Derivatives Defined

- A financial instrument whose return is derived from the return on another instrument
- Derivatives provide a means of managing financial risk
- By using derivatives, one party can transfer, for a price, (like an *insurance premium*), any undesired risk to other parties

Derivative Markets and Instruments

- Futures contract
  - Definition: a contract between two parties for one party to buy something from the other at a later date at a price agreed upon today; subject to a daily settlement of gains and losses and guaranteed against the risk that either party might default
  - Exclusively traded on a futures exchange

Option Terminology

- Price/premium
  - The price of an option contract which the ***buyer*** of the option pays to the option writer for the rights conveyed by the option contract
- Call
  - Option contract that gives the holder the right to ***buy*** the underlying security at a specified price for a certain fixed period of time

Option Terminology

- Put
  - An option contract that gives the holder the right to ***sell*** the underlying security at a specified price for a certain fixed period of time
- Exercise price/strike price/striking price
  - The ***stated price per share*** for which the underlying security may be purchased (in the case of a call) or sold (in the case of a put) by the option holder upon the exercise of the option contract

Call options

- Objective of a call- the buyer profits from the underlying asset's appreciation
  - In-the-money
    - A term describing an option that has intrinsic value. A ***call*** option is in-the-money if the underlying security is ***higher*** than the striking price of the call
  - Out-of-the-money
    - A ***call*** option is out-of-the-money if the strike price is ***greater than*** the market price of the underlying security

Option Terminology

- Expiration date
  - The day on which an option contract becomes void. Holders of options should indicate their desire to exercise, if they wish to do so, by this date.

Option Terminology

- At-the-money
  - An option is at-the-money if the strike price of the option is equal to the market price of the underlying security
  - Risk neutral investor or investment

    only concerned with an investment's expected return.

Put options

- Objective of a put- the buyer locks in stock price
  - In-the-money
    - A term describing an option that has intrinsic value. A ***put*** option is in-the-money if the underlying security is ***below*** the striking price of the call
  - Out-of-the-money
    - A ***put*** option is out-of-the-money if the strike price is ***less*** than the market price of the underlying security

Conventional pricing assumptions

- Returns follow a normal (Gaussian) distribution
- Market completeness – perfect information
- Arbitrage-free trades – no arbitrage transactions can occur
- Investors are risk neutral

1.8 Copula methods in finance: a primer

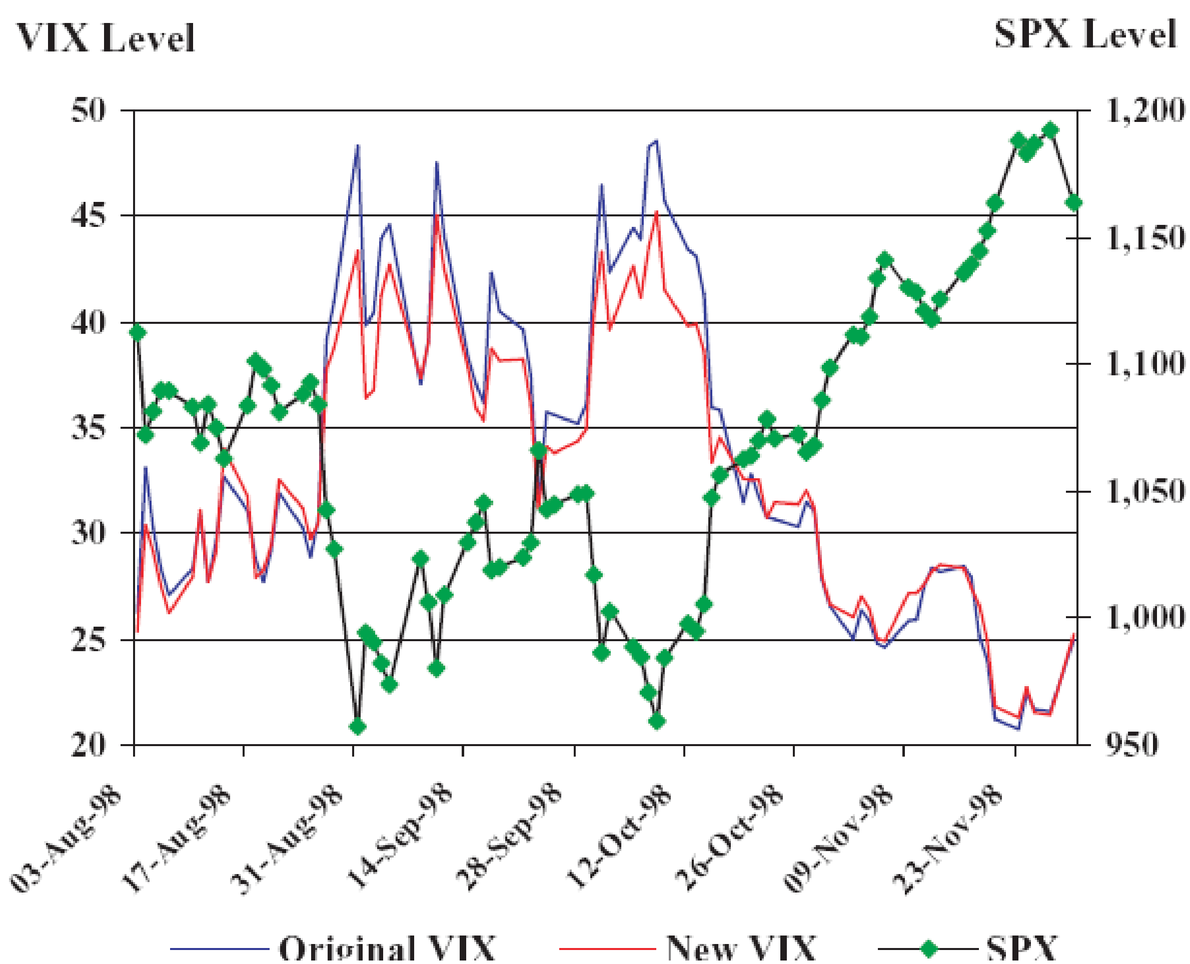
Three main frontier problems in derivative pricing:

❖Departure from normality

❖Emerging from the smile effect

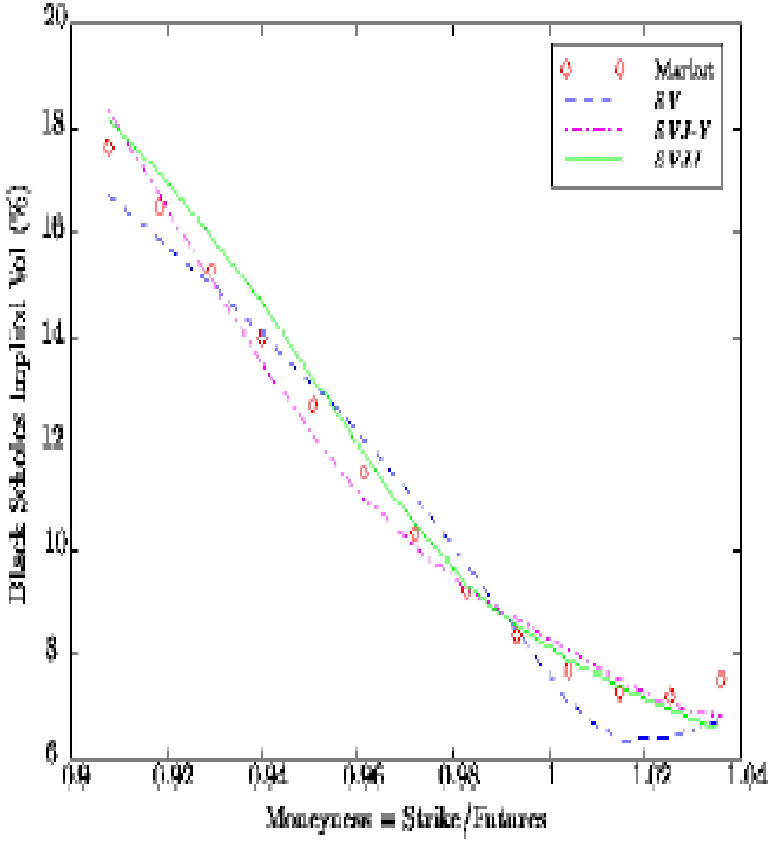
❖Market incompleteness, corresponding to hedging error, and credit risk, linked to the bivariate relationship in OTC transactions

Departure from normality



Smile effect

Anyway, Stochastic volatility can explain the smile effect:



How can we explain the smile effect?

- The main contradiction in the conventional Black and Scholes model is:
- that volatility is not constant.
- We then have to shift to stochastic volatility.
- The need for stochastic volatility has been recognized by several authors.

Market completeness

- Conventional models have depended on full information for all participants
  - However, this has proven not to be the case because no *perfect hedge exists*
  - Therefore, new strategies are developed based on market incompleteness

Market incompleteness

- Because no perfect hedge exists, financial products exist to earn money from their misalignment

  -Thus, the core of the derivatives market has shifted away from the underlying asset toward contingent claims on illiquid assets

1.8.1 Joint probabilities, marginal probabilities and copula functions

- For the pricing problem, a joint probability distribution can be expressed as a function of the marginal ones. So, the bivariate product is priced consistently with information from the univariate ones. Or, conversely, any copula function taking univariate distributions as arguments yields a joint distribution

The price of this digital ***put*** option in a complete market setting is:

- DP =exp[-r(T-t)] $Q(K_{NKY}\ K_{sp})$
  - Where $Q(K_{NKY}\ K_{sp})$ is the joint risk-neutral probability that both the Japanese and US market indexes are below the corresponding strike prices
  - In order to compare the price of our bivariate product with that of the univariate ones, we could write the price as:
- DP =exp[-r(T-t)] $C(Q_{NKY}\ Q_{sp})$

  with C(x,y) a bivariate function

1.8.2 Copula functions duality

The price of this digital ***call*** option in a complete market setting is:

- DC =exp[-r(T-t)] $\underline{\mathbf{Q}}(K_{NKY}\, K_{sp})$
  - Where $\underline{\mathbf{Q}}(K_{NKY}\, K_{sp})$ is the joint risk-neutral probability that both the Japanese and US market indexes are above the corresponding strike prices
  - Then, we can recover a copula function
- DP =exp[-r(T-t)] $\underline{\mathbf{C}}(\underline{\mathbf{Q}}(K_{NKY}),\underline{\mathbf{Q}}(Ksp)]$

with C(x,y) a bivariate function known as survival copula

The *survival copula* is shown below

$\underline{\mathbf{C}}[\underline{\mathbf{Q}}(K_{NKY}), \underline{\mathbf{Q}}(K_{sp})] =$

$1 - Q(K_{NKY}) - Q(Ksp) + C[Q(K_{NKY}), Q(K_{sp})]$

1.8.3 Examples of copula functions

- the main advantage from the use of copula functions is the ability to preserve the dependence structure typical of a multivariate normal distribution by modifying only the marginal distributions, which may be allowed to display skewness and fat-tails consistently with data observed from the market

Frechet bounds

The joint probability is constrained within the bounds:

$\text{Max}(Q_{NKY}+ Qsp -1, 0) \leq Q(K_{NKY}, K_{sp}) \leq \min(Q_{NKY}, Q_{sp})$

The upper bound corresponds to the case of perfect positive dependence between the two markets and the lower bound represents perfect negative dependence

Copula functions

- Min (x,y) = C(x,y) = maximum copula = perfect positive dependence
- Max (x+y-1,0) = minimum copula = Frechet lower bound C(x,y) = perfect negative dependence
- Imperfect dependence = possible strategy -use linear combination of above cases

1.8.4 Copula functions and market comovements

- Non-linear relationships can be measured by the non-parametric measures spearman's rho and Kendall's tau. These non-parametric measures do not depend on the shape of the marginal probability distributions
- The relationship between the non-parametric dependence measures and copula functions can be applied to recover a first calibration technique of the Frechet copula function itself:
  Spearman's rho = alpha -beta

$\iint f(x) = \int\left[\frac{d}{dx} f(x)\right] dx = \frac{d}{dx}\left[\int f(x) dx\right]$

Non-parametric features

- Both Spearman's rho and Kendall's tau do not require the specific shape of the marginal distributions

-taking the double integral C(u,v) dC(u,v) does not require the variances and covariances of the marginal distributions which linear correlations require

1.8.5 Tail dependence

- Non-normality at the univariate level is associated with skewness and leptokurtosis, or the fat-tail problem
- In the multivariate setting, the fat-tail problem can be referred both to the marginal univariate distributions or to the joint probability of large market movements. This concept is called tail dependence.
- Copula functions enables us to model fat-tails and tail dependence separately

Tail dependence

- For example, the likelihood that one event with probability lower than v occurs in the first variable, given that an event with probability lower than v occurs in the second one

Lamda = C(v,v)/v

For very small values of v, the lower tail index is

$lamda_L$= lim v->0+ C (v,v)/v