Stat 571
10-31-13

<u>Hotelling's T^2 example</u>

(1)

X_1	X_2
32	36
36	35
41	49
43	41
50	47
74	74

X_1 = reaction time in msec for left eye
X_2 = " " " " " right "

$H_0: \vec{\mu} = \begin{bmatrix} 40 \\ 40 \end{bmatrix}$ $\quad H_1: \vec{\mu} \neq \begin{bmatrix} 40 \\ 40 \end{bmatrix}$

$$\bar{X} = \begin{bmatrix} 46 \\ 47 \end{bmatrix} \quad S = \begin{bmatrix} 226 & 207.6 \\ 207.6 & 206.8 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} .0568 & -.0570 \\ -.0570 & .0621 \end{bmatrix} \quad |S| = 3639.04$$

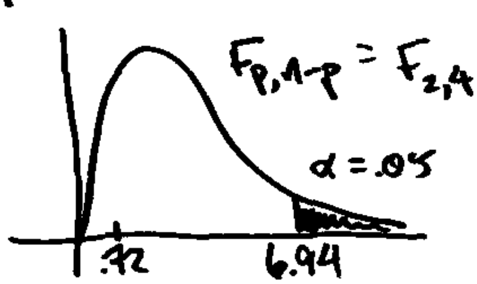
(2)

$$T^2 = n(\bar{X} - \vec{\mu}_0)' S^{-1} (\bar{X} - \vec{\mu}_0)$$

$$= 6 \begin{bmatrix} 6 & 7 \end{bmatrix} \begin{bmatrix} .0568 & -.0570 \\ -.0570 & .0621 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

$$= 1.7982$$

$$\frac{n-p}{p(n-1)} T^2 = \frac{6-2}{2(5)} T^2 = .72$$ <u>Test stat</u>



Fail to reject H_0

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Confidence regions for $\vec{\mu}$

Let $X_{n \times p}$ be the data matrix

Let $R(X)$ be a subset of $\mathbb{R}^p$ determined by X.

<u>Defn</u>: $R(X)$ is a $100(1-\alpha)\%$ confidence region for $\vec{\Theta}$ if, before observing the sample,

$$P[R(X) \text{ contains } \vec{\Theta}] = 1-\alpha.$$

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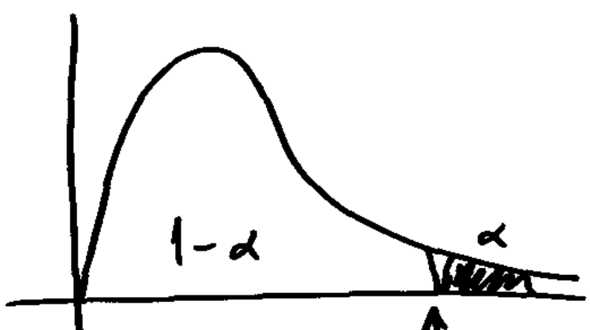
Know: $T^2 \sim \frac{p(n-1)}{n-p} F_{p,n-p}$



$F_{p,n-p}(\alpha) = f$

$$P\left[T^2 < \frac{p(n-1)}{n-p} f\right] = 1-\alpha$$

(5)

So $n(\bar{x}-\vec{\mu})' S^{-1} (\bar{x}-\vec{\mu}) < \frac{p(n-1)}{n-p} f$

is an event whose probability is $1-\alpha$

Thus, after observing $\bar{x}$ and S, the values of $\vec{\mu}$ that satisfy this inequality form a $100(1-\alpha)\%$ confidence region for $\vec{\mu}$.

Note: $(\bar{x}-\vec{\mu})' S^{-1} (\bar{x}-\vec{\mu}) < \frac{p(n-1)}{n(n-p)} f$ is an ellipsoid whose axes are the eigenvectors of S,

(6)

and whose half-lengths are

$$\sqrt{\lambda_i}\sqrt{\frac{p(n-1)}{n(n-p)} f}$$

Suppose we want <u>simultaneous</u> confidence intervals for all parameters of the form $\vec{a}'\vec{\mu}$, where $\vec{a}$ is a vector of constants.

<u>Theorem</u>: (Scheffé) $\vec{a}'\bar{X} \pm \sqrt{\frac{p(n-1)}{n(n-p)} F_{p,n-p}(\alpha)\, \vec{a}' S \vec{a}}$

will contain <u>all possible</u> $\vec{a}'\vec{\mu}$, simultaneously, with probability $1-\alpha$.

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Proof: Consider $\dfrac{[\vec{a}'(\bar{x}-\vec{\mu})]^2}{\vec{a}' S \vec{a}}$.

This is of the form $\dfrac{(\vec{x}'\vec{d})^2}{\vec{x}' B \vec{x}}$.

Using maximization lemma III (Oct 8),

the maximum value is $\vec{d}' B^{-1} \vec{d}$,

occurring when $\vec{x} = c B^{-1} \vec{d}$ for any c.

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That is,

$$\frac{[\vec{a}'(\bar{x}-\vec{\mu})]^2}{\vec{a}' S \vec{a}} \le (\bar{x}-\vec{\mu})' S^{-1} (\bar{x}-\vec{\mu}),$$

with equality when $\vec{a} = c S^{-1}(\bar{x}-\vec{\mu})$

$$\frac{n[\vec{a}'(\bar{x}-\vec{\mu})]^2}{\vec{a}' S \vec{a}} \le n(\bar{x}-\vec{\mu})' S^{-1}(\bar{x}-\vec{\mu}) = T^2$$

⑨

Claim: $\left(T^2 \le k\right) \Longleftrightarrow$

$$\left(\frac{n[\vec{a}'(\bar{x}-\vec{\mu})]^2}{\vec{a}' S \vec{a}} \le k \quad \forall \vec{a} \right)$$

$\Rightarrow$: If $T^2 \le k$, then $\dfrac{n[\vec{a}'(\bar{x}-\vec{\mu})]^2}{\vec{a}' S \vec{a}}$, since it is $\le T^2$, must also be $\le k$ $\forall \vec{a}$.

$\Leftarrow$: If $\dfrac{n[\vec{a}'(\bar{x}-\vec{\mu})]^2}{\vec{a}' S \vec{a}} \le k \ \forall \vec{a}$, then it must be true for $\vec{a} = S^{-1}(\bar{x}-\vec{\mu})$

⑩

$$\frac{n[(\bar{x}-\vec{\mu})' S^{-1} (\bar{x}-\vec{\mu})]^2}{(\bar{x}-\vec{\mu})' S^{-1} \cancel{S} \cancel{S^{-1}} (\bar{x}-\vec{\mu})} \le k$$

$$n(\bar{x}-\vec{\mu})' S^{-1} (\bar{x}-\vec{\mu}) \le k$$

$$T^2 \le k$$

So $1-\alpha = P\left[T^2 \le \frac{p(n-1)}{n-p} F_{p,n-p}(\alpha)\right]$

$$= P\left[\frac{n[\vec{a}'(\bar{x}-\vec{\mu})]^2}{\vec{a}' S \vec{a}} \le \frac{p(n-1)}{n-p} F_{p,n-p}(\alpha) \ \forall a\right]$$

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(11)

Apply Scheffé's Theorem to the individual components of $\vec{\mu}$:

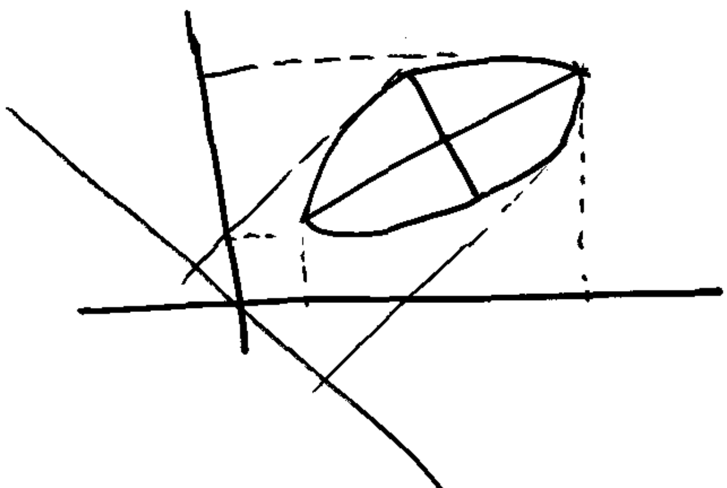
Let $\vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, …, $\vec{a}_p = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

Then simultaneous confidence intervals for $\mu_1, \dots, \mu_p$:

$$\mu_i: \quad \bar{x}_i \pm \sqrt{\frac{p(n-1)}{n(n-p)} F_{p,n-p}^{(\alpha)}}\, s_{ii}$$

$$\bar{x}_i \pm \sqrt{\frac{p(n-1)}{n-p} F_{p,n-p}^{(\alpha)}}\, \frac{s_i}{\sqrt{n}}$$

(12)



The projections of the confidence ellipsoid onto the individual axes are the individual Scheffé confidence intervals.

Midterm Nov 7

- Spectral decomps
- Properties of E and V
- Maximization lemmas
- $|S|$, $tr(S)$
- properties of the m.v. normal
- conditional & marginal distributions
- T^2, $\Lambda^{2/n}$
- Confidence regions & Scheffé intervals