HW

1. (Due 10 April 2024)

   Let p be prime and or every $n \in \mathbb{Z}_{\geq 0}$, let $\zeta_n = e^{\frac{2\pi i}{p^n}}$ and $L_n = \mathbb{Q}(\zeta_n)$. Let's write L for the smallest subfield of $\mathbb{C}$ that contains all the L_n. From class, we know that $L/\mathbb{Q}$ is Galois. Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence of integers such that for all $n \in \mathbb{Z}_{\geq 1}$,

   $$a_n + p^n\mathbb{Z} \in (\mathbb{Z}/p^n\mathbb{Z})^{\times} \qquad \text{and} \qquad a_{n+1} \equiv a_n \pmod{p^n}.$$

   Prove that there is an automorphism $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ such that for every $n \in \mathbb{Z}_{\geq 0}$, we have that $\sigma(\zeta_n) = \zeta_n^{a_n}$.

   Proof. From last year, we know that our first hypothesis on the a_n ensures there is a unique element of $\mathrm{Gal}(L_n/\mathbb{Q})$ that maps ζ_n to $(\zeta_n)^{a_n}$; let's name this automorphism σ_n.

   We know from class that $L = \cup_{n=0}^{\infty}$; let's define $\sigma\colon L \to L$ as follows: for any $\alpha \in L$, choose some n with $\alpha \in L_n$ and define $\sigma(\alpha) := \sigma_n(\alpha)$.

   To see this σ is well-defined, suppose that $j, k \in \mathbb{Z}_{\geq 0}$ with $j < k$, and note that our second hypothesis on the a_n ensures that there is some $m \in \mathbb{Z}$ such that $a_k = a_j + mp^j$; thus,

   $$\sigma_k(\zeta_j) = \sigma_k\left((\zeta_k)^{p^{k-j}}\right) = \sigma_k(\zeta_k)^{p^{k-j}} = (\zeta_k)^{a_k p^{k-j}} = (\zeta_k)^{(a_j+mp^j)p^{k-j}} = (\zeta_k)^{a_j p^{k-j}} = (\zeta_j)^{a_j} = \sigma_j(\zeta_j),$$

   so we know from last year that $\sigma_k|_{L_j} = \sigma_j$.

   Finally, we prove that σ is a surjective field automorphism:

   - To see that σ is surjective, choose any $\alpha \in L$, then find some n with $\alpha \in L_n$. Since $\sigma_n \in \mathrm{Gal}(L_n/\mathbb{Q})$, we know that σ_n is surjective, so there is some $\beta \in L_n$ with $\sigma_n(\beta) = \alpha$; hence, $\sigma(\beta) = \alpha$.
   - To see that σ is a field homomorphism, choose any $\alpha, \beta \in L$, then find some n with $\alpha, \beta \in L_n$ and note that

     $$\sigma(\alpha\beta) = \sigma_n(\alpha\beta) = \sigma_n(\alpha)\sigma_n(\beta) = \sigma(\alpha)\sigma(\beta) \quad \text{and} \quad \sigma(\alpha+\beta) = \sigma_n(\alpha+\beta) = \sigma_n(\alpha)+\sigma_n(\beta) = \sigma(\alpha)+\sigma(\beta).$$

   □

2. (Due 10 April 2024)

   Suppose that $a \in \mathbb{Z}$ with $a \equiv 1 \pmod 4$ and $a \not\equiv 1 \pmod 8$.

   (a) Prove that for all $k \in \mathbb{Z}_{\geq 0}$:

   $$a^{(2^k)} \equiv 1 \left(\text{mod } 2^{k+2}\right) \qquad \text{and} \qquad a^{(2^k)} \not\equiv 1 \left(\text{mod } 2^{k+3}\right)$$

   (b) For all $n \in \mathbb{Z}_{\geq 2}$, write $\pi_n\colon (\mathbb{Z}/2^n\mathbb{Z})^{\times} \to (\mathbb{Z}/4\mathbb{Z})^{\times}$ for the natural projection. Prove

   $$\langle a + 2^n\mathbb{Z}\rangle = \{b + 2^n\mathbb{Z} \mid b \in \mathbb{Z} \quad \text{and} \quad b \equiv 1 \pmod 4\} = \ker(\pi_n).$$

3. (Due 17 April 2024)

   Prove that if L/K is a Galois extension an with intermediate field E, then L/E is also Galois.

4. (Due 17 April 2024)

   Suppose X, Y are topological spaces and β is a base for the topology on Y. Let $f\colon X \to Y$ be any function. Prove:

   $$f \text{ continuous} \qquad \text{if and only if} \qquad \text{for all } U \in \beta, \text{ we have } f^{-1}(U) \text{ is open.}$$

5. (Due 24 April 2024)

   Suppose that X, Y are topological spaces, and $\pi_X: X \times Y \to X$ and $\pi_Y: X \times Y \to Y$ are the natural projection. Next, suppose that Z is a third topological space and $f: Z \to X$ and $g: Z \to Y$ are continuous maps. We know from class that there is a unique function $h: Z \to X \times Y$ such that $\pi_X \circ h = f$ and $\pi_Y \circ h = g$. Prove that h is continuous.

6. (Due 24 April 2024)

   Suppose that G is a topological group and H is a closed normal subgroup of G. Prove that G/H, with the quotient topology, is a topological group.

7. (Due 1 May 2024)

   Let X be the following subset of $\mathbb{R}^2$, equipped with the subspace topology:

   $$X = \{(0,0)\} \bigcup \left\{ \left(t, \sin\left(t^{-1}\right)\right) \mid t \in (0,1] \right\}.$$

   Prove that X is connected.

8. (Due 1 May 2024)

   Suppose that X, Y are topological spaces and $\phi: X \to Y$ a continuous function. Prove that if X is compact, then $\phi(X)$ is compact.