Stat 576
10-31-17

①

From last time,

$$E[E[X|Y]] = E[X] \text{ and}$$

$$V[X] = E_1 V_2[X] + V_1 E_2[X].$$

Apply this to $\hat{t}$:

$$E[\hat{t}] = E_1 E_2[\hat{t}] = E_1 E_2\left[\frac{N}{n}\sum_{i=1}^{n} \hat{t}_i\right]$$

$$= N E_1\left(\frac{1}{n}\sum_{i=1}^{n} E_2[\hat{t}_i]\right)$$

②

$$= N E_1\left(\frac{1}{n}\sum_{i=1}^{n} E_2[M_i \bar{y}_i]\right)$$

$$= N E_1\left(\frac{1}{n}\sum_{i=1}^{n} M_i E_2[\bar{y}_i]\right)$$

$$= N E_1\left(\frac{1}{n}\sum_{i=1}^{n} M_i \bar{Y}_i\right)$$

$$= N E_1\left(\frac{1}{n}\sum_{i=1}^{n} t_i\right) = N E_1(\bar{t})$$

$$= N \bar{T} = N \frac{t}{N} = t$$

③

$$V[\hat{t}] = \underbrace{E_1 V_2[\hat{t}]}_{②} + \underbrace{V_1 E_2[\hat{t}]}_{①}$$

$$①: V_1 E_2[\hat{t}] = V_1 E_2\left[\frac{N}{n}\sum_{i=1}^{n}\hat{t}_i\right]$$

$$= N^2 V_1\left(\frac{1}{n}\sum_{i=1}^{n} E_2[\hat{t}_i]\right)$$

$$= N^2 V_1\left(\frac{1}{n}\sum_{i=1}^{n} t_i\right) = N^2 V_1(\bar{t})$$

④

$$= N^2 \frac{S_t^2}{n}\left(1 - \frac{n}{N}\right)$$

$$② \; E_1 V_2[\hat{t}] = E_1 V_2\left(\frac{N}{n}\sum_{i=1}^{n}\hat{t}_i\right)$$

$$= \frac{N^2}{n^2} E_1\left(\sum_{i=1}^{n} V_2[\hat{t}_i]\right)$$

Note: there are no covariance terms since independent samples are taken in each cluster

$$= \frac{N^2}{n^2} E_1\left(\sum_{i=1}^{n} V_2[M_i \bar{y}_i]\right)$$

$$= \frac{N^2}{n^2} E_1\left(\sum_{i=1}^{n} M_i^2 V_2[\bar{y}_i]\right)$$

(5)

$$= \frac{N^2}{n^2} E_1\left(\sum_{i=1}^{n} M_i^2 \frac{S_i^2}{m_i}\left(1 - \frac{m_i}{M_i}\right)\right)$$

$$= \frac{N^2}{n} E_1\left(\frac{1}{n}\sum_{i=1}^{n} M_i^2 \frac{S_i^2}{m_i}\left(1 - \frac{m_i}{M_i}\right)\right) \qquad \left[E(\bar{a}) = \bar{A} \right.$$

$$= \frac{N^2}{n}\left(\frac{1}{N}\sum_{i=1}^{N} M_i^2 \frac{S_i^2}{m_i}\left(1 - \frac{m_i}{M_i}\right)\right)$$

$$= \frac{N}{n}\sum_{i=1}^{N} M_i^2 \frac{S_i^2}{m_i}\left(1 - \frac{m_i}{M_i}\right)$$

We have shown: (6)

$$V[\hat{t}] = N^2 \frac{S_t^2}{n}\left(1 - \frac{n}{N}\right) + \frac{N}{n}\sum_{i=1}^{N} M_i^2 \frac{S_i^2}{m_i}\left(1 - \frac{m_i}{M_i}\right)$$

How to estimate this?

Try this:

$$\hat{V}[\hat{t}] = N^2 \underbrace{\frac{s_t^2}{n}\left(1 - \frac{n}{N}\right)}_{1} + \underbrace{\frac{N}{n}\sum_{i=1}^{n} M_i^2 \frac{s_i^2}{m_i}\left(1 - \frac{m_i}{M_i}\right)}_{2}$$

⑦

① : $E[s_{\hat{t}}^2] = E_1 E_2 [s_{\hat{t}}^2]$

$$= E_1 E_2 \left[\frac{1}{n-1} \left(\sum_{i=1}^{n} \hat{t}_i^2 - n \bar{\hat{t}}^2 \right) \right]$$

$$= \frac{1}{n-1} E_1 \left[\sum_{i=1}^{n} E_2 [\hat{t}_i^2] - n E_2 (\bar{\hat{t}}^2) \right]$$

$$= \frac{1}{n-1} E_1 \left[\sum_{i=1}^{n} \left[V_2 [\hat{t}_i] + (E_2 [\hat{t}_i])^2 \right] - n \left(V_2 (\bar{\hat{t}}) + (E_2 (\bar{\hat{t}}))^2 \right) \right]$$

⑧

$$= \frac{1}{n-1} E_1 \left[\sum_{i=1}^{n} M_i^2 \frac{s_i^2}{m_i} \left(1 - \frac{m_i}{M_i}\right) + t_i^2 - n \left(\frac{1}{n^2} \sum_{i=1}^{n} M_i^2 \frac{s_i^2}{m_i} \left(1 - \frac{m_i}{M_i}\right) + \left(\frac{1}{n} \sum_{i=1}^{n} t_i\right)^2 \right) \right]$$

$$= \frac{1}{n-1} E_1 \left[\underbrace{\left(1 - \frac{1}{n}\right)}_{\frac{n-1}{n}} \sum_{i=1}^{n} M_i^2 \frac{s_i^2}{m_i} \left(1 - \frac{m_i}{M_i}\right) + \underbrace{t_i^2 - \frac{1}{n} \left(\sum_{i=1}^{n} t_i\right)^2}_{(n-1) s_t^2} \right]$$

$$= E_1 \left[\frac{1}{n} \sum_{i=1}^{n} M_i^2 \frac{s_i^2}{m_i} \left(1 - \frac{m_i}{M_i}\right) + s_t^2 \right]$$

$$= \frac{1}{N} \sum_{i=1}^{N} M_i^2 \frac{S_i^2}{m_i} \left(1 - \frac{m_i}{M_i}\right) + S_t^2$$ to be continued...