analysis, the solution of the mixed method (6)–(7) solves the minimization problem

$$\min_{r \in S_h} \mathcal{F}(r),$$

where

$$\mathcal{F}(r) = \sup_{v \in W_h} \left(\frac{1}{2} \int_0^1 |r|^2 \, \mathrm{d}x - \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} v \, r' \, \mathrm{d}x + \int_0^1 f v \, \mathrm{d}x \right).$$

Now since $S_h = \{\hat{r} \in \hat{S}_h : [\hat{r}] = 0\}$, the above minimization can be viewed as a constrained minimization problem with $[\hat{r}] = 0$ as the constraint. Hence, its solution can be obtained by finding the saddle point of the Lagrangian functional

$$\min_{\hat{r} \in \hat{S}_h} \sup_{\boldsymbol{\mu} \in \mathbb{R}^{N-1}} \mathcal{F}(\hat{r}) + \boldsymbol{\mu} \cdot [\hat{r}].$$

Equation (11) (and indeed the two remaining equations of the hybridized method) then arises naturally as the critical point equation of the Lagrangian functional. Thus, the hybridized mixed method (9)–(11) is equivalent to the original mixed method (6)–(7) in the sense that $\hat{q}_h$ and $\hat{u}_h$ coincide with the solution components of the mixed method q_h and u_h, respectively (so we will drop the superscript $\hat{\ }$ from now on).

Now let us concede that we *seem* to have done a terrible thing: Instead of the simple and elegant conforming method (4) with $N-1$ unknowns, we have proposed the hybridized method (9)–(11) with about three times as many unknowns. It does not even appear to compare favorably with the mixed method (6)–(7). These appearances are deceiving. In fact, the hybridized formulation is superior to both the conforming method and the mixed method for the following reasons:

R$_1$. Instead of solving the system (9)–(11) with its $3N$ degrees of freedom simultaneously, it is possible to find $\boldsymbol{\lambda}_h$ by solving one matrix system with $N-1$ unknowns. The remaining variables can immediately be computed from $\boldsymbol{\lambda}_h$ locally.

R$_2$. This matrix system for $\boldsymbol{\lambda}_h$ is sparse, symmetric, and positive definite (while the mixed method in the form (6)–(7) gives an indefinite matrix).

R$_3$. The Lagrange multiplier $\boldsymbol{\lambda}_h$ of the hybridized method contains further information about the solution that is not contained in the mixed method.

We substantiate these reasons in the remainder of this section.

The assertions R_1 and R_2 above follow because it is possible to eliminate q_h and u_h from the equations (9)–(11) and get one equation for the Lagrange multiplier. This is best seen by writing the equations in a matrix or operator form. Define operators $B : S_h \mapsto W_h$ and $C : S_h \mapsto \mathbb{R}^{N-1}$ and the function $F \in W_h$ by

$$(Bq, v) = \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} vq' \, \mathrm{d}x, \qquad Cq = -[q], \qquad (F, v) = (f, v),$$

for all $q \in S_h$ and $v \in W_h$. Here $(\cdot, \cdot)$ denotes the $L^2(0,1)$–innerproduct. Then the equations (9)–(11) take the following form:

$$\begin{pmatrix} I & B^t & C^t \\ B & 0 & 0 \\ C & 0 & 0 \end{pmatrix} \begin{pmatrix} q_h \\ u_h \\ \boldsymbol{\lambda}_h \end{pmatrix} = \begin{pmatrix} 0 \\ F \\ 0 \end{pmatrix}, \tag{12}$$

where I denotes the identity map (on S_h), and B^t and C^t denotes the adjoints of B and C with respect to the $L^2(0,1)$ and Euclidean innerproducts, respectively. Our perspective on hybridized methods is based on a characterization of the $\boldsymbol{\lambda}_h$–component of such systems.

It is convenient to now formulate this result more generally. Let S, W, and M be arbitrary finite dimensional spaces with innerproducts $(\cdot, \cdot)_S$, $(\cdot, \cdot)_W$, and $(\cdot, \cdot)_M$, respectively.

Assumption 2.1 *Assume that*

1. $\mathcal{A} : S \mapsto S$ *is a symmetric and positive definite operator,*
2. $\mathcal{B} : S \mapsto W$ *is surjective, and*
3. $\mathcal{D} : W \mapsto W$ *is a symmetric negative semidefinite operator.*

Consider the following general system, which includes (12) as a particular case:

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}^t & \mathcal{C}^t \\ \mathcal{B} & \mathcal{D} & 0 \\ \mathcal{C} & 0 & 0 \end{pmatrix} \begin{pmatrix} p \\ v \\ \lambda \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}. \tag{13}$$

Here the superscript t indicates the adjoint with respect to the innerproducts under consideration. We can formulate a result for such systems using the maps $\mathfrak{Q}$, $\mathfrak{U}$, Q , U , $\mathcal{Q}$, and $\mathcal{U}$, defined by

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}^t \\ \mathcal{B} & \mathcal{D} \end{pmatrix} \begin{pmatrix} \mathfrak{Q}\alpha \\ \mathfrak{U}\alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \ \begin{pmatrix} \mathcal{A} & \mathcal{B}^t \\ \mathcal{B} & \mathcal{D} \end{pmatrix} \begin{pmatrix} \mathsf{Q}\beta \\ \mathsf{U}\beta \end{pmatrix} = \begin{pmatrix} 0 \\ \beta \end{pmatrix}, \ \begin{pmatrix} \mathcal{Q}\lambda \\ \mathcal{U}\lambda \end{pmatrix} = - \begin{pmatrix} \mathfrak{Q}(\mathcal{C}^t\lambda) \\ \mathfrak{U}(\mathcal{C}^t\lambda) \end{pmatrix}. \tag{14}$$

We have the following theorem, whose proof is an exercise in linear algebra – see [14, Appendix A].

Theorem 2.1 [14, Appendix A] *Suppose Assumption* 2.1 *holds. Then (13) holds for some* $(p, v, \lambda) \in S \times W \times M$ *and* $(\alpha, \beta, \gamma) \in S \times W \times M$*, if and only if*

$$p = \mathfrak{Q}\alpha \, + \mathsf{Q}\beta \, + \mathcal{Q}\lambda \,, \tag{15}$$

$$v = \mathfrak{U}\alpha \, + \mathsf{U}\beta \, + \mathcal{U}\lambda \,, \tag{16}$$

and λ *satisfies*

$$a(\lambda, \mu) = b(\mu),$$

for all $\mu \in M$*, where*

$$a(\lambda, \mu) = (\mathcal{A}\mathcal{Q}\lambda \,, \mathcal{Q}\mu \,)_S - (\mathcal{D}\,\mathcal{U}\lambda \,, \, \mathcal{U}\mu \,)_S,$$
$$b(\mu) = -(\alpha, \mathcal{Q}\mu \,)_S - (\beta, \, \mathcal{U}\mu \,)_W - (\gamma, \mu)_M.$$

We apply this result to our system (12) with $S = S_h$, $W = W_h$ and $M = \mathbb{R}^{N-1}$ (where S_h and W_h are endowed with the L^2-innerproduct and M with the Euclidean innerproduct). First observe that $\mathcal{Q}\boldsymbol{\mu}$ and $\mathcal{U}\boldsymbol{\mu}$ for any $\boldsymbol{\mu} \in \mathbb{R}^{N-1}$ can be computed locally by solving the following mixed problem on each element $E_i \equiv (x_i, x_{i+1})$ independently:

$$\int_{E_i} (\mathcal{Q}\boldsymbol{\mu})\, r \, \mathrm{d}x - \int_{E_i} (\mathcal{U}\boldsymbol{\mu})\, r' \, \mathrm{d}x = -\boldsymbol{\mu} \cdot [r], \tag{17}$$

$$\int_{E_i} v\, (\mathcal{Q}\boldsymbol{\mu})' \, \mathrm{d}x = 0. \tag{18}$$

This follows immediately from the definition (14). Since the conditions of Theorem 2.1 are obviously satisfied in our case, it follows that the $\boldsymbol{\lambda}_h \in \mathbb{R}^{N-1}$ that solves (12) is the unique solution of the variational equation

$$\int_0^1 \mathcal{Q}\boldsymbol{\lambda}_h \cdot \mathcal{Q}\boldsymbol{\mu} \; \mathrm{d}x = \int_0^1 f \; \mathcal{U}\boldsymbol{\mu} \; \mathrm{d}x, \quad \text{for all } \boldsymbol{\mu} \in \mathbb{R}^{N-1}. \tag{19}$$

Let $\boldsymbol{\mu}^{(i)} \in \mathbb{R}^{N-1}$ denote the vector whose i-th component is one and all other components are zero, so $\{\boldsymbol{\mu}^{(i)}\}$ forms the standard basis for $\mathbb{R}^{N-1}$. Using this basis, we can transform the variational equation (19) into a matrix system: If A is the $(N-1) \times (N-1)$ matrix defined by $\mathsf{A}_{ij} = (\mathcal{Q}\boldsymbol{\mu}^{(j)}, \mathcal{Q}\boldsymbol{\mu}^{(i)})$, then (19) implies that

$$\mathsf{A}\,\boldsymbol{\lambda}_h = \mathsf{b} \tag{20}$$

where b is the vector whose i-th component is $(f, \mathcal{U}\boldsymbol{\mu}^{(i)})$. Clearly, since the kernel of the operator $\mathcal{Q} : M_h \mapsto V_h$ is the zero function, A is a symmetric and positive definite matrix. It is also sparse because the maps $\mathcal{Q}$ and $\mathcal{U}$ are local. Moreover, once $\boldsymbol{\lambda}_h$ is computed, the other unknowns q_h and u_h can be computed element by element using the formulae (15) and (16) of Theorem 2.1 (because the remaining maps in these formulae are also local). Hence we have substantiated the statements we labeled R_1 and R_2 previously.

In order to explain the third reason R_3 for preferring the hybridized method, we study the variational equation (19) a bit more. An elementary computation using (17) and (18) shows that for the standard basis $\{\boldsymbol{\mu}^{(i)}\}$, we have

$$\mathcal{Q}(\boldsymbol{\mu}^{(i)}) = \begin{cases} 1/h_i & \text{on } E_i, \\ -1/h_{i-1} & \text{on } E_{i-1}, \\ 0 & \text{elsewhere,} \end{cases} \qquad \mathcal{U}(\boldsymbol{\mu}^{(i)}) = \begin{cases} 1/2 & \text{on } E_i \cup E_{i+1}, \\ 0 & \text{elsewhere,} \end{cases}$$

where $h_i = |x_{i+1} - x_i|$. Observe that $\mathcal{Q}(\boldsymbol{\mu}^{(i)}) = \phi_i'$ where ϕ_i is the nodal basis for the H^1-conforming finite element space $\mathring{S}_h$, i.e., $\phi_i \in \mathring{S}_h$ is the function that is one at x_i and zero at all other nodes. This implies that the matrix A for the Lagrange multiplier system is identical to the matrix of the H^1-conforming method. Moreover, if f is constant on each element E_i, then the i-th component of b in (20), namely $(f, \mathcal{U}\boldsymbol{\mu}^{(i)})$, also coincides with the analogous term in the matrix equation for the H^1-conforming method, namely (f, ϕ_i). Thus, when $f \in W_h$, the *H^1-conforming method is "contained" in the hybridized method*. This shows that the Lagrange multiplier $\boldsymbol{\lambda}_h$ truly contains more information not present in the non-hybridized form of the mixed method. Indeed, when $f \in W_h$, the Lagrange multiplier at the node x_i coincides with the H^1-conforming solution $\mathring{u}_h$ there, and hence by virtue of the superconvergence result (5), it coincides with the exact solution at x_i.

Comparing (5) and (8), it may appear at first sight that the mixed method, notwithstanding its ability to give nice flux approximations, gives inferior approximations to the primal variable u. But as we have shown, the simple process of hybridization corrects this deficiency. One way to utilize the new information provided by the Lagrange multiplier is to construct local postprocessing techniques that give approximations to u that are better than u_h. E.g., in the case $f \in W_h$, if $\widetilde{u}_h$ is the function in $\mathring{S}_h$ whose values at the interior nodes x_i equal the corresponding values of $\boldsymbol{\lambda}_h$, then

$$\|\widetilde{u}_h - u\|_{L^2(0,1)} \le \mathrm{C}h^2 |u|_{H^2(0,1)},$$

since $\widetilde{u}_h$ is the linear interpolant of u. Thus, mere linear interpolation of the Lagrange multiplier values gives an approximate solution which is one order higher than u_h in accuracy. It is possible to increase the accuracy further by using more sophisticated postprocessing techniques.

3 The two dimensional Dirichlet problem

In this section we will see that much of what we saw for the one dimensional example in Section 2 continues to hold in two dimensions. But there are important differences which we shall highlight. By means of Theorem 2.1, we will compare a few existing methods and reveal old and new connections between them. The basic hybridization technique for the Dirichlet problem was first exploited in [2], wherein the Lagrange multipliers were utilized for postprocessing to get higher order solutions. We shall not discuss this and other important ideas such as hybridization techniques for certain finite element methods in elasticity, which are eminently presented in [2].

The generalized Dirichlet problem is to solve for u satisfying $\operatorname{div}(-\boldsymbol{a}(\boldsymbol{x})\,\mathbf{grad}\,u) = f$, together with a Dirichlet boundary condition. Writing the partial differential equation as a first order system, we consider the following problem of finding the vector flux function $\boldsymbol{q}(\boldsymbol{x}) := -\boldsymbol{a}(\boldsymbol{x})\,\mathbf{grad}\,u$ and the primal variable $u(\boldsymbol{x})$ satisfying

$$\boldsymbol{c}(\boldsymbol{x})\boldsymbol{q} + \mathbf{grad}\,u = 0, \quad \text{on } \Omega, \tag{21}$$

$$\operatorname{div}\,\boldsymbol{q} = f \quad \text{on } \Omega, \tag{22}$$

$$u = g \quad \text{on } \partial\Omega. \tag{23}$$

Here $\boldsymbol{c}(\boldsymbol{x}) = \boldsymbol{a}^{-1}(\boldsymbol{x})$, Ω is a polygonal plane domain, f is in $L^2(\Omega)$, g is in $H^{1/2}(\partial\Omega)$, and $\boldsymbol{a}(\boldsymbol{x})$ is a symmetric matrix function that is uniformly positive definite on Ω with components in $L^\infty(\Omega)$.

Mixed methods and the hybridized mixed methods for this problem can be derived as in the one dimensional case. They are motivated by the following well known variational formulation of (21)–(23), obtained as in the one dimensional case, by multiplying (21) and (22) by test functions ($\boldsymbol{r}$ and v) and integrating by parts: Find $u \in L^2(\Omega)$ and $\boldsymbol{q} \in H(\operatorname{div},\Omega)$ satisfying

$$(\boldsymbol{c}\,\boldsymbol{q},\boldsymbol{r})_\Omega - (u,\operatorname{div}\boldsymbol{r})_\Omega = (g,\boldsymbol{r}\cdot\boldsymbol{n})_{\partial\Omega} \qquad \text{for all } \boldsymbol{r} \in H(\operatorname{div},\Omega),$$

$$(v,\operatorname{div}\boldsymbol{q})_\Omega = (f,v)_\Omega \qquad \text{for all } v \in L^2(\Omega).$$

Here and elsewhere $(\cdot,\cdot)_D$ denotes the $L^2(D)$-innerproduct (or its analogue for vector functions with components in $L^2(D)$) and $\boldsymbol{n}$ denotes the outward unit normal of the domain under consideration. A classical mixed method for this problem is the Raviart-Thomas (RT) mixed method [33]. Mesh Ω by a triangulation $\mathcal{T}$ satisfying the usual finite element assumptions [7, 12]. Then the RT method defines approximate solutions $u_h \in W_h$ and $\boldsymbol{q}_h \in \bar{V}_h$ satisfying

$$(\boldsymbol{c}\,\boldsymbol{q}_h,\boldsymbol{r}_h)_\Omega - (u_h,\operatorname{div}\boldsymbol{r}_h)_\Omega = (g,\boldsymbol{r}_h\cdot\boldsymbol{n})_{\partial\Omega} \qquad \text{for all } \boldsymbol{r}_h \in \bar{V}_h, \tag{24}$$

$$(v_h,\operatorname{div}\boldsymbol{q}_h)_\Omega = (f,v_h)_\Omega \qquad \text{for all } v_h \in W_h, \tag{25}$$

where the finite element spaces $\bar{V}_h \subset H(\operatorname{div},\Omega)$ and $W_h \subset L^2(\Omega)$ are defined as follows: Let $P_k(K)$ denote the set of all polynomials of degree at most k on an element K. Then the RT space on an element K, namely $R_k(K)$, is the space of all vector polynomials of the form $\boldsymbol{x}p_k(\boldsymbol{x}) + \boldsymbol{p}_k(\boldsymbol{x})$ for some $p_k \in P_k(K)$ and $\boldsymbol{p}_k \in P_k(K)\times P_k(K)$, or in short,

$$R_k(K) = \boldsymbol{x}P_k(K) \oplus P_k(K)^2.$$

Now set

$$V_h = \{\boldsymbol{r} : \boldsymbol{r}|_K \in R_k(K)\}, \qquad \bar{V}_h = V_h \cap H(\operatorname{div},\Omega), \qquad W_h = \{v : v|_K \in P_k(K)\}.$$

It is well known [31, 33] that the requirement that functions in $\bar{V}_h$ be in $H(\operatorname{div},\Omega)$ translates into the continuity of the normal component of its functions, so an equivalent characterization of $\bar{V}_h$ is

$$\bar{V}_h = \{\boldsymbol{r} \in V_h : [\![\boldsymbol{r}\cdot\boldsymbol{n}]\!] = 0\},$$

where the jump $[\![\cdot]\!]$ is defined similarly as in the one dimensional case on element interfaces, which are now mesh edges: Let $\mathcal{E}$ denote the set of all edges of the mesh. For any mesh edge $e \in \mathcal{E}$ shared by two triangles K_e^+ and K_e^- with outward unit normals $\boldsymbol{n}_e^+$ and $\boldsymbol{n}_e^-$, respectively (see Figure 1), setting $\boldsymbol{r}_e^{\pm}(\boldsymbol{x}) = \lim_{\epsilon \downarrow 0} \boldsymbol{r}(\boldsymbol{x} - \epsilon\, \boldsymbol{n}_e^{\pm})$, we define

$$[\![\boldsymbol{r} \cdot \boldsymbol{n}]\!]_e = \boldsymbol{r}_e^+ \cdot \boldsymbol{n}_e^+ + \boldsymbol{r}_e^- \cdot \boldsymbol{n}_e^-,$$

while for boundary edges $e \subset \partial\Omega$ we define $[\![\boldsymbol{r} \cdot \boldsymbol{n}]\!]_e = \boldsymbol{r} \cdot \boldsymbol{n}$. By $[\![\boldsymbol{v} \cdot \boldsymbol{n}]\!]$ (without subscripts) we mean the function that is defined on the union of all the mesh edges and equals $[\![\boldsymbol{v} \cdot \boldsymbol{n}]\!]_e$ on each mesh edge e. This completes the definition of the RT method.

Other standard methods for the Dirichlet problem include the two dimensional H^1-conforming method and the P_1-nonconforming method, both of which are based on an alternate variational formulation: Find $u \in H_0^1(\Omega)$ satisfying

$$(\boldsymbol{a}\, \mathbf{grad}\, u, \mathbf{grad}\, v)_\Omega = (f, v)_\Omega \quad \text{for all } v \in H_0^1(\Omega),$$

where we have considered the case $g = 0$ for simplicity. The H^1-conforming method is developed using finite element subspaces of $H_0^1(\Omega)$ as in Section 2. The P_1-nonconforming method defines an approximation $\hat{u}_h$ in the space

$$\hat{W}_h = \{v : v|_K \in P_1(K) \text{ and } v \text{ is continuous at the midpoints of each edge } e \in \mathcal{E}\}$$

satisfying

$$\sum_{K \in \mathcal{T}} (\boldsymbol{a}\, \mathbf{grad}\, \hat{u}_h, \mathbf{grad}\, \hat{v}_h)_K = (f, \hat{v}_h)_\Omega \quad \text{for all } \hat{v}_h \in \hat{W}_h. \tag{26}$$

The RT mixed method is usually preferred over these methods whenever there is a need for good flux approximations.

Let us now consider the hybridized form of the mixed method (24)–(25). As per the hybridization paradigm, we relax the continuity constraints of the space $\bar{V}_h$ (to get V_h), but require that the flux solution $\boldsymbol{q}_h$ satisfy normal continuity by means of the additional equation $[\![\boldsymbol{q} \cdot \boldsymbol{n}]\!] = 0$, or equivalently,

$$([\![\boldsymbol{q}_h \cdot \boldsymbol{n}]\!], \mu)_e = 0 \quad \text{for all } e \in \mathcal{E}_0$$

and for all μ in the space $M_h = \{ [\![\boldsymbol{r} \cdot \boldsymbol{n}]\!] : \boldsymbol{r} \in V_h\}$. Here $\mathcal{E}_0$ denotes the set of interior mesh edges. From the properties of the RT spaces it is easy to prove that

$$M_h = \{\mu : \mu|_e \in P_k(e) \text{ for all edges } e \in \mathcal{E}_0\}.$$

Thus the hybridized method defines an approximate solution triple $(\boldsymbol{q}_h, u_h, \lambda_h) \in V_h \times W_h \times M_h$ satisfying

$$(\boldsymbol{c}\, \boldsymbol{q}_h, \boldsymbol{r})_\Omega - \sum_{K \in \mathcal{T}} (u_h, \operatorname{div} \boldsymbol{r})_K + \sum_{e \in \mathcal{E}_0} (\lambda_h,\, [\![\boldsymbol{r} \cdot \boldsymbol{n}]\!])_e = -(g, \boldsymbol{r} \cdot \boldsymbol{n})_{\partial\Omega}, \tag{27}$$

$$\sum_{K \in \mathcal{T}} (w, \operatorname{div} \boldsymbol{q}_h)_K = (f, w)_\Omega, \tag{28}$$

$$\sum_{e \in \mathcal{E}_0} (\mu,\, [\![\boldsymbol{q}_h \cdot \boldsymbol{n}]\!])_e = 0, \tag{29}$$

for all $\boldsymbol{r} \in V_h$, $w \in W_h$ and $\mu \in M_h$. It is well known [2] that the above equations uniquely define the solution triple $(\boldsymbol{q}_h, u_h, \lambda_h)$ and that $\boldsymbol{q}_h$ and u_h so obtained solves the mixed method equations (24)–(25).

As in the one dimensional case, while it may appear at first sight that hybridization has resulted in a method with too many unknowns, this is not a valid criticism. It is possible to eliminate both $\boldsymbol{q}_h$ and u_h and obtain a single equation for λ_h. This is best seen by applying Theorem 2.1 again.

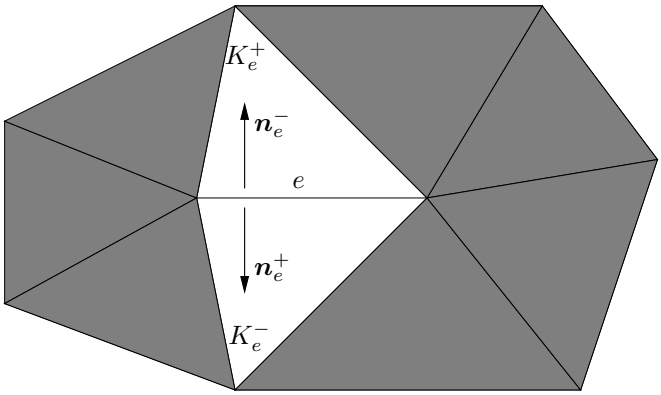

Fig. 1 Support of the liftings from an edge e

Equations (27)–(29) can be written in the form (13) with the obvious definitions of the operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, spaces, and innerproducts. Then Theorem 2.1 implies that λ_h is the unique function in M_h satisfying

$$a(\lambda_h, \mu) = (f, \mathcal{U}\boldsymbol{\mu})_\Omega + (g, \boldsymbol{n}\cdot\boldsymbol{\Omega}\mu)_{\partial\Omega} \quad \text{for all } \mu \in M_h, \tag{30}$$

where $a(\lambda,\mu) = (\boldsymbol{\Omega}\lambda, \boldsymbol{\Omega}\mu)_\Omega$. As in the one dimensional case, the *liftings* $\boldsymbol{\Omega}\mu$ and $\mathcal{U}\mu$ can be computed locally by solving mixed problems element by element: The restriction of $(\boldsymbol{\Omega}\mu, \mathcal{U}\mu)$ to each element K can be obtained independently as the only element of $R_k(K)\times P_k(K)$ that solves the following mixed problem on K

$$(\boldsymbol{c}\boldsymbol{\Omega}\mu, \boldsymbol{r})_K - (\mathcal{U}\mu, \operatorname{div}\boldsymbol{r})_K = -(\mu, \boldsymbol{r}\cdot\boldsymbol{n})_{\partial K\setminus\partial\Omega}, \qquad \text{for all } \boldsymbol{r}\in R_k(K), \tag{31}$$

$$(w, \operatorname{div}\boldsymbol{\Omega}\mu)_K = 0, \qquad \text{for all } w \in P_k(K). \tag{32}$$

Theorem 2.1 also asserts that once λ_h is found from (30), the other solution components $\boldsymbol{q}_h$ and u_h follow by purely local computations, as described by the formulae (15)–(16).

The advantages of hybridization in the two dimensional case are already evident now. Instead of solving for $\boldsymbol{q}_h$ and u_h using the mixed method (24)–(25), which will result in an indefinite system, we can instead solve the symmetric and positive definite system (30) and then recover $\boldsymbol{q}_h$ and u_h locally using λ_h. Moreover, Equation (30) obviously results in a smaller system than either (24)–(25) or (27)–(29). It is possible to easily conclude various other properties of the Lagrange multiplier equation (30) from the properties of the bilinear form: For instance, if we employ a nodal basis for M_h consisting of functions that are supported on an edge, e.g., properly scaled Legendre polynomials of degree at most k on each edge, then each edge can "interact" with *at most four* other edges on *any* mesh (see Figure 1), so the matrix equation resulting from (30) is sparse, with a precisely characterizable sparsity structure: The matrix is a matrix of $(k+1)\times(k+1)$ blocks with at most four off-diagonal blocks. Finally, we note that it is possible to use the standard "finite element technology" in implementing (30): One can assemble the stiffness matrix of the bilinear form $a(\cdot,\cdot)$ with respect to a local basis for M_h using local element stiffness matrices on a reference element and then mapping back using the Piola map. These and other details are discussed at length in [13].

In the one dimensional example of Section 2, we saw that the Lagrange multiplier contains additional information not contained in the mixed method. Let us now investigate if this is true in the two dimensional case as well. Consider the lowest order case $k=0$. Let $e\in\mathcal{E}_0$ be an edge shared by two elements K_e^+ and K_e^- of the mesh $\mathcal{T}_h$. Let χ_e be the characteristic function of the edge e. Obviously χ_e for all edges $e\in\mathcal{E}_0$ forms a basis for M_h. From (31) and (32), one can easily calculate the liftings explicitly:

$$\boldsymbol{\Omega}(\chi_e) = \begin{cases} -\dfrac{|e|}{|K_\pm|}\bar{\boldsymbol{c}}_\pm^{-1}\boldsymbol{n}_e^\pm & \text{on } K_e^\pm, \\ 0 & \text{elsewhere,} \end{cases} \qquad \mathcal{U}(\chi_e) = \begin{cases} \dfrac{|e|(\boldsymbol{m}_e - \boldsymbol{b}_K^\pm)\cdot\boldsymbol{n}_e^\pm}{2|K_\pm|} & \text{on } K_e^\pm, \\ 0 & \text{elsewhere,} \end{cases} \tag{33}$$

where $|\cdot|$ denotes measure (length or area, as appropriate), $\bar{\boldsymbol{c}}_\pm = |K_\pm|^{-1}\int_{K_\pm} \boldsymbol{c}\,\mathrm{d}x$, $\boldsymbol{m}_e$ is the midpoint of the edge e, and $\boldsymbol{b}_K^\pm = \bar{\boldsymbol{c}}^{-1}|K_e^\pm|^{-1}\int_{K_e^\pm} \boldsymbol{c}\,\boldsymbol{x}\,\mathrm{d}x$. The notations are illustrated in Figure 1. From (33), it now immediately follows that whenever $\boldsymbol{c}(\boldsymbol{x})$ is a scalar constant on each triangle of the mesh, we have

$$\boldsymbol{\Omega}(\chi_e) = -\boldsymbol{c}_\pm^{-1}\,\mathbf{grad}\,\psi_e \qquad \text{on } K_e^\pm,$$

where ψ_e is the nodal basis function of the edge e in the P_1-nonconforming method, i.e, ψ_e is the unique function that is linear on K_+ and K_-, is one at the midpoint of edge e, and is zero at the midpoint of all other edges of $\mathcal{E}$. All these considerations, of course, hold for any other edge $\ell \in \mathcal{E}$ as well. Hence,

$$a_h(\chi_e,\chi_\ell) = (\boldsymbol{c}\,\boldsymbol{\Omega}(\chi_e),\boldsymbol{\Omega}(\chi_\ell))_\Omega = \sum_{K\in\mathcal{T}} (\boldsymbol{a}\,\mathbf{grad}\,\psi_e,\mathbf{grad}\,\psi_\ell)_K$$

Thus we find that the bilinear form determining the multiplier λ_h is equal to the bilinear form of the P_1-nonconforming method (26). Moreover, if f is constant on each mesh element and $g \equiv 0$, then it is easily verified using the latter half of (33) that the right hand side of (26) and (30) coincide. Thus the P_1-nonconforming solution $\hat{u}_h$ given by (26) and the Lagrange multiplier λ_h of the hybridized method coincide at the midpoints of all interior mesh edges.

Recapitulating, in contrast to the one dimensional case where the H^1-conforming method was contained in the mixed method (see Section 2), in the two dimensional case we see that the P_1-*nonconforming method is "contained" in the hybridized method* whenever $k = 0$, $f \in W_h$, $g \equiv 0$, and $\boldsymbol{c}(\boldsymbol{x})$ is a scalar constant on each mesh element. Since the P_1-nonconforming method is not equivalent to the mixed method, we find that the Lagrange multiplier truly contains new information not contained in the mixed method. This fact is essentially contained (but derived by different arguments) in [2] and exploited in many papers including [6, 29]. It is possible, as in the one dimensional example, to use this new information in λ_h to locally postprocess the computed u_h to obtain a new approximation of one higher order – see [2].

We have now seen twice that Theorem 2.1 is a useful tool in comparing hybridized methods to other methods. We shall now give one more comparison of two classical methods leading to a previously unknown result. The hybridized version of the Brezzi-Douglas-Marini (BDM) mixed method [8] is another classical method that uses the same variational equations as the hybridized RT method, namely (27)–(29), but with V_h and W_h replaced by the BDM finite element spaces

$$V_h^{\mathrm{BDM}} = \{\boldsymbol{r} : \boldsymbol{r}|_K \in P_k(K)^2 \text{ for all mesh elements } K \in \mathcal{T}\},$$
$$W_h^{\mathrm{BDM}} = \{w : w|_K \in P_{k-1}(K), \text{ for all mesh elements } K \in \mathcal{T}\},$$

respectively. The Lagrange multiplier spaces are identical for both the methods. The BDM and RT methods were independently developed [8, 33] and error estimates for both methods are well known. Nevertheless, we shall now establish that there is an intimate relationship between these methods. In comparing RT and BDM methods, for the sake of readability, we shall superscript the notations previously introduced in connection with the Raviart-Thomas method by "RT". When superscripted by "BDM", such notations are to be understood as defined exactly as before except that the RT spaces are replaced by the BDM spaces, e.g., the solutions of the RT and BDM methods are denoted by $(\boldsymbol{q}_h^{\mathrm{BDM}}, u_h^{\mathrm{BDM}}, \lambda_h^{\mathrm{BDM}})$ and $(\boldsymbol{q}_h^{\mathrm{RT}}, u_h^{\mathrm{RT}}, \lambda_h^{\mathrm{RT}})$, respectively. Let Π_k^W denote the $L^2(\Omega)$-orthogonal projection into W_h (the space of functions which are polynomials of degree k on each mesh element $K \in \mathcal{T}$). Then we have the following theorem:

Theorem 3.1 [13] *If $(\Pi_k^W - \Pi_{k-1}^W)f = 0$ then the Lagrange multipliers of the RT and BDM hybridized mixed methods coincide:* $\lambda_h^{\mathrm{RT}} = \lambda_h^{\mathrm{BDM}}$.

P r o o f. Let us sketch the main argument of the proof. Theorem 2.1 applies to both the hybridized RT and the BDM method. Hence their Lagrange multipliers are characterized by

$$a_h^{\mathrm{RT}}(\lambda_h^{\mathrm{RT}},\mu) = b_h^{\mathrm{RT}}(\mu), \quad \text{for all } \mu \in M_h, \tag{34}$$
$$a_h^{\mathrm{BDM}}(\lambda_h^{\mathrm{BDM}},\mu) = b_h^{\mathrm{BDM}}(\mu), \quad \text{for all } \mu \in M_h. \tag{35}$$

where bilinear forms are as defined in Theorem 2.1 with the liftings specific to each method. Now observe that the flux liftings $\mathbf{\Omega}^{\mathrm{RT}}\mu$ and $\mathbf{\Omega}^{\mathrm{BDM}}\mu$ of both methods are divergence free on each element – see (32). It is elementary to show using the properties of polynomials that the divergence free subspaces of the RT and BDM spaces on an element coincide i.e.,

$$R_k^0(K) := \{\boldsymbol{r} \in R_k(K) : \operatorname{div} \boldsymbol{r} = 0\} = \{\boldsymbol{p} \in P_k(K)^2 : \operatorname{div} \boldsymbol{p} = 0\}. \tag{36}$$

Hence both the liftings satisfy the same equations (cf (31)):

$$\begin{aligned}(\boldsymbol{c}\,\mathbf{\Omega}^{\mathrm{RT}}\lambda\,,\boldsymbol{r})_K &= -(\lambda, \boldsymbol{r}\cdot\boldsymbol{n})_{\partial K}, \quad \text{for all } \boldsymbol{r} \in R_k^0(K),\\ (\boldsymbol{c}\,\mathbf{\Omega}^{\mathrm{BDM}}\lambda\,,\boldsymbol{r})_K &= -(\lambda, \boldsymbol{r}\cdot\boldsymbol{n})_{\partial K}, \quad \text{for all } \boldsymbol{r} \in R_k^0(K),\end{aligned}$$

for all mesh elements K. Therefore $\mathbf{\Omega}^{\mathrm{RT}}\mu = \mathbf{\Omega}^{\mathrm{BDM}}\mu$. Thus, the left hand sides of (34) and (35) coincide.

Finally, if $(\Pi_k^W - \Pi_{k-1}^W)f = 0$, then one can show the linear forms on the right hand sides of (34) and (35) also coincide [13]. Consequently, the Lagrange multipliers of both the methods coincide. □

Notwithstanding the fact that RT and BDM methods have been studied for many decades, the coincidence of their Lagrange multipliers seem to have gone unnoticed, even numerically. The coincidence occurs, e.g., when f is a polynomial of degree $k-1$ on every element of the mesh, and in several applications of practical interest, e.g., incompressible flow in porous media, where $f = 0$. Note that the coincidence of Lagrange multipliers imply that the flux and primal solution components of the methods are also related, since it is possible to compute $\boldsymbol{q}_h$ and u_h of both methods locally once λ_h is known. Thus, one may implement the relatively inexpensive BDM method and recover the higher order RT solution u_h locally. Another interesting point to note concerns preconditioning: During the proof of Theorem 3.1 above, we actually showed that the stiffness matrices of $a_h^{\mathrm{RT}}(\cdot,\cdot)$ and $a_h^{\mathrm{BDM}}(\cdot,\cdot)$ are always the same (without any condition on f). Hence any preconditioner for the Lagrange multiplier equation of the RT method is also a preconditioner for the BDM method and vice versa.

The characterization of Lagrange multipliers via our variational approach also facilitates the design and analysis of preconditioners for mixed methods. Most of the standard preconditioners [4, 5, 6, 28, 34] for the RT and BDM methods are for matrices arising from the non-hybridized form of the method. The only two earlier works on preconditioning the hybridized form of the mixed method that we are aware of are [18, 35]. But these papers considered systems that couple the Lagrange multiplier unknowns together with the primal variable. We have already established that best strategy for implementing the hybridized mixed method is by eliminating the flux as well as the primal variable thereby obtaining one system for the Lagrange multiplier. Thus what is of interest in practice is a preconditioner for this Lagrange multiplier equation. It is possible to show that the matrix arising from (30) via a local basis has spectral condition number that grows like $O(h^{-2})$ for quasiuniform meshes of mesh size h [24]. In this respect, this equation has no advantages over (but is no worse than) other local discretizations for the Dirichlet problem. Nonetheless, in view of our variational characterization, it is possible to design good preconditioners for this equation by studying the spectral properties of the mesh dependent bilinear form $a(\cdot,\cdot)$. A pertinent result in this direction is the following norm equivalence theorem which asserts that $a(\cdot,\cdot)^{1/2}$ is equivalent to a more transparent norm $|\!|\!|\cdot|\!|\!|_h$ defined by

$$|\!|\!|\lambda|\!|\!|_h^2 = \sum_{K\in\mathcal{T}} \|\lambda - m_K(\lambda)\|_{L^2(\partial K)}^2 \frac{1}{|\partial K|}, \qquad \text{where} \quad m_K(\lambda) = \frac{1}{|\partial K|}\int_{\partial K} \lambda \,\mathrm{d}s.$$

Theorem 3.2 [24] *There are positive constants* C_1 *and* C_2 *(depending on* $\boldsymbol{c}(\boldsymbol{x})$ *and the minimal angle of mesh elements, but independent of* h*) such that*

$$\mathrm{C}_1|\!|\!|\lambda|\!|\!|_h^2 \le a(\lambda,\lambda) \le \mathrm{C}_2|\!|\!|\lambda|\!|\!|_h^2, \quad \textit{for all } \lambda \in M_h,$$

This theorem is proved in [24], where we use it to also analyze a Schwarz overlapping domain decomposition preconditioner, which when used to precondition (30), eliminates the asymptotic growth in condition number.

4 Variable degree methods via hybridization

In this section we will show how hybridization facilitates the construction and implementation of variable degree mixed methods. In order to successfully solve practical problems with singularities, it is well known that one must use different polynomial degrees for finite elements in different regions of the computational domain. Such variable degree spaces are obvious to construct in the case of H^1-conforming methods – one just needs to implement continuity constraints on interfaces of two mesh elements with non-equal polynomial spaces. However, they are not so immediate for mixed methods. The main difficulty in the construction and analysis of variable degree versions of mixed methods is in ensuring that the variation of the polynomial degree does not destroy the delicate stability of the methods as manifested in the inf-sup condition.

Consider the problem of constructing a variable degree analogue of (24)–(25). We endow each mesh element K with the RT pair of spaces of index $k(K)$ (recall that the RT space of index ℓ is $R_\ell(K) = \boldsymbol{x}P_\ell(K) \oplus \ P_\ell(K)^N$ in N space dimensions) and replace $\bar{V}_h$ and W_h in (24)–(25) by their variable degree analogues $\bar{V}_\mathcal{T}$ and $W_\mathcal{T}$, respectively, defined as follows:

$$V_\mathcal{T} = \{\boldsymbol{r} : \boldsymbol{r}|_K \in R_{k(K)}(K)\}, \quad \bar{V}_\mathcal{T} = V_\mathcal{T} \cap H(\mathrm{div}, \Omega) \tag{37}$$

$$W_\mathcal{T} = \{w : w|_K \in P_{k(K)}(K)\}. \tag{38}$$

An immediate question that arises is if the new method is well defined, i.e., if the new equations admit a unique solution. The traditional approach to settle such questions relied on using a projector with a commuting diagram property. In the early paper of [9], one finds a variable degree extension of their original BDM method using the so-called "transition elements". But they could obtain the required projector only under the assumption that the maximum variation of the polynomial degrees of two adjacent elements be one. This restriction was removed recently in [19] where projectors for BDM type spaces with arbitrarily varying polynomial degrees are constructed. No variable degree RT method for triangles with arbitrarily varying polynomial degrees exists in the literature.

We take a different approach to the construction of variable degree methods via hybridization, which allows us to construct a variable degree RT method with ease. To formulate a variable degree version of (27)–(29), we simply use a Lagrange multiplier to impose the required continuity constraints. In other words, we replace V_h, W_h, and M_h in (27)–(29) by $V_\mathcal{T}$, $W_\mathcal{T}$ and

$$M_\mathcal{T} = \{\mu : \mu|_e \in P_{\overline{k}(e)}(e) \text{ for all mesh edges } e\},$$

respectively, where $\overline{k}(e) = \max(k(K_e^+), k(K_e^-))$. Where we differ from the previous works is in choosing the Lagrange multiplier to have the maximum of the two degrees from the adjacent elements (unlike the minimum – see e.g. [9]). This results in a new hybridized method for which the cardinal question now is whether it admits a unique solution. This question is easily answered in the case when the Lagrange multiplier is set to have the minimal degree from either sides. In our case however, the Lagrange multiplier has more degrees of freedom since it has the maximal degree, so the answer is not immediate. Note that despite the increase in degrees of freedom, our method with the maximal degree Lagrange multiplier has many attractive features: We do not need to implement "transition elements" or the "minimum degree rule". It suffices to maintain a set of shape functions for the full RT space on a reference element. Moreover, minimal degree multipliers do not give sufficiently high order of convergence for the postprocessing techniques that utilize the extra information in the multipliers to construct higher order approximations.

We will now show that the new variable degree RT method does admit a unique solution. We shall be more general than in the previous sections and consider the following Dirichlet problem in an N-dimensional polyhedral domain Ω ($N \geq 2$):

$$\boldsymbol{q} + \boldsymbol{a}(\boldsymbol{x})\ \mathbf{grad}\, u = 0, \quad \text{on } \Omega, \tag{39}$$

$$\mathrm{div}\ \boldsymbol{q} + d(\boldsymbol{x})\, u = f \quad \text{on } \Omega, \tag{40}$$

$$u = g \quad \text{on } \partial\Omega. \tag{41}$$

Note that now we have added a lower order term $d(\boldsymbol{x})u(\boldsymbol{x})$ to the differential equation. We assume that $d(\boldsymbol{x})$ is a non-negative function in $L^\infty(\Omega)$. Now the domain is meshed by a simplicial finite element

mesh $\mathcal{T}$. Let $\mathcal{E}$ denotes the set of all mesh faces (of dimension $N-1$). As before, functions in $M_{\mathcal{T}}$ are defined to be zero on boundary faces.

Theorem 4.1 [14] *There exists a unique solution* $(\boldsymbol{q}_{\mathcal{T}}, u_{\mathcal{T}}, \lambda_{\mathcal{T}}) \in V_{\mathcal{T}} \times W_{\mathcal{T}} \times M_{\mathcal{T}}$ *satisfying the following equations of the variable degree RT method for every* $(\boldsymbol{v}, w, \mu) \in V_{\mathcal{T}} \times W_{\mathcal{T}} \times M_{\mathcal{T}}$*:*

$$(\boldsymbol{c}\,\boldsymbol{q}_{\mathcal{T}}, \boldsymbol{v})_\Omega - \sum_{K\in\mathcal{T}} (u_{\mathcal{T}}, \operatorname{div} \boldsymbol{v})_K + \sum_{e\in\mathcal{E}} (\lambda_{\mathcal{T}},\, [\![\boldsymbol{v}\cdot\boldsymbol{n}]\!])_e = -(g,\, [\![\boldsymbol{v}\cdot\boldsymbol{n}]\!])_{\partial\Omega} \tag{42}$$

$$\sum_{K\in\mathcal{T}} (w, \operatorname{div} \boldsymbol{q}_{\mathcal{T}})_K + (d\,u_{\mathcal{T}},\, w)_\Omega = (f, w)_\Omega \tag{43}$$

$$\sum_{e\in\mathcal{E}} (\mu,\, [\![\boldsymbol{q}_{\mathcal{T}}\cdot\boldsymbol{n}]\!])_e = 0. \tag{44}$$

Finally, we note that as in the uniform degree case, we can apply the characterization of Lagrange multipliers, namely Theorem 2.1, to the method (42)–(44) to obtain a single variational equation for $\lambda_{\mathcal{T}}$. After solving it, $\boldsymbol{q}_{\mathcal{T}}$ and $u_{\mathcal{T}}$ can be recovered element by element as in the uniform degree case, but now using the RT spaces of the appropriate degree on each element. Details are in [14].

5 Error analysis

In this section we exploit hybridization as a theoretical tool for error analysis. The traditional method of error analysis uses the inf-sup condition to get error estimates for the flux and primal variables and then uses them to obtain error estimates for the Lagrange multiplier. We proceed in the reverse order. As we shall see, our approach will lead to new error estimates for the old methods as well as the new variable degree RT method introduced in Section 4.

Consider the problem of finding *a priori* error estimates for the variable degree method defined by (42)–(44) for the N-dimensional Dirichlet problem (39)–(41), letting $d(\boldsymbol{x}) \equiv 0$ for simplicity. The traditional approach would be as follows: (i) First, prove an inf-sup condition for this method. (ii) Second, conclude estimates for $\|\boldsymbol{q}-\boldsymbol{q}_{\mathcal{T}}\|_{H(\operatorname{div},\Omega)}$ and $\|u-u_{\mathcal{T}}\|_{L^2(\Omega)}$ from the Babuška-Brezzi theory. (iii) Third, prove superconvergence estimates for $\|\Pi^W u - u_{\mathcal{T}}\|_{L^2(\Omega)}$ where Π^W denotes the $L^2(\Omega)$-orthogonal projection into $W_{\mathcal{T}}$. (iv) Finally, use the superconvergence estimate and flux error estimate to prove an error estimate for the Lagrange multiplier $\lambda_{\mathcal{T}}$. In the last step one typically proves, e.g., in two dimensions [2], that for every edge e of a mesh triangle K,

$$\|\lambda_{\mathcal{T}} - \Pi^M u\|_{L^2(e)} \le \mathrm{C}_*\big(h_K^{1/2}\|\boldsymbol{q}-\boldsymbol{q}_{\mathcal{T}}\|_{L^2(K)} + h_K^{-1/2}\|\Pi^W u - u_{\mathcal{T}}\|_{L^2(K)}\big), \tag{45}$$

so the superconvergence estimate from the previous step is essential to obtain optimal order estimates for $\lambda_{\mathcal{T}}$. Here and elsewhere Π^M denotes the L^2-projection into $M_{\mathcal{T}}$ defined by $(\Pi^M v, \eta)_e = (v,\eta)_e$ for all $\eta \in M_{\mathcal{T}}$ and for all $e \in \mathcal{E}$, h_K denotes $\operatorname{diam}(K)$, and C (with or without subscripts) denotes a generic constant independent of mesh sizes.

It is difficult, if not impossible, to modify the above mentioned last step of the traditional analyses to obtain optimal *hp*-type estimates for the Lagrange multiplier, because inequalities like (45) are derived using finite dimensional arguments [2, 8] with constants (e.g., C_* above) that are hard to trace. Other annoyances resulting from the above type of analysis include the fact that one needs at least $H^3(\Omega)$-regularity of u to obtain superconvergence and hence for the Lagrange multiplier estimate [2, Corollary 1.5], [8, Lemma 4.1]. Such stringent regularity requirements creates difficulties when attempting duality arguments and in applications such as analysis of multigrid methods. To avoid these problems, we fundamentally rethink the error analysis.

Again, our new approach is motivated by the characterization of the Lagrange multiplier as given by Theorem 2.1. It is easy to check that the theorem applies to our variable degree RT method of Section 4. Denote by $a_{\mathcal{T}}(\cdot,\cdot)$ and $b_{\mathcal{T}}(\cdot)$ the forms given by Theorem 2.1 for this case, e.g., $a_{\mathcal{T}}(\lambda,\mu) = (\boldsymbol{c}\,\boldsymbol{\mathcal{Q}}\lambda\,, \boldsymbol{\mathcal{Q}}\mu\,)_\Omega$ where now the $\boldsymbol{\mathcal{Q}}$-lifting is to be taken into the variable degree elements. Then $\lambda_{\mathcal{T}}$ is the only element of $M_{\mathcal{T}}$ satisfying

$$a_{\mathcal{T}}(\lambda_{\mathcal{T}}, \mu) = b_{\mathcal{T}}(\mu) \qquad \text{for all } \mu \in M_{\mathcal{T}}.$$

This result shapes our approach to error estimation: Because of it, it is reasonable to expect an error estimate in the "energy" norm $a_{\mathcal{T}}(\mu,\mu)^{1/2}$. But it turns out that the "consistency error"

$$\tau(\mu) := a_{\mathcal{T}}(u,\mu) - b_{\mathcal{T}}(\mu),$$

where u is the exact solution of (39)–(41), is nonzero, in general. This and the mesh dependent nature of $a_{\mathcal{T}}(\cdot,\cdot)$ makes the error analysis reminiscent of finite element methods with variational crimes – see e.g., [7, 12]. Where we depart from the usual analyses of such consistency errors is in showing that the above $\tau(\mu)$ admits a particularly simple bound in terms of a projection satisfying a commuting diagram property.

Let Π^V denote any projector into $\bar{V}_{\mathcal{T}} := V_{\mathcal{T}} \cap H(\mathrm{div},\Omega)$ for which the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\ \mathrm{div}\ } & L^2(\Omega) \\ {\scriptstyle \Pi^V}\big\downarrow & & \big\downarrow{\scriptstyle \Pi^W} \\ \bar{V}_{\mathcal{T}} & \xrightarrow{\ \mathrm{div}\ } & W_{\mathcal{T}}, \end{array} \tag{46}$$

Here the domain of definition of Π^V, namely V, is a subspace of $H(\mathrm{div},\Omega)$ (usually of slightly smoother functions). The existence of such a projector, while well known in the uniform degree case, is not trivial to establish for the variable degree spaces, but is known [14, Appendix B],[19]. We tacitly assume that the exact flux $\boldsymbol{q}$ is in V so we can apply Π^V to it. Define

$$\|\boldsymbol{r}\|_{\boldsymbol{c}} = \left(\int_\Omega \boldsymbol{c}\,\boldsymbol{r}\cdot\boldsymbol{r}\,\mathrm{d}x \right)^{1/2} \quad \text{and} \quad \|\mu\|_a = a_{\mathcal{T}}(\mu,\mu)^{1/2}.$$

The following theorem exemplifies our approach to error analysis. It holds without any uniformity assumptions on the mesh and contains no unknown constants.

Theorem 5.1 [14] *For any $\mu \in M_{\mathcal{T}}$,*

$$a_{\mathcal{T}}(\lambda_{\mathcal{T}} - \Pi^M u, \mu) = \int_\Omega \boldsymbol{c}\,(\Pi^V\boldsymbol{q} - \boldsymbol{q})\cdot\boldsymbol{\Omega}\mu\ \mathrm{d}x. \tag{47}$$

Consequently,

$$|\!|\!| \lambda_{\mathcal{T}} - \Pi^M u |\!|\!|_a \ \le\ \|\boldsymbol{q} - \Pi^V\boldsymbol{q}\|_{\boldsymbol{c}}. \tag{48}$$

P r o o f. Let us first write the variable degree RT method (42)–(44) in the operator form as

$$\begin{pmatrix} A & B^t & C^t \\ B & 0 & 0 \\ C & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{q}_{\mathcal{T}} \\ u_{\mathcal{T}} \\ \lambda_{\mathcal{T}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{G} \\ F \\ 0 \end{pmatrix} \tag{49}$$

with the obvious definitions of the operators and the right hand side functions, e.g., $\boldsymbol{G} \in V_{\mathcal{T}}$ is defined by $(\boldsymbol{G},\boldsymbol{r})_\Omega = -(g,\,[\![\boldsymbol{r}\cdot\boldsymbol{n}]\!])_{\partial\Omega}$. Next, observe that (39) and (40) imply that the exact solution u and flux $\boldsymbol{q}$ satisfy

$$\begin{aligned} (\boldsymbol{c}\,\boldsymbol{q},\boldsymbol{r})_\Omega - \sum_{K\in\mathcal{T}} (u, \mathrm{div}\,\boldsymbol{r})_K + \sum_{e\in\mathcal{E}_0} (u,\,[\![\boldsymbol{r}\cdot\boldsymbol{n}]\!])_e = -(g,\,[\![\boldsymbol{r}\cdot\boldsymbol{n}]\!])_{\partial\Omega}, \qquad & \text{for all } \boldsymbol{r}\in V_{\mathcal{T}}, \\ \sum_{K\in\mathcal{T}} (w, \mathrm{div}\,\boldsymbol{q})_K = (f,w)_\Omega, \qquad & \text{for all } w \in W_{\mathcal{T}}. \end{aligned}$$

Here $\mathcal{E}_0$ denotes the collection of interior mesh faces. By the commuting diagram (46), $\Pi^W \mathrm{div}\,\boldsymbol{q} = \mathrm{div}\,\Pi^V\boldsymbol{q}$. Using this in the second equation above and rewriting in operator notation we have

$$\begin{pmatrix} A & B^t & C^t \\ B & 0 & 0 \\ C & 0 & 0 \end{pmatrix} \begin{pmatrix} \Pi^V\boldsymbol{q} \\ \Pi^W u \\ \Pi^M u \end{pmatrix} = \begin{pmatrix} \boldsymbol{G}+\boldsymbol{\delta} \\ F \\ 0 \end{pmatrix}, \tag{50}$$

where $\boldsymbol{\delta}$ is the L^2-projection of $\boldsymbol{c}(\Pi^V \boldsymbol{q} - \boldsymbol{q})$ into $V_{\mathcal{T}}$. Hence, defining the discrete error functions $e_{\boldsymbol{q}} = \Pi^V \boldsymbol{q} - \boldsymbol{q}_{\mathcal{T}}$, $e_u = \Pi^W u - u_{\mathcal{T}}$ and $e_\lambda = \Pi^M u - \lambda_{\mathcal{T}}$, and subtracting (49) from (50) we find that

$$\begin{pmatrix} A & B^t & C^t \\ B & 0 & 0 \\ C & 0 & 0 \end{pmatrix} \begin{pmatrix} e_{\boldsymbol{q}} \\ e_u \\ e_\lambda \end{pmatrix} = \begin{pmatrix} \boldsymbol{\delta} \\ 0 \\ 0 \end{pmatrix}, \tag{51}$$

Now the proof is immediately completed: Equation (47) follows from an application of Theorem 2.1 to (51). Finally, (48) is obtained from (47) by an application of Cauchy-Schwarz inequality. □

The error estimate of Theorem 5.1 is truly different from the previously known estimates. E.g., the previous difficulty of having to assume $H^3(\Omega)$-regularity is no longer present now. If the components of the exact flux $\boldsymbol{q} = -\boldsymbol{a}\,\mathbf{grad}\,u$ are in $H^1(\Omega)$, the mesh $\mathcal{T}$ is a quasi-uniform mesh of mesh-size h, and if Π^V preserves constants, then

$$|\!|\!| \lambda_{\mathcal{T}} - \Pi^M u |\!|\!|_a \leq \mathrm{C} h |\boldsymbol{q}|_{H^1(\Omega)}. \tag{52}$$

The nature of the norm on the left hand side above may not appear transparent at the first sight. But, analogous to Theorem 3.2, we have a norm equivalence in the variable degree case as well. Therefore, it is possible to obtain error estimates with more transparent integral norms (without the liftings) in place of the norm on the left hand side of (52). It is also possible to develop an Aubin-Nitsche type duality argument to obtain error estimates for the Lagrange multiplier in weaker L^2-like norms. Finally, it is possible to use Theorem 5.1 to prove error estimates for the remaining variables. The key again lies in the error equation (51). In proving Theorem 5.1 we only used a part of Theorem 2.1 applied to (51). By using the characterization of the remaining error components in (51) in terms of the Lagrange multiplier as stated in Theorem 2.1, we can obtain error estimates for the flux and primal variables from our error estimate for the Lagrange multiplier. Details are in [14].

6 Stokes equations: Discontinuous Galerkin method

This section is devoted to describing the ideas proposed in [11] for hybridizing a discontinuous Galerkin method using divergence-free approximate velocities for the two dimensional Stokes problem

$$\begin{aligned} -\boldsymbol{\Delta}\boldsymbol{u} + \mathbf{grad}\, p &= \boldsymbol{f}, && \text{on } \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0, && \text{on } \Omega, \\ \boldsymbol{u} &= \boldsymbol{g}, && \text{on } \partial\Omega. \end{aligned}$$

Here we assume that Ω is a bounded connected domain with polygonal boundary $\partial\Omega$, the data $\boldsymbol{f}$ is in $L^2(\Omega)^2$ and $\boldsymbol{g} \in H^{1/2}(\partial\Omega)^2$. As usual, we require the data $\boldsymbol{g}$ to satisfy

$$(g_n, 1)_{\partial\Omega} = 0,$$

where $g_n = \boldsymbol{g} \cdot \boldsymbol{n}$ and $\boldsymbol{n}$ is the outward unit normal on $\partial\Omega$.

6.1 A discontinuous Galerkin method for the vorticity-velocity formulation

To introduce the local discontinuous Galerkin (LDG) method under consideration, we begin by rewriting the Stokes system as

$$\omega - \operatorname{curl} \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{53}$$

$$\mathbf{curl}\, \omega + \mathbf{grad}\, p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{54}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{55}$$

$$\boldsymbol{u}_{\mathsf{T}} = \boldsymbol{g}_{\mathsf{T}}, \qquad \text{on } \partial\Omega, \tag{56}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = g_n, \qquad \text{on } \partial\Omega, \tag{57}$$

where ω is the scalar vorticity, $\omega = \operatorname{curl} \boldsymbol{u} = \partial_1 u_2 - \partial_2 u_1$, and $\mathbf{curl}\, \omega$ is the vector-valued curl given by $\mathbf{curl}\, \omega = (\partial_2 \omega, -\partial_1 \omega)$. Here, $\boldsymbol{g}_{\mathsf{T}} := \boldsymbol{g} - (\boldsymbol{g} \cdot \boldsymbol{n})\boldsymbol{n}$ denotes the tangential component of $\boldsymbol{g}$.

Next, we multiply the equations (53) and (54) by test functions (σ and $\boldsymbol{v}$) and then integrate over the triangle K. After integrating by parts, we get

$$
\begin{aligned}
&(\omega, \sigma)_K - (\boldsymbol{u}, \mathbf{curl}\, \sigma)_K - (\boldsymbol{u}, \sigma \times \boldsymbol{n})_{\partial K} = 0, \\
&(\omega, \operatorname{curl} \boldsymbol{v})_K + (\omega, \boldsymbol{v} \times \boldsymbol{n})_{\partial K} = (\boldsymbol{f}, \boldsymbol{v})_K .
\end{aligned}
$$

Here, we continue to denote by $\boldsymbol{n}$ the unit outward normal on ∂K since no confusion can arise, and we have made use of the identities $\boldsymbol{a} \times \boldsymbol{b} = a_1 b_2 - a_2 b_1, c \times \boldsymbol{a} = c\,(-a_2, a_1)$, and $\boldsymbol{a} \times c = -c \times \boldsymbol{a}$. We have also used a velocity test function $\boldsymbol{v}$ which is divergence-free and has zero normal component on the boundary of Ω. This is why the pressure p does not appear in the equations.

Before adding on the triangles K, we need to introduce some notation. On every interior edge $e \in \mathcal{E}_0$ shared by two triangles K_e^+ and K_e^- we define

$$
[\![\boldsymbol{n} \times \boldsymbol{v}]\!]_e = \boldsymbol{n}_e^+ \times \boldsymbol{v}_e^+ + \boldsymbol{n}_e^- \times \boldsymbol{v}_e^- , \tag{58}
$$

where $\boldsymbol{n}_e^+$ and $\boldsymbol{n}_e^-$ denote the outward unit normals on the boundaries of K_e^+ and K_e^- and $\boldsymbol{v}_e^{\pm}(\boldsymbol{x}) = \lim_{\epsilon \downarrow 0} \boldsymbol{v}(\boldsymbol{x} - \epsilon\, \boldsymbol{n}_e^{\pm})$. On edges $e \subset \partial\Omega$, we set $[\![\boldsymbol{n} \times \boldsymbol{v}]\!]_e = \boldsymbol{n} \times \boldsymbol{v}$. By $[\![\boldsymbol{n} \times \boldsymbol{v}]\!]$ (without subscripts) we mean the function that is defined on the union of all the edges and equals $[\![\boldsymbol{n} \times \boldsymbol{v}]\!]_e$ on each face $e \in \mathcal{E}$. (The jumps of scalar functions are denoted by the same notation, i.e., $[\![\boldsymbol{n} \times \sigma]\!]$ is defined similarly.)

Thus, adding on the triangles, and incorporating the equation (57), we get

$$
\begin{aligned}
&\sum_{K \in \mathcal{T}} (\omega, \sigma)_K - \sum_{K \in \mathcal{T}} (\boldsymbol{u}, \mathbf{curl}\, \sigma)_K - \sum_{e \in \mathcal{E}} (\boldsymbol{u},\, [\![\sigma \times \boldsymbol{n}]\!])_e = 0, \\
&\sum_{K \in \mathcal{T}} (\omega, \operatorname{curl} \boldsymbol{v})_K + \sum_{e \in \mathcal{E}} (\omega,\, [\![\boldsymbol{v} \times \boldsymbol{n}]\!])_e = (\boldsymbol{f}, \boldsymbol{v})_\Omega, \\
&(\boldsymbol{u} \cdot \boldsymbol{n}, q)_{\partial\Omega} = (g_n, q)_{\partial\Omega} .
\end{aligned}
$$

Here we have used the continuity of the tangential components of $\boldsymbol{u}$ and ω. Notice that only the information about the *tangential* component of the Dirichlet boundary condition $\boldsymbol{g}$ appears in the first equation whereas the information about its *normal* component is contained in the third equation. Notice also that the last equation is trivially satisfied when q is a constant. Indeed, we have

$$
(\boldsymbol{u} \cdot \boldsymbol{n}, 1)_{\partial\Omega} = (\operatorname{div} \boldsymbol{u}, 1)_\Omega = (g_n, 1)_{\partial\Omega} = 0.
$$

We now define the LDG approximation $(\omega_h, \boldsymbol{u}_h)$ in the finite dimensional space $W_h \times \mathcal{V}_h(g_n)$ by requiring that

$$
\sum_{K \in \mathcal{T}} (\omega_h, \sigma)_K - \sum_{K \in \mathcal{T}} (\boldsymbol{u}_h, \mathbf{curl}\, \sigma)_K - \sum_{e \in \mathcal{E}} (\widehat{\boldsymbol{u}}_h,\, [\![\sigma \times \boldsymbol{n}]\!])_e = 0 \qquad \text{for all } \sigma \in W_h, \tag{59}
$$

$$
\sum_{K \in \mathcal{T}} (\omega_h, \operatorname{curl} \boldsymbol{v})_K + \sum_{e \in \mathcal{E}} (\widehat{\omega}_h,\, [\![\boldsymbol{v} \times \boldsymbol{n}]\!])_e = (\boldsymbol{f}, \boldsymbol{v})_\Omega \qquad \text{for all } \boldsymbol{v} \in \mathcal{V}_h(0), \tag{60}
$$

$$
\sum_{e \in \mathcal{E}, e \subset \partial\Omega} (\boldsymbol{u}_h \cdot \boldsymbol{n}, q)_e = (g_n, q)_{\partial\Omega} \qquad \text{for all } q \in \mathcal{Q}_h/\mathbb{R}, \tag{61}
$$

where

$$
\begin{aligned}
W_h &= \{\, \sigma \in L^2(\Omega) :\, \sigma|_K \in P_{k-1}(K),\ K \in \mathcal{T} \,\}, \\
\mathcal{V}_h(b) &= \{\, \boldsymbol{v} \in H(\operatorname{div}, \Omega) : \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega, \boldsymbol{v} \cdot \boldsymbol{n} = b \text{ on } \partial\Omega,\ \boldsymbol{v}|_K \in P_k(K)^2,\ K \in \mathcal{T} \,\}, \\
\mathcal{Q}_h &= \{\, q \in L^2(\partial\Omega) :\ \ q|_e \in P_k(e), \text{ for all edges } e \subset \partial\Omega, \,\}.
\end{aligned}
$$

The quantities $\widehat{\boldsymbol{u}}_h$ and $\widehat{\omega}_h$ are approximations to the traces of $\boldsymbol{u}_h$ and ω_h which we define next. On interior edges, these discrete traces are chosen as

$$
\widehat{\omega}_h = \{\!\{ \omega_h \}\!\} + \mathsf{E} \cdot [\![\omega_h \times \boldsymbol{n}]\!] + \mathsf{D}\, [\![\boldsymbol{u}_h \times \boldsymbol{n}]\!], \qquad \widehat{\boldsymbol{u}}_h = \{\!\{ \boldsymbol{u}_h \}\!\} + \mathsf{E}\, [\![\boldsymbol{u}_h \times \boldsymbol{n}]\!]. \tag{62}
$$

Here $\{\!\{\varphi\}\!\} = \frac{1}{2}(\varphi^+ + \varphi^-)$ where $\varphi^\pm$ are the traces of the function φ. Similarly, on boundary edges, we take

$$\widehat{\omega}_h = \omega_h + \mathsf{D}\,(\boldsymbol{u}_h \cdot \boldsymbol{n} - g_n), \qquad \widehat{\boldsymbol{u}}_h = \boldsymbol{g}. \tag{63}$$

Here, D and $\mathbf{E}$ are functions defined on $\mathcal{E}$ and $\mathcal{E}_0$, respectively. Note that since the numerical flux $\widehat{\boldsymbol{u}}_h$ enters the equations only through the last term of the equation (59), it captures the boundary condition (56). Indeed, on $\partial\Omega$ we have

$$\widehat{\boldsymbol{u}}_h \cdot \sigma \times \boldsymbol{n} = \boldsymbol{g} \cdot \sigma \times \boldsymbol{n} = \boldsymbol{g}_{\mathsf{T}} \cdot \sigma \times \boldsymbol{n}.$$

This completes the definition of the LDG method.

Notice that the vorticity ω_h does not belong to a $H(\mathbf{curl}\,)$-conforming finite dimensional space – it does not have any continuity constraints across element interfaces. Thanks to this, and to the structure of the LDG method, it can be easily eliminated from the equations since, by equations (59), (62) and (63), it can be expressed in terms of the velocity in each element. However, the velocity space $\mathcal{V}_h$ does consist of functions with normal continuity constraints. The next step is to eliminate these constraints via hybridization.

6.2 The hybridized LDG method

Now we hybridize the above introduced LDG method to base the approximation of the velocities on the space of locally divergence-free functions given by

$$V_h = \{\, \boldsymbol{v} \in L^2(\Omega)^2 : \, \boldsymbol{v}|_K \in J_k(K),\ K \in \mathcal{T}_h \,\},$$

where $J_k(K) = \{\, \boldsymbol{v} \in P_k(K)^2 : \, \operatorname{div} \boldsymbol{v} = 0 \text{ on } K \,\}$. Although this space is bigger than $\mathcal{V}_h(g_n)$, its functions do not have to satisfy any continuity constraint across inter-element boundaries. We therefore anticipate the pressure to reappear in the equations once we use this space. The resulting approximate pressure will be taken in the space $P_h/\mathbb{R}$ where

$$P_h = \{\, q \in L^2(\mathcal{E}) : \, q|_e \in P_k(e),\ e \in \mathcal{E} \,\}.$$

Notice that since the exact pressure is defined up to a constant, the same should hold for its approximation. This is why we consider the space $P_h/\mathbb{R}$ and not simply P_h.

Thus, we define the LDG approximation $(\omega_h, \boldsymbol{u}_h, p_h) \in W_h \times V_h \times P_h/\mathbb{R}$ by requiring that

$$\sum_{K\in\mathcal{T}} (\omega_h, \sigma)_K - \sum_{K\in\mathcal{T}} (\boldsymbol{u}_h, \mathbf{curl}\,\sigma)_K - \sum_{e\in\mathcal{E}} (\widehat{\boldsymbol{u}}_h,\, [\![\sigma \times \boldsymbol{n}]\!])_e = 0 \tag{64}$$

$$\sum_{K\in\mathcal{T}} (\omega_h, \operatorname{curl} \boldsymbol{v})_e + \sum_{e\in\mathcal{E}} (\widehat{\omega}_h,\, [\![\boldsymbol{v} \times \boldsymbol{n}]\!])_e + \sum_{e\in\mathcal{E}} (\, [\![\boldsymbol{v} \cdot \boldsymbol{n}]\!], p_h)_e = (\boldsymbol{f}, \boldsymbol{v})_\Omega, \tag{65}$$

$$\sum_{e\in\mathcal{E}} (\, [\![\boldsymbol{u}_h \cdot \boldsymbol{n}]\!], q)_e = (g_n, q)_{\partial\Omega} \tag{66}$$

for all $(\sigma, \boldsymbol{v}, q) \in W_h \times V_h \times P_h/\mathbb{R}$. As we pointed out above, the pressure now reappears as p_h in equation (65) because $[\![\boldsymbol{v} \cdot \boldsymbol{n}]\!]$ is not necessarily equal to zero. We can compute a pressure p_h defined on the triangle K as the element of $P_{k-1}(K)$ such that

$$-(\mathsf{p}_h, \operatorname{div} \mathbf{v})_K = (\boldsymbol{f}, \mathbf{v})_K - (\omega_h, \operatorname{curl} \mathbf{v})_K - (\mathbf{v} \cdot \boldsymbol{n}, p_h)_{\partial K} - (\widehat{\omega}_h, \mathbf{v} \times \boldsymbol{n})_{\partial K} \tag{67}$$

for all $\mathbf{v}$ in $P_k^2(K)$. This defines uniquely the pressure p_h because $\operatorname{div} : P_k(K)^2 \mapsto P_{k-1}(K)$ is a surjection, and because, if $\operatorname{div} \mathbf{v} = 0$ for a $\mathbf{v} \in P_k(K)^2$, the right-hand side of the above equation is identically equal to zero by the equation (65). This idea of recovering pressure approximations *a posteriori* from approximations of other variables is not new (see e.g. [25]), but because hybridization provides p_h, we are able to compute p_h in an element-by-element fashion.

Thus our method can simultaneously provide approximations to the *velocity, vorticity,* and *pressure.* Moreover, these approximations converge in an optimal way as we see in the next result.

Theorem 6.1 [11] *Set* $\mathsf{E} = \mathbf{0}$ *and* $\mathsf{D}|_e = |e|$. *Let* $(\omega, \boldsymbol{u}, p)$ *be the exact solution of the Stokes system, and let* $(\omega_h, \boldsymbol{u}_h, p_h) \in W_h \times V_h \times P_h/\mathbb{R}$, *where* $k \geq 1$, *be the approximation given by the hybridized* LDG *method. Let* p *be the exact pressure of the Stokes system and* $\mathsf{p}_h \in W_h$ *is the approximation given by the post-processing method* (67). *Then, for* $s \geq 1$,

$$\|\omega - \omega_h\|_0 + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,h} + \|p - \mathsf{p}_h\|_{L^2(\Omega)/\mathbb{R}} \leq C\, h^{\min\{k,s\}} \left[\|\boldsymbol{u}\|_{s+1} + \|p\|_s\right],$$

where the constant C *is independent of the mesh size and the exact solution. Moreover,*

$$\operatorname{div} \boldsymbol{u}_h = 0,$$

in the $L^2(\Omega)$*-sense.*

7 Stokes equations: Hybridized mixed method

In this section we review the new hybridized method for the Stokes system [15, 16] in three dimensions. Just like the hybridized LDG method in the previous section, this method simultaneously yields an exactly divergence free numerical approximation of the fluid velocity and a discontinuous approximation of the pressure. Moreover, it provides an $H(\mathbf{curl})$-conforming numerical approximation of the vorticity. These three approximations are obtained in an element-by-element fashion after one global system for certain Lagrange multipliers arising from the hybridization is solved. This global system represents a new "tangential velocity–pressure" discretization of the Stokes system on the mesh faces.

7.1 The classical mixed method

The three dimensional Stokes problem is to find a fluid velocity field $\boldsymbol{u}$ and pressure p satisfying

$$-\boldsymbol{\Delta}\boldsymbol{u} + \mathbf{grad}\, p = \boldsymbol{f}, \qquad \text{on } \Omega, \tag{68}$$

$$\operatorname{div} \boldsymbol{u} = 0, \qquad \text{on } \Omega, \tag{69}$$

$$\boldsymbol{u} = \boldsymbol{g}, \qquad \text{on } \partial\Omega. \tag{70}$$

Here we assume that Ω is a bounded connected domain with polyhedral boundary $\partial\Omega$ such that Ω lies only on one side of $\partial\Omega$ locally, the data $\boldsymbol{f}$ is in $L^2(\Omega)^3$ and $\boldsymbol{g} \in H^{1/2}(\partial\Omega)^3$. We do not assume that Ω is simply connected. We also do not assume that $\partial\Omega$ is connected. We require the data $\boldsymbol{g}$ to satisfy

$$(g_n, 1)_{\partial\Omega} = 0,$$

where $g_n = \boldsymbol{g} \cdot \boldsymbol{n}$ and $\boldsymbol{n}$ is the outward unit normal on $\partial\Omega$. Under this assumption, it is well known that the Stokes problem has a unique solution.

Let us reformulate the Stokes problem by introducing vorticity $\boldsymbol{\omega} = \mathbf{curl}\, \boldsymbol{u}$. Using the identity

$$-\boldsymbol{\Delta}\boldsymbol{u} = \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{u} - \mathbf{grad}\operatorname{div} \boldsymbol{u},$$

the Stokes system (68)–(70) can be rewritten as in (53)–(57), where are the curls are now three dimensional. There is a well known weak problem based on this reformulation. Define $\mathcal{W} = H(\mathbf{curl}\,, \Omega)$, $\mathcal{V} = H(\operatorname{div}, \Omega)$, and

$$\mathcal{V}(b) = \{\boldsymbol{v} \in H(\operatorname{div}, \Omega) : \operatorname{div} \boldsymbol{v} = 0 \text{ and } \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\Omega} = b\}.$$

for any $b \in H^{-1/2}(\partial\Omega)$. Then $(\boldsymbol{\omega}, \boldsymbol{u})$ is the only element of $\mathcal{W} \times \mathcal{V}(g_n)$ satisfying

$$(\boldsymbol{\omega}, \boldsymbol{\tau})_\Omega - (\boldsymbol{u}, \mathbf{curl}\, \boldsymbol{\tau})_\Omega = (\boldsymbol{g}_{\mathsf{T}}, \boldsymbol{\tau})_{\partial\Omega} \qquad \text{for all } \boldsymbol{\tau} \in \mathcal{W}, \tag{71}$$

$$(\boldsymbol{v}, \mathbf{curl}\, \boldsymbol{\omega})_\Omega = (\boldsymbol{f}, \boldsymbol{v})_\Omega \qquad \text{for all } \boldsymbol{v} \in \mathcal{V}(0). \tag{72}$$

Here $(\cdot, \cdot)_\Omega$ denotes the $L^2(\Omega)$ (or $L^2(\Omega)^3$) innerproduct. Note that, as expected, the pressure has disappeared in this mixed formulation.

To approximate this formulation using a variable degree mixed method, we associate to each tetrahedron K, a degree $k(K)$, and the following pair of spaces:

$$W(K) = P_{k(K)}(K)^3 \oplus S_{k(K)+1}(K),$$
$$V(K) = \{\boldsymbol{v} \in P_{k(K)}(K)^3 : \ \mathrm{div}\, \boldsymbol{v} = 0\},$$

where $S_\ell(K)$ is the set of all vector functions $\boldsymbol{p}_\ell(\boldsymbol{x})$ whose components are homogeneous polynomials of degree ℓ satisfying $\boldsymbol{p}_\ell(\boldsymbol{x}) \cdot \boldsymbol{x} = 0$ (so $W(K)$ is the well known Nédélec space [31] on the element K).

Now the approximate solution is sought in the finite element subspaces of the above defined spaces:

$$\mathcal{W}_h = \{w \in \mathcal{W} : w|_K \in W(K) \text{ for all } K \in \mathcal{T}\},$$
$$\mathcal{V}_h = \{\boldsymbol{v} \in \mathcal{V} : \boldsymbol{v}|_K \in V(K) \text{ for all } K \in \mathcal{T}\}.$$

Let $\mathcal{V}_h(b) = \mathcal{V}(b) \cap \mathcal{V}_h$ and $g_{n,h}$ be the $L^2(\partial\Omega)$-orthogonal projection of the boundary data g_n onto the space $\{\boldsymbol{v}_h \cdot \boldsymbol{n}|_{\partial\Omega} : \boldsymbol{v}_h \in \mathcal{V}_h\}$. Then the discrete mixed formulation seeks $(\boldsymbol{\omega}_h, \boldsymbol{u}_h)$ in $\mathcal{W}_h \times \mathcal{V}_h(g_{n,h})$ satisfying

$$(\boldsymbol{\omega}_h, \boldsymbol{\tau})_\Omega - (\boldsymbol{u}_h, \mathbf{curl}\, \boldsymbol{\tau})_\Omega = (\boldsymbol{g}_\mathsf{T}, \boldsymbol{\tau})_{\partial\Omega} \qquad \text{for all } \boldsymbol{\tau} \in \mathcal{W}_h, \tag{73}$$
$$(\boldsymbol{v}, \mathbf{curl}\, \boldsymbol{\omega}_h)_\Omega = (\boldsymbol{f}, \boldsymbol{v})_\Omega \qquad \text{for all } \boldsymbol{v} \in \mathcal{V}_h(0). \tag{74}$$

This mixed discretization, in the uniform degree case, was studied in [22, 32] where the existence of a unique solution is established. Note that in order to implement the method in the above form, we must face the difficult task of constructing bases for the finite dimensional space of globally divergence-free velocities $\mathcal{V}_h(0)$.

7.2 Two hybridizations

We hybridize the method in two steps. The first hybridization is similar to that applied to the LDG method in the previous section; its objective is thus to avoid having to construct finite element bases of exactly divergence-free velocities. The objective of the second hybridization, which is far more involved, is to avoid having to work with $H(\mathbf{curl}\,)$-conforming spaces. This will allow the elimination of the original variables from the equations and the reduction of the unknowns to variables defined only on the faces of the elements.

To carry out the *first hybridization*, we just proceed as in the case of the LDG method. Thus, instead of seeking velocity approximations in the space $\mathcal{V}_h$, we seek them in the space

$$V_h = \{\boldsymbol{v} : \boldsymbol{v}|_K \in V(K) \text{ and } \mathrm{div}(\boldsymbol{v}|_K) = 0 \text{ for all } K \in \mathcal{T}\}.$$

This is the variable degree analogue of the space that appeared in the previous section. We also introduce the space

$$P_h = \{p : p = [\![\boldsymbol{v} \cdot \boldsymbol{n}]\!] \text{ for some } \boldsymbol{v} \in V_h\}, \tag{75}$$

in which we seek the approximate pressure. The method then defines approximations $(\boldsymbol{\omega}_h, \boldsymbol{u}_h, p_h) \in \mathcal{W}_h \times V_h \times P_h$ satisfying

$$(\boldsymbol{\omega}_h, \boldsymbol{\tau}_h)_\Omega - (\boldsymbol{u}_h, \mathbf{curl}\, \boldsymbol{\tau}_h)_\Omega = (\boldsymbol{g}_\mathsf{T}, \boldsymbol{\tau}_h)_{\partial\Omega} \quad \text{for all } \boldsymbol{\tau}_h \in \mathcal{W}_h, \tag{76}$$
$$(\boldsymbol{v}_h, \mathbf{curl}\, \boldsymbol{\omega}_h)_\Omega + \sum_{F \in \mathcal{F}} (p_h, [\![\boldsymbol{v}_h \cdot \boldsymbol{n}]\!])_F = (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega \quad \text{for all } \boldsymbol{v}_h \in V_h, \tag{77}$$
$$\sum_{F \in \mathcal{F}} (q_h, [\![\boldsymbol{u}_h \cdot \boldsymbol{n}]\!])_F = (g_n, q_h)_{\partial\Omega} \quad \text{for all } q_h \in P_h. \tag{78}$$

It can be proved that the above discrete formulation has a unique solution [15].

Let us point out that, just as for the hybridized LDG method, we can easily recover an approximation for the pressure throughout the computational domain, p_h, in an element by element fashion. Indeed, the restriction of $\mathsf{p}_h$to the tetrahedron K is the element of $P_{k(K)}(K)$ such that

$$-(\mathsf{p}_h, \mathrm{div}\, \boldsymbol{v})_K = (\boldsymbol{f}, \boldsymbol{v})_K - (\mathbf{curl}\, \boldsymbol{\omega}_h, \boldsymbol{v})_K - (\boldsymbol{v} \cdot \boldsymbol{n}, p_h)_{\partial K}, \tag{79}$$

for all $\boldsymbol{v}$ in $P_{k(K)}(K)^3+\boldsymbol{x}\,P_{k(K)}(K)$, where $\boldsymbol{n}$ denotes the outward unit normal to K. That (79) uniquely defines p_h follows as for the LDG method.

Let us now carry out the *second hybridization*. We relax the continuity of the approximate vorticity $\boldsymbol{\omega}_h$ across mesh edges in the interior of the domain so that instead of considering approximations in the space $\mathcal{W}_h$, we approximate using the space

$$W_h = \{\boldsymbol{w} : \boldsymbol{w}|_K \in W(K) \text{ for all } K \in \mathcal{T}\}.$$

Multiplying (57) by a test function $\boldsymbol{\tau}$ and integrating by parts,

$$(\boldsymbol{\omega},\boldsymbol{\tau})_K - (\boldsymbol{u},\mathbf{curl}\,\boldsymbol{\tau})_K - (\boldsymbol{u}_{\mathsf{T}},\boldsymbol{n}\times\boldsymbol{\tau})_{\partial K} = 0.$$

Together with (77), we therefore require that the discrete approximations to vorticity and velocity, namely $\boldsymbol{\omega}_h$ and $\boldsymbol{u}_h$, respectively, satisfy

$$\begin{aligned}(\boldsymbol{\omega}_h,\boldsymbol{\tau})_K - (\boldsymbol{u}_h,\mathbf{curl}\,\boldsymbol{\tau})_K - (\boldsymbol{\lambda}_h,\boldsymbol{n}\times\boldsymbol{\tau})_{\partial K} &= 0,\\ (\boldsymbol{v},\mathbf{curl}\,\boldsymbol{\omega}_h)_K + (\boldsymbol{v}\cdot\boldsymbol{n},p_h)_{\partial K} &= (\boldsymbol{f},\boldsymbol{v})_K,\end{aligned}$$

where we now have two Lagrange multipliers $\boldsymbol{\lambda}_h \approx \boldsymbol{u}_{\mathsf{T}}$ and $p_h \approx p$.

The description of the method is completed by adding appropriate continuity conditions for $\boldsymbol{\omega}_h$ and $\boldsymbol{u}_h$ at the element interfaces. Since $\boldsymbol{\omega}_h$ and $\boldsymbol{u}_h$ are to approximate $\boldsymbol{\omega}$ and $\boldsymbol{u}$ in (71)–(72), the functional setting of (71)–(72) clarifies the continuity constraints to be put on $\boldsymbol{\omega}_h$ and $\boldsymbol{u}_h$. Let

$$M_h = \{\boldsymbol{\mu} : \quad \boldsymbol{\mu} = [\![\boldsymbol{n}\times\boldsymbol{\tau}]\!] \text{ for some } \boldsymbol{\tau} \in W_h\}, \tag{80}$$

where $[\![\boldsymbol{n}\times\boldsymbol{\tau}]\!]$ is the three dimensional analogue of (58) defined before, except that now we define $[\![\boldsymbol{n}\times\boldsymbol{\tau}]\!]|_F$ to be zero for all mesh faces F on the boundary $\partial\Omega$. Then we require that

$$\sum_{F\in\mathcal{F}} (\boldsymbol{\mu},\,[\![\boldsymbol{n}\times\boldsymbol{\omega}_h]\!])_F = 0, \quad \text{for all } \boldsymbol{\mu} \in M_h$$

so that the computed vorticity is in $H(\mathbf{curl}\,)$.

Thus we have motivated the following definition of our variable degree hybridized mixed method: Find $(\boldsymbol{\omega}_h,\boldsymbol{u}_h,\boldsymbol{\lambda}_h,p_h) \in W_h\times V_h\times M_h\times P_h$ satisfying

$$(\boldsymbol{\omega}_h,\boldsymbol{\tau}_h)_\Omega - (\boldsymbol{u}_h,\mathbf{curl}\,\boldsymbol{\tau}_h)_\Omega - \sum_{F\in\mathcal{F}} (\boldsymbol{\lambda}_h,\,[\![\boldsymbol{n}\times\boldsymbol{\tau}_h]\!])_F = (\boldsymbol{g}_{\mathsf{T}},\boldsymbol{n}\times\boldsymbol{\tau}_h)_{\partial\Omega}, \tag{81}$$

$$(\boldsymbol{v}_h,\mathbf{curl}\,\boldsymbol{\omega}_h)_\Omega + \sum_{F\in\mathcal{F}} (p_h,\,[\![\boldsymbol{v}_h\cdot\boldsymbol{n}]\!])_F = (\boldsymbol{f},\boldsymbol{v}_h)_\Omega, \tag{82}$$

$$\sum_{F\in\mathcal{F}} (q_h,\,[\![\boldsymbol{u}_h\cdot\boldsymbol{n}]\!])_F = (g_n,q_h)_{\partial\Omega}, \tag{83}$$

$$\sum_{F\in\mathcal{F}} (\boldsymbol{\mu}_h,\,[\![\boldsymbol{n}\times\boldsymbol{\omega}_h]\!])_F = 0, \tag{84}$$

for all $\boldsymbol{\tau}_h \in W_h, \boldsymbol{v}_h \in V_h, q_h \in P_h, \boldsymbol{\mu}_h \in M_h$. Then one can show that [16] there is a unique solution for this variable degree method. Moreover, the hybridized method is equivalent to the mixed method (73)–(74) in the sense that the velocity and vorticity approximations of both methods coincide.

It may appear at this point that our method has too many unknowns. But as we shall see in the next subsection, it is possible to eliminate all but the Lagrange multiplier variables from (81)–(84), thus making our formulation attractive.

7.3 A characterization of the Lagrange multipliers

To eliminate the vorticity as well as the velocity variables from our hybridized mixed method (81)–(84), we proceed as in the case of the Dirichlet problem. The crucial result there that allowed us to arrive

at a system of equations involving the Lagrange multipliers alone was Theorem 2.1. We will now state such a result for the Stokes system.

We define *lifting* maps that map functions defined on element interfaces into functions on Ω: Define $(\boldsymbol{w}(\boldsymbol{\lambda}), \boldsymbol{u}(\boldsymbol{\lambda})) \in W_h \times V_h$ and $(\mathcal{w}(p), \mathbf{u}(p)) \in W_h \times V_h$ element by element as follows:

$$(\boldsymbol{w}(\boldsymbol{\lambda}), \boldsymbol{\tau})_K - (\boldsymbol{u}(\boldsymbol{\lambda}), \mathbf{curl}\, \boldsymbol{\tau})_K = (\boldsymbol{\lambda}, \boldsymbol{n} \times \boldsymbol{\tau})_{\partial K}, \qquad \text{for all } \boldsymbol{\tau} \in W(K), \tag{85}$$

$$(\boldsymbol{v}, \mathbf{curl}\ \boldsymbol{w}(\boldsymbol{\lambda}))_K = 0, \qquad \text{for all } \boldsymbol{v} \in V(K), \tag{86}$$

$$(\mathcal{w}(p), \boldsymbol{\tau})_K - (\mathbf{u}(p), \mathbf{curl}\, \boldsymbol{\tau})_K = 0, \qquad \text{for all } \boldsymbol{\tau} \in W(K), \tag{87}$$

$$(\boldsymbol{v}, \mathbf{curl}\ \mathcal{w}(p))_K = -(p, \boldsymbol{v} \cdot \boldsymbol{n})_{\partial K}, \qquad \text{for all } \boldsymbol{v} \in V(K). \tag{88}$$

In addition, define $(\mathbf{w}(\boldsymbol{f}), \mathbf{u}(\boldsymbol{f}))$ and $(\boldsymbol{w}(\boldsymbol{g}_{\mathsf{T}}), \boldsymbol{u}(\boldsymbol{g}_{\mathsf{T}}))$ in $W_h \times V_h$ by

$$(\mathbf{w}(\boldsymbol{f}), \boldsymbol{\tau})_K - (\mathbf{u}(\boldsymbol{f}), \mathbf{curl}\, \boldsymbol{\tau})_K = 0, \qquad \text{for all } \boldsymbol{\tau} \in W(K), \tag{89}$$

$$(\boldsymbol{v}, \mathbf{curl}\ \mathbf{w}(\boldsymbol{f}))_K = (\boldsymbol{f}, \boldsymbol{v})_K, \qquad \text{for all } \boldsymbol{v} \in V(K), \tag{90}$$

$$(\boldsymbol{w}(\boldsymbol{g}_{\mathsf{T}}), \boldsymbol{\tau})_K - (\boldsymbol{u}(\boldsymbol{g}_{\mathsf{T}}), \mathbf{curl}\, \boldsymbol{\tau})_K = (\boldsymbol{g}_{\mathsf{T}}, \boldsymbol{n} \times \boldsymbol{\tau})_{\partial K \cap \partial\Omega}, \qquad \text{for all } \boldsymbol{\tau} \in W(K), \tag{91}$$

$$(\boldsymbol{v}, \mathbf{curl}\ \boldsymbol{w}(\boldsymbol{g}_{\mathsf{T}}))_K = 0, \qquad \text{for all } \boldsymbol{v} \in V(K). \tag{92}$$

Note that all of the above local problems are uniquely solvable. Hence, these local maps are well defined.

Then the Lagrange multipliers are characterized as the unique solution of a variational equation involving the bilinear forms,

$$\begin{aligned} a(\boldsymbol{\lambda}, \boldsymbol{\mu}) &= (\boldsymbol{w}(\boldsymbol{\lambda}), \boldsymbol{w}(\boldsymbol{\mu}))_\Omega, \\ c(p, q) &= (\mathcal{w}(p), \mathcal{w}(q))_\Omega, \\ b(\boldsymbol{\mu}, p) &= -\sum_{K \in \mathcal{T}} (\boldsymbol{u}(\boldsymbol{\mu}), \mathbf{curl}\ \mathcal{w}(p))_K, \end{aligned}$$

and the functionals

$$\ell_1(\boldsymbol{\mu}) = (\boldsymbol{f}, \boldsymbol{u}(\boldsymbol{\mu}))_\Omega - (\boldsymbol{g}_{\mathsf{T}}, \boldsymbol{w}(\boldsymbol{\mu}))_{\partial\Omega} \tag{93}$$

$$\ell_2(q) = (\boldsymbol{f}, \mathbf{u}(q))_\Omega + (g_n, q)_{\partial\Omega} - (\boldsymbol{g}_{\mathsf{T}}, \mathcal{w}(q))_{\partial\Omega}. \tag{94}$$

Theorem 7.1 [16] *The Lagrange multiplier $(\boldsymbol{\lambda}_h, p_h) \in M_h \times P_h$ of the hybridized mixed method* (81)–(84) *is the unique solution of*

$$a(\boldsymbol{\lambda}_h, \boldsymbol{\mu}) + b(\boldsymbol{\mu}, p_h) = \ell_1(\boldsymbol{\mu}), \qquad \textit{for all } \boldsymbol{\mu} \in M_h \textit{ and} \tag{95}$$

$$b(\boldsymbol{\lambda}_h, q) - c(p_h, q) = \ell_2(q), \qquad \textit{for all } q \in P_h. \tag{96}$$

Moreover, the solution components $\boldsymbol{\omega}_h$ and $\boldsymbol{u}_h$ of the hybridized mixed method (81)–(84) *can be determined locally as follows:*

$$\boldsymbol{\omega}_h = \boldsymbol{w}(\boldsymbol{\lambda}_h) + \mathcal{w}(p_h) + \boldsymbol{w}(\boldsymbol{g}_{\mathsf{T}}) + \mathbf{w}(\boldsymbol{f}), \tag{97}$$

$$\boldsymbol{u}_h = \boldsymbol{u}(\boldsymbol{\lambda}_h) + \mathbf{u}(p_h) + \boldsymbol{u}(\boldsymbol{g}_{\mathsf{T}}) + \mathbf{u}(\boldsymbol{f}). \tag{98}$$

This theorem is proved via the approach introduced in [13] to prove results like Theorem 2.1: We first use the local maps to rewrite the first two equations of the hybridized method, namely (81) and (82). This yields (97) and (98). Next, the two remaining equations of the hybridized method, namely (83) and (84), are used to characterize the pressure and tangential velocity Lagrange multipliers of the method. For details see [15, 16].

It is clear from this theorem that in practice one should implement our hybridized mixed method not in its direct form (81)–(84), but rather in the reduced form (95)–(96). Let us end by pointing out that to implement this method, convenient local basis functions for the spaces P_h and W_h must be found. This is possible – see [15, 16] for details.

8 Conclusion

We have shown that hybridization is a potent tool that can be used to realize computational goals difficult or even impossible to achieve with other conventional techniques as well as for theoretical analysis and comparison of mixed methods.

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