Definition 1 (Conventions and notations in this course). The "base" inner product in a Hilbert space V (over $\mathbb{C}$ or $\mathbb{R}$) under consideration is denoted by $(\cdot,\cdot)_V$, or simply $(\cdot,\cdot)$ when no confusion can arise. The corresponding norm is $\|x\| = (x,x)^{1/2}$. If $V = \mathbb{C}^N$ or $\mathbb{R}^N$, then the base inner product is $(x,y) = y^* x$. The set of continuous linear operators from V to another Hilbert space W is denoted by $\mathcal{L}(V,W)$. When $V = W$, we abbreviate it to $\mathcal{L}(V)$, and denote its subset of bijective operators by $\mathcal{B}(V)$. An operator $A \in \mathcal{L}(V)$ is **self adjoint** if $(Ax,y) = (x,Ay)$ for all $x,y \in V$, and is **positive definite** if in addition $(Ax,x) > 0$ for all $0 \ne x \in V$. A self adjoint and positive definite operator M in $\mathcal{B}(V)$ defines another inner product $(x,y)_M = (Mx,y)$ on V and a corresponding norm $\|x\|_M = (x,x)_M^{1/2}$. (Do not confuse the different meanings of $(\cdot,\cdot)_S$ when S is a space and when S is an operator.)

Theorem 2. *If $A \in \mathcal{B}(\mathbb{R}^N)$ is self adjoint and positive definite and e_n is the error in the n-th iterate of the steepest descent algorithm for solving $Ax = b$, then*

$$\|e_n\|_A \le \left(\frac{\kappa(A)-1}{\kappa(A)+1}\right)^n \|e_0\|_A.$$

where $\kappa(A) = \max\sigma(A)/\min\sigma(A)$.

Theorem 3. *If $A \in \mathcal{B}(\mathbb{R}^N)$ is self adjoint and positive definite and e_n is the error in the n-th iterate of the conjugate gradient algorithm for solving $Ax = b$, then*

$$\|e_n\|_A \le 2\left(\frac{\sqrt{\kappa(A)}-1}{\sqrt{\kappa(A)}+1}\right)^n \|e_0\|_A.$$

Proposition 4. *Suppose $A \in \mathcal{L}(V)$ is bounded and self adjoint in $(\cdot,\cdot)$. Then its spectrum $\sigma(A) \subseteq \mathbb{R}$ and both $\inf\sigma(A)$ and $\sup\sigma(A)$ are in $\sigma(A)$. Moreover, for any $C_0, C_1 > 0$, the following are equivalent statements:*

(1) *$C_0(x,x) \le (Ax,x) \le C_1(x,x)$, for all $x \in V$.*
(2) *$C_0 \le \inf\sigma(A)$ and $\sup\sigma(A) \le C_1$.*

Proposition 5. *Suppose $A, B \in \mathcal{B}(V)$ are self adjoint and positive definite in $(\cdot,\cdot)$ and $C_0, C_1 > 0$. Then the following are equivalent statements:*

(1) *For all $v \in V$*
$$C_0(B^{-1}v,v) \le (Av,v) \le C_1(B^{-1}v,v).$$
(2) *For all $v \in V$,*
$$C_0(A^{-1}v,v) \le (Bv,v) \le C_1(A^{-1}v,v).$$
(3) *The spectrum of the product BA is real and satisfies*
$$C_0 \le \inf\sigma(BA) \quad \text{and} \quad \sup\sigma(BA) \le C_1.$$

Exercise 6. Suppose $A \in \mathcal{L}(V)$ is self adjoint and positive definite. Then $A \in \mathcal{B}(V)$ if and only if there exists a $C_0 > 0$ such that $C_0(x,x) \le (Ax,x)$ for all $x \in V$.

Exercise 7. Suppose $A \in \mathcal{L}(V)$ is self adjoint in $(\cdot,\cdot)$. Define

$$\rho(A) = \sup_{\lambda\in\sigma(A)} |\lambda|, \qquad \|A\| = \sup_{0\neq x\in V} \frac{(Ax,Ax)^{1/2}}{(x,x)^{1/2}}, \qquad r(A) = \sup_{0\neq x\in V} \frac{|(Ax,x)|}{(x,x)}.$$

Prove that $\rho(A) = \|A\| = r(A)$.

Definition 8. Given $x_0, b \in V$, suppose a sequence $x_n \in V$ is constructed by the one-step **iteration**

$$x_{n+1} = J(x_n, b) \tag{1}$$

using a continuous map $J : V \times V \to V$.

(1) If J is a linear map on the product space $V \times V$, i.e.,

$$J(\alpha x + \beta y, \alpha b + \beta d) = \alpha J(x,b) + \beta J(y,d)$$

then (1) is said to be a **linear iteration**.

(2) Iteration (1) is said to be **consistent** with the system $Ax = b$ for some $A \in \mathcal{L}(V)$ if x is a fixed point in the sense

$$x = J(x,b).$$

(3) Iteration (1) is **convergent** if $x_n \to x$ in V as $n \to \infty$.

Proposition 9. *Suppose $A \in \mathcal{B}(V)$, $b \in V$, and $x = A^{-1}b$. For any linear iteration* (1) *the following are equivalent statements:*

(1) *The iteration is consistent with $Ax = b$.*

(2) *There is a linear operator $E \in \mathcal{L}(V)$ such that the error $e_n = x - x_n$ satisfies*

$$e_{n+1} = Ee_n, \qquad \forall n = 0, 1, \ldots.$$

(The operator E is called the **reducer** *of the algorithm and $Ez = J(z,0)$.)*

(3) *There is a linear operator $B \in \mathcal{L}(V)$ such that*

$$x_{n+1} = x_n + B(b - Ax_n), \qquad \forall n = 0, 1, \ldots. \tag{2}$$

(The operator B is called the **iterator** *and $Bb = (I - E)A^{-1}b = J(0,b)$.)*

Proposition 10 (Iterator as a preconditioner)**.** *Consider the iteration* (2)*, suppose A and B are self adjoint in $(\cdot,\cdot)$. If $A \in \mathcal{B}(V)$ is positive definite and*

$$\eta = \|I - BA\|_A < 1,$$

then

(1) *B is positive definite,*

(2) *the iteration* (2) *is convergent,*

(3) *the condition number $\kappa(BA) = \dfrac{\sup\sigma(BA)}{\inf\sigma(BA)}$ satisfies $\kappa(BA) \leq \dfrac{1+\eta}{1-\eta}$,*

(4) *the asymptotic convergence rate of the conjugate gradient method for $Ax = b$ preconditioned by B is faster than the rate of convergence of* (2).

Definition 11. Suppose A is self adjoint in $(\cdot,\cdot)$. If B is also self adjoint in the same inner product, then the iteration

$$u_{n+1} = u_n + B(b - Au_n), \qquad \forall n = 0, 1, \ldots. \tag{3}$$

is called a **symmetric** iteration. When $B \neq B^t$, we often symmetrize the algorithm by revising it to compute u_{n+1} from u_n in these two steps:

$$u_{n+1/2} = u_n + B(b - Au_n) \tag{4a}$$

$$u_{n+1} = u_{n+1/2} + B^t(b - Au_{n+1/2}). \tag{4b}$$

These two steps define the **symmetrization** of (2).

Proposition 12. *Suppose $A \in \mathcal{L}(V)$ is self adjoint. Then iteration (4) defines a consistent and symmetric linear iteration. Its self adjoint iterator is given by*

$$\bar{B} = B^t + B - B^t AB.$$

If in addition $A \in \mathcal{B}(V)$ is also positive definite, then the reducer satisfies these:

(1) $\|I - \bar{B}A\|_A < 1 \iff \bar{B}$ *is in $\mathcal{B}(V)$ and is self adjoint and positive definite.*
(2) $\rho(I - BA)^2 \leq \rho(I - \bar{B}A) = \|I - \bar{B}A\|_A = \|I - BA\|_A^2 = 1 - \inf \sigma(\bar{B}A)$.

Definition 13. Recall that the Hilbert **adjoint** of $A \in \mathcal{L}(V, W)$ is the operator $A^t \in \mathcal{L}(W, V)$ satisfying $(Av, w)_W = (v, A^t w)_V$ for all $v \in V$ and $w \in W$. (This is related but not equal to the operator dual on Banach spaces.)

Exercise 14. Suppose X and Y are Hilbert and $A \in \mathcal{L}(X, Y)$. Prove that the following are equivalent statements:

(1) A is surjective.
(2) A^t is injective and $\operatorname{ran} A^t$ is closed.
(3) There exists an $\alpha > 0$ such that $\|A^t y\|_X \geq \alpha \|y\|_Y$ for all $y \in Y$.

Assumption 15. Suppose V and $\hat{V}$ are Hilbert spaces with inner products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_{\hat{V}}$, resp. Assume that $\hat{\Lambda} \in \mathcal{B}(\hat{V})$ is self adjoint and positive definite in $(\cdot,\cdot)_{\hat{V}}$ and $R \in \mathcal{L}(\hat{V}, V)$ is surjective.

Definition 16. In the setting of Assumption 15, the operator

$$B_2 = R\hat{\Lambda}^{-1}R^t \tag{5}$$

is called a **two-level** or **auxiliary space** preconditioner.

Assumption 17. Suppose $\hat{\Lambda}$ and R are as in Assumption 15 and assume additionally that there is an $A \in \mathcal{B}(V)$, self adjoint and positive definite in $(\cdot,\cdot)_V$, and satisfying

$$(AR\hat{v}, R\hat{v})_V \leq C_R(\hat{\Lambda}\hat{v}, \hat{v})_{\hat{V}}, \qquad \forall \hat{v} \in \hat{V}.$$

Assumption 18. Suppose $\hat{\Lambda}$ and R are as in Assumption 15 and assume additionally that we have an operator $S \in \mathcal{L}(V, \hat{V})$ such that $RS = I$ (where I is the identity on V) and

$$C_S(\hat{\Lambda}Sv, Sv)_{\hat{V}} \leq (Av, v)_V, \qquad \forall v \in V$$

for some $A \in \mathcal{B}(V)$ that is self adjoint and positive definite in $(\cdot,\cdot)_V$.

Lemma 19. *Suppose Assumption 15 holds. Then the two-level operator B_2 in* (5) *is a bijection in $\mathcal{B}(V)$ and the operator $T = \hat{\Lambda}^{-1} R^t B_2^{-1} : V \to \hat{V}$ is a continuous right inverse of R (i.e., $RT = I$).*

Lemma 20. *Suppose Assumption 15 holds. Then, for all $v \in V$,*

$$(B_2^{-1} v, v)_V = \inf_{\hat{v} \in R^{-1}\{v\}} (\hat{\Lambda} \hat{v}, \hat{v})_{\hat{V}}$$

where the set $R^{-1}\{v\} = \{\hat{v} \in \hat{V} : R\hat{v} = v\}$.

Lemma 21. *Assumptions 15 and 17 imply that the B_2 in* (5) *satisfies*

$$(Av, v)_V \leq C_R (B_2^{-1} v, v)_V, \qquad \forall v \in V.$$

Lemma 22. *Assumptions 15 and 18 imply that the B_2 in* (5) *satisfies*

$$C_S (B_2^{-1} v, v)_V \leq (Av, v)_V \qquad \forall v \in V.$$

Theorem 23 (Fictitious space lemma of Nepomnyaschikh)**.** *Suppose Assumptions 15, 17, and 18 hold. Then $B_2 = R \hat{\Lambda}^{-1} R^t$ satisfies*

$$C_S (B_2^{-1} v, v)_V \leq (Av, v)_V \leq C_R (B_2^{-1} v, v)_V, \qquad \forall v \in V,$$

and consequently $\kappa(B_2 A) \leq C_R / C_S$.

Example 24. Use an enclosing fictitious domain to precondition the Neumann problem.

Corollary 25. *Suppose Assumptions 15 and 17 hold with $V \subset \hat{V}$ and with $R : \hat{V} \to V$ equal to a projection onto V. Then*

$$\sigma(B_2 A) \subseteq [1, C_R].$$

Assumption 26 (Subspace correction setting). Let $A \in \mathcal{B}(V)$ be self adjoint and positive definite. Suppose V_i, $i = 1, \ldots, J$, are closed subspaces of the Hilbert space $\{V, (\cdot,\cdot)\}$ and suppose $\Lambda_i \in \mathcal{B}(V_i)$.

Definition 27. In setting of Assumption 26, let $Q_i : V \to V_i$ denote the $(\cdot,\cdot)$-orthogonal projection onto V_i. The operator

$$B_a = \sum_{i=1}^{J} \Lambda_i^{-1} Q_i \tag{6}$$

is called the **additive preconditioner** based on subspaces V_i and operators $\Lambda_i \in \mathcal{B}(V_i)$.

Algorithm 28 (Additive Schwarz Method/Parallel Subspace Correction). Given an approximation $u_n \in V$ to $u = A^{-1} f$, compute u_{n+1} as follows:

(1) Project the residual onto V_j and compute $r_j = Q_j r = Q_j (f - A u_n)$.
(2) Find $\varepsilon_j \in V_j$ by solving $\Lambda_j \varepsilon_j = r_j$.
(3) Correct u_n on each subspace by $u_{n+1} = u_n + \omega \sum_{i=1}^{J} \varepsilon_j$,
where $\omega > 0$ is a "relaxation" parameter.

Proposition 29. *Suppose Λ_i are self adjoint. Then Algorithm 28 is a linear symmetric consistent iteration that can be rewritten as*

$$u_{n+1} = u_n + \omega B_a(f - Au_n) \tag{7}$$

whose iterator and reducer are given, respectively, by

$$\omega B_a = \omega \sum_{i=1}^{J} \Lambda_i^{-1} Q_i, \qquad E = I - \omega B_a A = I - \omega \left(\sum_{j=1}^{J} T_j \right), \qquad \text{where } T_j = \Lambda_j^{-1} Q_j A.$$

Algorithm 30 (Multiplicative Schwarz Method/Successive Subspace Correction). Given an approximation $u_n \in V$ to $u = A^{-1} f$, compute u_{n+1} as follows:

(1) Set $u_n^{(0)} = u_n$.
(2) For $j = 1, \dots, J$ do:
  (a) Solve for $\varepsilon_j \in V_j$ satisfying $\Lambda_j \varepsilon_j = Q_j(f - Au_n^{(j-1)})$.
  (b) Compute $u_n^{(j)} = u_n^{(j-1)} + \varepsilon_j$.
(3) Set $u_{n+1} = u_n^{(J)}$.

Proposition 31. *Algorithm 30 is a linear consistent iteration that can be rewritten as*

$$u_{n+1} = u_n + B_m(f - Au_n),$$

whose iterator and reducer are given by

$$B_m = (I - E)A^{-1}, \qquad\qquad E = (I - T_J)(I - T_{J-1}) \cdots (I - T_1). \tag{8}$$

Definition 32. B_m is called the **multiplicative preconditioner** based on subspaces V_i and operators $\Lambda_i \in \mathcal{B}(V_i)$. Also define

$$\bar{B}_m = B_m^t + B_m - B_m^t A B_m, \qquad A_{ij} = Q_i A I_j, \qquad M_j = \Lambda_j^t + \Lambda_j - A_{jj},$$

where $I_j : V_j \to V$ denotes the natural embedding.

Assumption 33. In the subspace correction setting of Assumption 26, assume further that $\hat{V} = V_1 \times V_2 \times \cdots \times V_J$ with inner product $(\hat{v}, \hat{w})_{\hat{V}} = \sum_{j=1}^{J} ([\hat{v}]_j, [\hat{w}]_j)_V$ and set $R : \hat{V} \to V$ by $R\hat{v} = \sum_{j=1}^{J} [\hat{v}]_j$. Write elements of $\hat{V}$ as column vectors of its V_j-components and write an operator on $\hat{V}$ as a matrix of operators on the component spaces, e.g.,

$$\begin{bmatrix} [\hat{v}]_1 \\ \vdots \\ [\hat{v}]_J \end{bmatrix} \equiv \hat{v} \in \hat{V}, \quad R = \begin{bmatrix} I_1 & I_2 & \cdots & I_J \end{bmatrix}, \quad R^t = \begin{bmatrix} Q_1 \\ \vdots \\ Q_J \end{bmatrix}.$$

Continuing in such notations of matrices of operators, set

$$\mathbb{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1J} \\ A_{21} & A_{2,2} & & A_{2J} \\ \vdots & & \ddots & \vdots \\ A_{J1} & \cdots & A_{J,J-1} & A_{JJ} \end{bmatrix}, \quad \mathbb{L} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ A_{21} & 0 & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A_{J1} & \cdots & A_{J,J-1} & 0 \end{bmatrix}, \quad \mathbb{D} = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & A_{JJ} \end{bmatrix},$$

$$\mathbb{\Lambda} = \operatorname{diag}(\Lambda_1, \dots, \Lambda_J), \qquad \mathbb{B}_a = \mathbb{\Lambda}^{-1}, \qquad \mathbb{B}_m = (\mathbb{\Lambda} + \mathbb{L})^{-1},$$

$$\bar{\mathbb{B}}_m = \mathbb{B}^t + \mathbb{B} - \mathbb{B}^t \mathbb{A} \mathbb{B}, \qquad \mathbb{M} = \mathbb{\Lambda}^t + \mathbb{\Lambda} - \mathbb{D}, \qquad \mathbb{U} = (\mathbb{\Lambda} + \mathbb{L})^t.$$

Lemma 34 (B_m and B_a take the form of B_2)**.** *Suppose Assumptions 26 and 33 hold. Then* $\mathbb{A} = R^t A R$ *is self adjoint in* $(\cdot,\cdot)_{\hat{V}}$ *and the following identities hold:*

$$B_a = R\,\mathbb{B}_a R^t$$
$$B_m = R\,\mathbb{B}_m R^t$$
$$\bar{B}_m = R\,\bar{\mathbb{B}}_m R^t.$$

Example 35. The classical Jacobi and Gauss-Seidel iteration for a symmetric positive definite matrix fits Algorithm 28 with $V = \mathbb{R}^N$ and $V_i = \mathrm{Span}(e_i)$.

Lemma 36. *Suppose* $\sum_{j=1}^J V_j$ *is dense in* V *and* Λ_i *are self adjoint and positive definite. Then the following are equivalent:*

(1) *The sum of the subspaces is closed, i.e.,*

$$V = \sum_{j=1}^J V_j. \tag{9}$$

(2) B_a *is a bijection on* V.

Theorem 37. *Suppose* Λ_i *are self adjoint and positive definite, Assumption 26 and* (9) *hold. Then*

$$(B_a^{-1}v, v) = \inf_{\{\sum_{i=1}^J v_i = v\}} \sum_{i=1}^J (\Lambda_i v_i, v_i),$$

where the infimum is taken over all decompositions of form $v = v_1 + v_2 + \cdots + v_J$ *with* $v_i \in V_i$.

Assumption 38. Suppose Assumption 26 holds and

$$\|I - \Lambda_i^{-1} A_{ii}\|_{A_{ii}} < 1, \qquad \forall i = 1, \ldots, J.$$

Lemma 39. *Suppose Assumptions 26, 33, 38 and* (9) *hold. Then* $\bar{\mathbb{B}}_m$ *and* $\mathbb{M}$ *are bijections on* $\hat{V}$ *and the following identities hold:*

$$\bar{\mathbb{B}}_m^{-1} = \mathbb{U}^t \mathbb{M}^{-1} \mathbb{U},$$
$$\bar{\mathbb{B}}_m^{-1} = \mathbb{A} + (\mathbb{L}^t + \mathbb{D} - \mathbb{\Lambda})^t \mathbb{M}^{-1} (\mathbb{L}^t + \mathbb{D} - \mathbb{\Lambda}).$$

Theorem 40. *Suppose Assumptions 26, 38 and* (9) *hold. Then* $\bar{B}_m$ *is a self adjoint and positive definite operator in* $\mathcal{B}(V)$*, and for all* $v \in V$,

$$(\bar{B}_m^{-1} v, v) = \inf_{\{\sum_{i=1}^J v_i = v\}} \sum_{i=1}^J \left\| \Lambda_i^t v_i + Q_i A \sum_{j=i+1}^J v_j \right\|_{M_i^{-1}}^2 = \|v\|_A^2 + \inf_{\{\sum_{i=1}^J v_i = v\}} \sum_{i=1}^J \left\| Q_i A \left(\sum_{j=i}^J v_j \right) - \Lambda_i v_i \right\|_{M_i^{-1}}^2$$

where the infimum is taken over all decompositions of form $v = v_1 + v_2 + \cdots + v_J$ *with* $v_i \in V_i$.

Corollary 41 (XZ identity)**.** *Suppose Assumptions 26, 38 and* (9) *hold. Then*

$$\|(I - T_J)\cdots(I - T_2)(I - T_1)\|_A^2 = 1 - \frac{1}{1 + c_0}$$

where

$$c_0 = \sup_{\|v\|_A = 1} \inf_{\{\sum_{i=1}^J v_i = v\}} \sum_{i=1}^J \left\| Q_i A \left(\sum_{j=i}^J v_j \right) - \Lambda_i v_i \right\|_{M_i^{-1}}^2.$$

Definition 42. Let $P_i : V \to V_i$ be the A-orthogonal projector defined by

$$(P_i v, w_i)_A = (v, w_i)_A, \qquad \forall w_i \in V_i$$

for any self adjoint and positive definite A. Note that $A_{ii} P_i = Q_i A$.

Corollary 43 (XZ identity for A-orthogonal projectors of a subspace decomposition)**.** *Suppose V_i, $i = 1, \dots, J$, are closed subspaces of the Hilbert space V satisfying* (9) *and A is a self adjoint and positive definite operator in $\mathcal{B}(V)$. Then*

$$\|(I - P_J)\cdots(I - P_2)(I - P_1)\|_A^2 = 1 - \frac{1}{1 + c_1} \tag{10}$$

where

$$c_1 = \sup_{\|v\|_A = 1} \inf_{\{\sum_{i=1}^J v_i = v\}} \sum_{i=1}^J \left\| P_i \left(\sum_{j=i+1}^J v_j \right) \right\|_A^2 .$$

Definition 44. In the setting of Assumption 26, define

$$\mathcal{J} = \left[\sum_{i=1}^J P_i \right] A^{-1}, \qquad \mathcal{G} = \Big[I - (I - P_J)\cdots(I - P_2)(I - P_1) \Big] A^{-1}.$$

Note that $\mathcal{J}$ and $\mathcal{G}$ coincides with B_a and B_m, respectively, if we set $\Lambda_i = A_{ii}$ for all i.

Condition 45 (**Strengthened Cauchy Schwarz inequality**)*.* $\beta > 0$ is a number such that for all v_i and w_i in V_i,

$$\sum_{i=1}^J \sum_{j=1}^J \left| (w_i, v_j)_A \right| \le \beta^{1/2} \left(\sum_{i=1}^J \|w_i\|_A^2 \right)^{1/2} \left(\sum_{j=1}^J \|v_i\|_A^2 \right)^{1/2} .$$

Lemma 46. *In the subspace correction setting of Assumption 26,*

$$\begin{aligned} \textit{Condition 45} \implies & \sum_{i=1}^J \left\| P_i \left(\sum_{i=1}^J v_j \right) \right\|_A^2 \le \beta \sum_{i=1}^J \|v_i\|_A^2, && \forall\ v_i \in V_i, \\ \implies & (\bar{\mathcal{G}}^{-1} v, v) \le \beta (\mathcal{J}^{-1} v, v), && \forall v \in V, \end{aligned}$$

which also implies $(\mathcal{J} A v, v)_A \le \beta (\bar{\mathcal{G}} A v, v)_A$ for all $v \in V$. Here $\bar{\mathcal{G}} = \mathcal{G}^t + \mathcal{G} - \mathcal{G}^t A \mathcal{G}$.

Condition 47 (**Stable Decomposition**)*.* $\exists\ \alpha > 0$ such that $\forall v \in V$, a decomposition

$$v = \sum_{i=1}^J v_i, \qquad \text{with } v_i \in V_i,$$

exists and satisfies

$$\sum_{i=1}^J \|v_i\|_A^2 \le \alpha \|v\|_A^2.$$

Theorem 48. *In the subspace correction setting of Assumption 26, if Conditions 45 and 47 are verified, then*

$$\alpha^{-1}(v, v)_A \le (\mathcal{J} A v, v)_A \le \beta^{1/2} (v, v)_A,$$

or equivalently

$$\beta^{-1/2}(Av, v) \le \mathcal{J}^{-1} v, v) \le \alpha (Av, v),$$

for all $v \in V$.

Exercise 49 (Case of $\Lambda_i \neq A_{ii}$). Suppose $\beta_1 > 0$ is a number such that for all v_i and w_i in V_i,

$$\sum_{i=1}^{J}\sum_{j=1}^{J} \left|(w_i, v_j)_A\right| \le \beta_1^{1/2} \left(\sum_{i=1}^{J} \|w_i\|^2_{\Lambda_i}\right)^{1/2} \left(\sum_{j=1}^{J} \|v_i\|^2_{\Lambda_i}\right)^{1/2}. \tag{11}$$

and suppose $\exists\ \alpha_1 > 0$ such that $\forall v \in V$, a decomposition $v = \sum_{i=1}^{J} v_i$ with $v_i \in V_i$ exists and satisfies

$$\sum_{i=1}^{J} \|v_i\|^2_{\Lambda_i} \le \alpha_1 \|v\|^2_A. \tag{12}$$

Then show that

$$\beta_1^{-1/2}(Av, v) \le (B_a^{-1}v, v) \le \alpha_1(Av, v), \qquad \forall v \in V. \tag{13}$$

Theorem 50. *In the subspace correction setting of Assumption 26, if Conditions 45 and 47 are verified, then*

$$\|I - \mathcal{G}A\|_A^2 \le 1 - \frac{1}{1+\alpha\beta},$$

and moreover, for the Jacobi case, setting relaxation parameter ω *such that* $0 < \omega < 2/\beta^{1/2}$,

$$\sigma(I - \omega\mathcal{J}A) \subseteq [-\theta, \gamma] \subseteq (-1, 1)$$

where $-\theta \equiv 1 - \omega\beta^{1/2} \le \gamma \equiv 1 - (\omega/\alpha)$.

Assumption 51 (A setting using Lagrange finite elements). Set V = Lagrange finite element subspace of $H_0^1(\Omega)$, of order $p \ge 1$, on a simplicial quasiuniform mesh $\mathcal{T}_h$ (of mesh size h) subdividing a domain $\Omega \subseteq \mathbb{R}^d$, and

$$(u, v)_V = \int_\Omega uv, \qquad (Au, v) = \int_\Omega \operatorname{grad} u \cdot \operatorname{grad} v \tag{14}$$

for all $u, v \in V$. Let $\{\Omega_i : i = 1, \ldots, J\}$ be a finite cover of Ω such that each $\bar{\Omega}_i$ is a union of elements of $\mathcal{T}_h$. Assume that there is a W^1_∞ partition of unity $\{\theta_i\}$ subordinate to the covering and suppose $\Theta > 0$ is a number satisfying $\|\operatorname{grad}\theta_i\|_{L^\infty(\Omega)} \le \Theta$ for all $i = 1, 2, \ldots, J$. Assume that the cover has the **limited overlap property**: there is an integer r such that each point of Ω is contained in no more than r of the sets Ω_i. Set

$$V_i = \{v \in V : \operatorname{supp}(v) \subseteq \Omega_i\}, \qquad i = 1, 2, \ldots, J. \tag{15}$$

Theorem 52. *Assumption 51* $\Longrightarrow$

(1) *Condition 45 holds with* $\beta = r^2$,
(2) *Condition 47 also holds:* $\exists C > 0$ *independent of* h *such that* $\forall v \in V$, *there is a decomposition* $v = \sum_{i=1}^{J} v_i$ *with* $v_i \in V_i$ *and*

$$\sum_{i=1}^{J} \|v_i\|_A^2 \le Cr\left(\Theta^2\|v\|^2 + \|v\|_A^2\right).$$

Exercise 53. For the A and V set in Assumption 51, prove that there is a mesh-independent constant $C > 0$ such that $\rho(A) \le Ch^{-2}$.

Example 54. Block Gauss-Seidel and Jacobi iterations with the overlapping blocks obtained when Ω_i is set to vertex patches.

Definition 55 (Real method of interpolation). If X_0 and X_1 are normed linear space are subspaces of a larger linear space, then $X \equiv [X_0, X_1]$ is called a **compatible pair** of spaces. If $Y \equiv [Y_0, Y_1]$ is also a compatible pair, then two bounded linear operators $L_i \in \mathcal{L}(X_i, Y_i)$, $i = 0, 1$, are called a pair of **compatible operators** whenever

$$L_0 u = L_1 u \qquad \forall\, u \in X_0 \cap X_1.$$

The K**-functional** (of Peetre) for the compatible pair X is defined by

$$K(t,u) = \inf_{u_0+u_1=u} \left(\|u_0\|_{X_0}^2 + t^2\|u_1\|_{X_1}^2\right)^{1/2}, \qquad \forall u \in X_0 + X_1,\ \forall t > 0.$$

Define, for $0 < s < 1$,

$$\|u\|_{X_s} = \left(2s(1-s)\int_0^\infty t^{-2s} K(t,u)^2 \,\frac{dt}{t}\right)^{1/2}$$

and define the **interpolation space** $X_s \equiv [X_0, X_1]_s \equiv \{u \in X_0 + X_1 :\ \|u\|_{X_s} < \infty\}$.

Theorem 56. *If* $[X_0, X_1]$ *is a compatible pair, then for all* $0 < s < 1$,

$$X_0 \cap X_1 \hookrightarrow X_s \hookrightarrow X_0 + X_1,$$

and moreover:

$$\begin{aligned} \|u\|_{X_s} \le \|u\|_{X_0}^{1-s}\|u\|_{X_1}^{s} \le \|u\|_{X_0\cap X_1} &\qquad \forall u \in X_0 \cap X_1,\\ K(t,u) \le t^s \|u\|_{X_s} &\qquad \forall u \in X_s,\\ \|u\|_{X_0+X_1} \le \|u\|_{X_s} &\qquad \forall u \in X_s. \end{aligned}$$

Theorem 57 (Interpolation of operators)**.** *Suppose* $[X_0, X_1]$, $[Y_0, Y_1]$ *and* $L_i \in \mathcal{L}(X_i, Y_i)$, $i = 0, 1$, *are compatible. Then, for all* $0 < s < 1$, $\exists!$ *linear operator* $L_s \in \mathcal{L}(X_s, Y_s)$ *satisfying*

$$L_s u = L_0 u = L_1 u \qquad \forall u \in X_0 \cap X_1.$$

Moreover, if $C_i > 0$ *are such that*

$$\|L_i u\|_{Y_i} \le C_i \|u\|_{X_i}, \qquad i = 0, 1,$$

then

$$\|L_s u\|_{Y_s} \le C_0^{1-s} C_1^s \|u\|_{X_s}, \qquad \forall u \in X_s.$$

Fact 58. *On any nonempty open* $\Omega \subseteq \mathbb{R}^d$, $d \ge 1$, *for all* $s_0, s_1 \in \mathbb{R}$,

$$[H^{s_0}(\Omega), H^{s_1}(\Omega)]_s = H^\sigma(\Omega)$$

where $0 < s < 1$ *and* $\sigma = (1-s)s_0 + s s_1$.

Example 59. The error in $L^2(\Omega)$ projection of functions in $H^s(\Omega)$ into the Lagrange finite element space.

Lemma 60. *Let $K_{02}(t,u)$ denote the K-functional for the compatible pair $[L^2(\mathbb{R}^d), H^2(\mathbb{R}^d)]$. Then, $\forall \eta > 0,\ \eta \neq 1,\ \exists C > 0$ such that*

$$\sum_{\ell=-\infty}^{\infty} \eta^{-\ell} K_{02}(\eta^\ell, u)^2 \le C \|u\|^2_{H^1(\mathbb{R}^d)}, \qquad \forall u \in H^1(\mathbb{R}^d).$$

The same result holds if $\mathbb{R}^d$ is replaced by a bounded open $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary.

Lemma 61 (Stability of the L^2 projection in H^1)**.** *Suppose V is the Lagrange finite element subspace of $H^1_0(\Omega)$, of order $p \ge 1$ on a quasiuniform mesh $\mathcal{T}_h$ (of mesh size h) subdividing a domain $\Omega \subset \mathbb{R}^d$. Let $Q_h : L^2(\Omega) \to V$ denote the $L^2(\Omega)$ orthogonal projector into V. Then there is a $C > 0$ independent of h such that*

$$\|v - Q_h v\|_{L^2(\Omega)} + h\|Q_h v\|_{H^1(\Omega)} \le Ch\|v\|_{H^1(\Omega)}, \qquad \forall v \in H^1(\Omega).$$

Lemma 62 (Elliptic projection under full regularity)**.** *Suppose, in addition to the assumptions of Lemma 61, that Ω is convex. Let $P_h : H^1(\Omega) \to V$ denote the orthogonal projector in the $(A\cdot,\cdot)$ inner product set in (14). Then there is a $C > 0$ independent of h such that*

$$\|v - P_h v\|_{L^2(\Omega)} + h\|P_h v\|_{H^1(\Omega)} \le Ch\|v\|_{H^1(\Omega)}, \qquad \forall v \in H^1(\Omega).$$

Assumption 63 (A setting for Overlapping Schwarz Methods). Suppose Ω is subdivided by a simplicial quasiuniform "coarse" mesh $\mathcal{T}_H$ (of mesh size H) with J elements K_j, as well as by a simplicial quasiuniform "fine" mesh $\mathcal{T}_h$ that is a refinement of $\mathcal{T}_H$ (so $h \ll H$). For each coarse element K_j, set Ω_j to be the domain formed by K_j and all its neighboring elements, i.e., $\bar{\Omega}_j = \bigcup\{K \in \mathcal{T}_H : K \cap K_j \text{ is nonempty}\}$. Set V = Lagrange finite element space of order p on $\mathcal{T}_h$, set $(\cdot,\cdot)$, $(A\cdot,\cdot)$ by (14) and set V_i by (15) for all $i = 1,\dots,J$. In addition, we now also set a **coarse space** V_{J+1} to be the Lagrange finite element space of order p on the coarse mesh $\mathcal{T}_H$.

Definition 64. For the subspace decomposition including the coarse space,

$$V = V_1 + V_2 + \cdots + V_J + V_{J+1},$$

define the additive and multiplicative overlapping Schwarz preconditioners by

$$B_a^{\mathrm{OS}} = \left[\sum_{i=1}^{J+1} P_i\right] A^{-1}, \qquad B_m^{\mathrm{OS}} = \Big[I - (I - P_{J+1})(I - P_J)\cdots(I - P_2)(I - P_1)\Big] A^{-1}.$$

Theorem 65 (Uniform preconditioning & convergence of Overlapping Schwarz method)**.** *Assumption 63 $\implies \exists C_1, C_2 > 0$ independent of H and h such that*

$$\kappa(B_a^{\mathrm{OS}} A) \le C_1,$$
$$\|I - B_m^{\mathrm{OS}} A\|_A^2 \le 1 - \frac{1}{C_2}.$$

Assumption 66 (General geometric multilevel setting). Suppose $A \in \mathcal{B}(V)$ be self adjoint and positive definite and suppose we have a nested sequence of closed subspaces

$$V_1 \subset V_2 \subset \cdots \subset V_L \equiv V.$$

Furthermore, suppose each V_k has closed subspaces $V_{k,i}$ such that

$$V_k = \sum_{i=1}^{J_k} V_{k,i}. \tag{16}$$

Definition 67. The decomposition (16) of V_k is often called a **micro decomposition** of a multilevel space, while the sum

$$V = V_1 + V_2 + \cdots + V_L \tag{17}$$

is called a **macro decomposition**. Let Q_k and $Q_{k,i}$ be the $(\cdot,\cdot)$-projections into V_k and $V_{k,i}$, respectively. Let P_k and $P_{k,i}$ be $(\cdot,\cdot)_A$-projection into V_k and $V_{k,i}$, respectively. Let $A_k \in \mathcal{B}(V_k)$ be defined by $(A_k v, w) = (v, w)_A$ for all $w \in V_k$ and similarly let $A_{k,i} \in \mathcal{B}(V_{k,i})$ be defined by $(A_{k,i} v, w) = (v, w)_A$ for all $w \in V_{k,i}$. The **BPX preconditioner** (also known as the additive multigrid preconditioner) based on the **full multilevel subspace decomposition**

$$V = \sum_{k=1}^{L} \sum_{i=1}^{J_k} V_{k,i}, \tag{18}$$

is defined by

$$B_{\text{BPX}} = \left[\sum_{k=1}^{L} \sum_{i=1}^{J_k} P_{k,i} \right] A^{-1} = \sum_{k=1}^{L} \sum_{i=1}^{J_k} A_{k,i}^{-1} Q_{k,i}.$$

It is the same as the additive preconditioner B_a (see (6) and Algorithm 28) obtained by setting the subspaces $\{V_i\}$ to $\{V_{k,i}\}$ and operators $\{A_i\}$ to $\{A_{k,i}\}$.

Algorithm 68 (The **\cycle**: $u_{n+1} = \text{Slash}_L(u_n, f)$). We define the map $\text{Slash}_k : V_k \times V_k \to V_k$ for all $1 \le k \le L$, inductively, namely $w = \text{Slash}_k(v, g)$ is set as follows:

(1) If $k = 1$, set $w = A_1^{-1} g$.
(2) If $k > 1$, set $w = \text{Slash}_k(v, g)$ recursively using $\text{Slash}_{k-1}(\cdot,\cdot)$, as follows:
  (a) Set $v^{(0)} = v$.
  (b) For $i = 1, \ldots, J_k$, do:
    (i) $v^{(i)} = v^{(i-1)} + A_{k,i}^{-1} Q_{k,i}(g - A v^{(i-1)})$
  (c) Set output $w = v^{(J_k)} + \text{Slash}_{k-1}(0, Q_{k-1}(g - A v^{(J_k)}))$.

Exercise 69. Show that $u_{n+1} = \text{Slash}_L(u_n, f)$ can be written as $u_{n+1} = u_n + B_{\backslash\text{cycle}}(f - A u_n)$ where $B_{\backslash\text{cycle}}$ is the same as the multiplicative preconditioner B_m (see Algorithm 30 and (8)) obtained by setting the subspaces $\{V_i\}$ to $\{V_{k,i}\}$ and operators $\{A_i\}$ to $\{A_{k,i}\}$.

Assumption 70 (Multilevel Lagrange finite element setting). Suppose a bounded $\Omega \subset \mathbb{R}^d$ is subdivided by a simplicial quasiuniform mesh $\mathcal{T}_1$ (of meshsize h_1). Suppose that $\mathcal{T}_k$ (of meshsize h_k) for $1 < k \le L$, is obtained by a uniform refinement of $\mathcal{T}_{k-1}$. Set V_k to the linear ($p = 1$) Lagrange finite element subspace of $H_0^1(\Omega)$ on $\mathcal{T}_k$. Let $\Omega_{k,i}$ denote the vertex patch composed of all elements of $\mathcal{T}_k$ connected to ith vertex of $\mathcal{T}_k$ and set $V_{k,i} = \{v \in V_k : \text{supp}(v) \subseteq \Omega_{k,i}\}$. Finally, set $(\cdot,\cdot)$, $(A\cdot,\cdot)$ by (14).

Definition 71. Let $\mathcal{H} \in \mathcal{B}(V)$ be defined by

$$(\mathcal{H}^{-1}v, v) = \inf_{\sum_{k=1}^{L} v_k = v} \sum_{k=1}^{L} h_k^{-2}(v_k, v_k),$$

i.e., $\mathcal{H}$ is the same as the additive operator B_a (see Theorem 37) obtained using the macro decomposition (17) and setting $\Lambda_k v = h_k^{-2} v$ for all $v \in V_k$.

Lemma 72. *Assumption 70* $\Longrightarrow$ *There are* L*-independent constants* $C_1, C_2 > 0$ *such that*

$$C_1(\mathcal{H}^{-1}v, v) \le (B_{\mathrm{BPX}}^{-1}v, v) \le C_2(\mathcal{H}^{-1}v, v) \qquad \forall v \in V.$$

Lemma 73. *Assumption 70* $\Longrightarrow$ $\exists\ C_1 > 0$, *and* $0 \le \delta < 1$, *both independent of* $\{h_m\}$, *such that whenever* $k \le l$,

$$(w_k, v_l)_A \le C_1 \delta^{l-k} \|w_k\|_A \left(h_l^{-1}\|v_l\|\right), \qquad \forall w_k \in V_k,\ \forall v_l \in V_l.$$

Hence, the condition (11) *holds with* $\Lambda_k = h_k^{-2} I$ *and an* $\{h_k\}$*-independent* β_1.

Lemma 74. *Suppose Assumption 70 holds and* $\partial\Omega$ *is Lipschitz. If there is a* $C_\Pi > 0$ *and linear operators* $\Pi_k : L^2(\Omega) \to V_k$ *such that for all* k,

$$\|\Pi_k u\|_{L^2(\Omega)} \le C_\Pi \|u\|_{L^2(\Omega)} \qquad \forall u \in L^2(\Omega), \tag{19a}$$

$$\|(I - \Pi_k)v\|_{L^2(\Omega)} \le C_\Pi h_k^2 \|v\|_{H^2(\Omega)} \qquad \forall v \in H^2(\Omega), \tag{19b}$$

then there is a $C > 0$, *depending only on* C_Π, Ω, *and* h_1, *such that*

$$\sum_{k=2}^{L} h_k^{-2} \|(\Pi_k - \Pi_{k-1})v\|_{L^2(\Omega)}^2 \le C\|v\|_{H^1(\Omega)}^2 \qquad \forall v \in H^1(\Omega).$$

In particular, since Q_k *satisfies* (19), *taking* $L \to \infty$,

$$\sum_{k=2}^{\infty} h_k^{-2} \|(Q_k - Q_{k-1})v\|_{L^2(\Omega)}^2 \le C\|v\|_{H^1(\Omega)}^2 \qquad \forall v \in H^1(\Omega).$$

Hence the condition (12) *holds with* $\Lambda_k = h_k^{-2} I$ *and an* $\{h_k\}$*-independent constant* α_1.

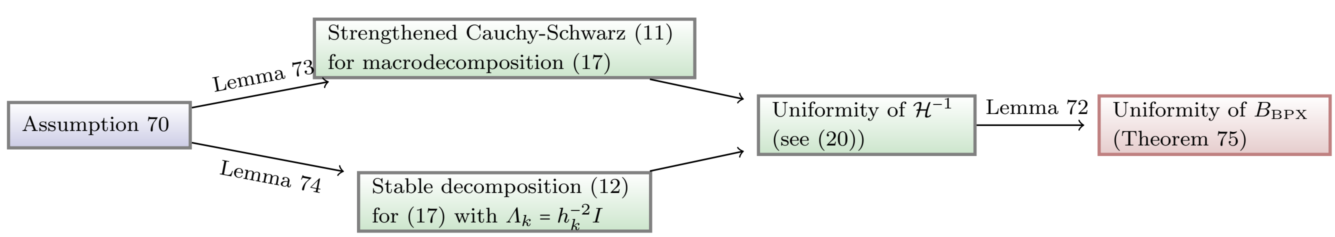
Theorem 75 (Uniformity of BPX preconditioner)**.** *In the setting of Assumption 70, additionally assume that* $\partial\Omega$ *is Lipschitz. Then (applying Exercise 49), there are* L*-independent positive constants* α_1 *and* β_1 *such that*

$$\beta_1^{-1/2}(Av, v) \le (\mathcal{H}^{-1}v, v) \le \alpha_1(Av, v), \qquad \forall v \in V. \tag{20}$$

Hence $\exists C_1 > 0$ *independent of* L *such*

$$\kappa(B_{\mathrm{BPX}}A) \le C_1.$$

Remark 76. Chain of arguments in regularity-free multigrid theory:



Lemma 77. *Assumption 70* $\implies$ $\exists C > 0$*, depending only on the* C_1 *and* δ *in Lemma 73, such that*

$$\sum_{k=1}^{L} \|(P_k - Q_k)v\|_A^2 \;\le\; C \sum_{k=1}^{L} h_k^{-2} \|(Q_k - Q_{k-1})v\|^2$$

Theorem 78 (Uniform convergence of \cycle)**.** *In the setting of Assumption 70, additionally assume that* $\partial\Omega$ *is Lipschitz. Then the* c_1 *in the XZ identity* (10) *for the subspace decomposition* (18) *is bounded independent of* L *and hence*

$$\|I - B_{\backslash\text{cycle}} A\|_A^2 \;\le\; 1 - \frac{1}{1 + c_1}.$$

Remark 79. Ingredients in the proof of Theorem 78:

$$\boxed{\text{Assumption 70}} \xrightarrow[\text{Lemma 73}]{\text{Lemma 77}} \boxed{\begin{array}{l}\text{XZ identity for the full}\\ \text{multilevel decomposition (18)}\end{array}} \xrightarrow{\text{Corollary 41}} \boxed{\begin{array}{l}\text{Uniform convergence of \textbackslash cycle}\\ \text{(Theorem 78)}\end{array}}$$

Algorithm 80 (The **Vcycle**: $u_{n+1} = \mathrm{Vcycle}_L(u_n, f)$). Given "smoothers" $S_k \in \mathcal{L}(V_k)$, for all $1 \le k \le L$, we define the map $\mathrm{Vcycle}_k : V_k \times V_k \to V_k$ inductively. Set $w = \mathrm{Vcycle}_k(v, g)$ as follows:

(1) If $k = 1$, set $w = A_1^{-1} g$.
(2) If $k > 1$, set $w = \mathrm{Vcycle}_k(v, g)$ recursively:
 (a) Pre-smoothing step: $v' = v + S_k(g - A_k v)$.
 (b) Coarse correction: $v'' = v' + \mathrm{Vcycle}_{k-1}(0, Q_{k-1}(g - Av'))$.
 (c) Post-smoothing step: $w = v'' + S_k^t(g - A_k v'')$.

Exercise 81. Show that the symmetrization (see Definition 11) of $u_{n+1} = \mathrm{Slash}_L(u_n, f)$ is the Vcycle algorithm with S_k set to $\mathcal{G}$ at each V_k.

Proposition 82. *Algorithm 80 is a consistent linear iteration whose reducer* $E \equiv E_L$ *is given recursively by* $E_1 = 0$ *and*

$$E_k = K_k^*(I - P_{k-1} + E_{k-1} P_{k-1}) K_k, \qquad \forall k > 1,$$

where $K_k = I - S_k A_k$ *and* K_k^* *is the* $(\cdot,\cdot)_A$*-adjoint of* K_k.

Condition 83 (Regularity & Approximation Property). $\exists \alpha_0 > 0$ such that for all $k \ge 1$ and for all $u \in V_k$,

$$\|(I - P_{k-1}) K_k u\|_A^2 \le \alpha_0 \left(\|u\|_A^2 - \|K_k u\|_A^2 \right).$$

Remark 84. Condition 83 quantifies the following folkloric prerequisite for V-cycle to work: Errors undamped by smoothing at any refinement level must be well representable at the next coarser level. Interpret

$$\| \underbrace{(I - P_{k-1}) K_k e}_{\substack{V_k\text{-component of error}\\ \text{after smoothing by } K_k}} \|_A^2 \;\le\; \alpha_0 \,(\underbrace{\|e\|_A^2 - \|K_k e\|_A^2}_{\substack{\text{quantifies damping of}\\ \text{error } e \text{ by } K_k}}).$$

Clearly, if $\|e\|_A \approx \|K_k e\|$ (i.e., if e is left undamped), then the above implies that $K_k e$ must almost be in V_{k-1}. Condition 83 is usually verified using regularity estimates.

Theorem 85. *In the geometric multilevel setting of Assumption 66, Condition 83 implies*

$$0 \le (E_k v, v)_A \le \delta(v,v)_A, \qquad \forall v \in V_k,$$

with $\delta = \dfrac{\alpha_0}{1+\alpha_0}$.

Lemma 86. *Suppose V_k is finite dimensional and suppose there are constants $0 \le \theta < 1$ and $C_1 > 0$ such that S_k satisfies these properties for all k:*

$$S_k \text{ is self adjoint in } (\cdot,\cdot), \tag{21a}$$

$$\sigma(I - S_k A_k) \subseteq [-\theta, 1), \tag{21b}$$

$$(S_k^{-1} e, e) \le C_1 (Ae, e), \quad \forall e \in (I - P_{k-1})V_k. \tag{21c}$$

Then, Condition 83 holds with $\alpha_0 = C_1 \max(1, \theta^2/(1-\theta))$. (Note that (21a) *and* (21b) *imply that S_k is a bijection, so S_k^{-1} makes sense in* (21c)*.)*

Lemma 87. *Suppose there is a $C_1 > 0$ such that S_k satisfies these properties for all k:*

$$\|I - S_k A_k\|_A < 1, \tag{22a}$$

$$(\bar{S}_k^{-1} e, e) \le C_1 (Ae, e), \qquad \forall e \in (I - P_{k-1})V_k, \tag{22b}$$

where $\bar{S}_k = S_k + S_k^t - S_k^t A_k S_k$ (which is a bijection by Proposition 12). Then Condition 83 holds with $\alpha_0 = C_1$.

Definition 88. In the setting of Assumption 70 define the Gauss-Seidel smoother

$$\mathcal{G}_k = \Big[I - (I - P_{k,J_k}) \cdots (I - P_{k,1})(I - P_{k,1}) \Big] A_k^{-1}$$

and the Jacobi smoother

$$\mathcal{J}_k = \left[\sum_{i=1}^{J_k} P_{k,i} \right] A_k^{-1}.$$

Theorem 89 (Braess-Hackbusch)**.** *Suppose Assumption 70 holds and suppose $\Omega \subset \mathbb{R}^d$ is convex. Set S_k in the Vcycle (Algorithm 80) to be either the Gauss-Seidel smoother $\mathcal{G}_k$ or the damped Jacobi smoother $\omega \mathcal{J}_k$ with $0 < \omega < 2/(d+1)$. Then there is a $0 < \delta < 1$ independent of L such that*

$$0 \le (E_k v, v)_A \le \delta(v,v)_A, \qquad \forall v \in V_k,$$

so the Vcycle converges at a rate independent of number of refinements.

Remark 90. Chain of arguments in regularity-based multigrid theory: