Stat 571
10-1-13

Review of linear algebra

①

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} \qquad L_{\vec{x}} = \|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

$$\vec{x} \cdot \vec{y} = \vec{x}\,'\vec{y} = x_1y_1 + x_2y_2 + \dots + x_ny_n$$

Fact: $\vec{x} \cdot \vec{y} = \|\vec{x}\| \, \|\vec{y}\| \cos\theta$

②

Defn: The vectors $\vec{x}_1, \dots, \vec{x}_k$ are linearly dependent if $\exists$ $c_1, \dots, c_k$, not all 0, such that

$$c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_k\vec{x}_k = \vec{0}$$

Defn: The matrix A is symmetric if $A' = A$.

Defn: The identity matrix is $I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{n \times n}$

<u>Defn</u>: The matrix A^{-1} is the inverse of A if $A^{-1}A = AA^{-1} = I$

<u>Fact</u>: The matrix A has an inverse iff its columns are linearly independent.

<u>Defn</u>: The matrix A is <u>orthogonal</u> if $A' = A^{-1}$.

Note: If A is orthogonal, then $A'A = I$

Suppose $A = [\vec{x}_1 \mid \vec{x}_2 \mid \dots \mid \vec{x}_n]$

Then
$$\underbrace{A'A}_{I} = \begin{bmatrix} \vec{x}_1{}' \\ \hline \vec{x}_2{}' \\ \hline \vdots \\ \hline \vec{x}_n{}' \end{bmatrix} [\vec{x}_1 \mid \vec{x}_2 \mid \dots \mid \vec{x}_n]$$

$$= \begin{bmatrix} \vec{x}_1{}'\vec{x}_1 & \vec{x}_1{}'\vec{x}_2 & \cdots & \vec{x}_1{}'\vec{x}_n \\ \vec{x}_2{}'\vec{x}_1 & - & - & - \\ \vdots & & & \\ \vec{x}_n{}'\vec{x}_1 & - & - & - \end{bmatrix}$$

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So, if A is orthogonal,
each of its columns has unit length
and its columns are pairwise orthogonal

Eigenvalues & eigenvectors

For a square matrix A, λ is an eigenvalue
and $\vec{e}$ is its corresponding eigenvector if

$$A\vec{e} = \lambda\vec{e}$$

Note: $\vec{e}$ is only unique up to scalar multiples

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We will usually select our eigenvectors
to have unit length.

<u>Spectral decomposition</u>

Suppose that $A_{n \times n}$ has an inverse.

Suppose that we know $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$, the eigenvectors of A.

$$\text{Let } P = \left[\underset{n\times 1}{\vec{e}_1} \middle| \vec{e}_2 \middle| \cdots \middle| \vec{e}_n\right]_{n\times n}$$

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$$AP = A\left[\vec{e}_1 \,|\, \vec{e}_2 \,|\, \cdots \,|\, \vec{e}_n\right]$$
$$= \left[A\vec{e}_1 \,|\, A\vec{e}_2 \,|\, \cdots \,|\, A\vec{e}_n\right]$$
$$= \left[\lambda_1\vec{e}_1 \,|\, \lambda_2\vec{e}_2 \,|\, \cdots \,|\, \lambda_n\vec{e}_n\right]$$
$$= \left[\vec{e}_1 \,|\, \vec{e}_2 \,|\, \cdots \,|\, \vec{e}_n\right]\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$
$$= P\Lambda$$

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So $AP = P\Lambda$

$$\underline{\underline{A = P\Lambda P^{-1}}}$$

(Note: A having an inverse $\Rightarrow$ P does)

Defn: This is the <u>spectral</u> <u>decomposition</u> of A.

Note: $A^2 = P\Lambda \cancel{P^{-1}}\,\cancel{P}\Lambda P^{-1}$
$= P\Lambda^2 P^{-1}$

Fact: If A is symmetric, then

$$\underset{n\times n}{A} = P \Lambda P' \qquad (P \text{ is orthogonal})$$

$$= \left[\underset{n\times 1}{\vec{e}_1} \middle| \cdots \middle| \vec{e}_n\right] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \underset{1\times n}{\vec{e}_1'} \\ \hline \vdots \\ \hline \vec{e}_n' \end{bmatrix}$$

$$= \lambda_1 \vec{e}_1 \vec{e}_1' + \lambda_2 \vec{e}_2 \vec{e}_2' + \cdots + \lambda_n \vec{e}_n \vec{e}_n'$$