Stat 571

10-24-13

(1)

Let $\vec{X}_1, \vec{X}_2, \dots, \vec{X}_n$ be

iid $N_p(\vec{\mu}, \Sigma)$

Test $H_0: \vec{\mu} = \vec{\mu}_0$ vs. $H_1: \vec{\mu} \neq \vec{\mu}_0$

Likelihood Ratio Test:

Reject H_0 if $\Lambda = \dfrac{\sup_{H_0} L(\theta)}{\sup L(\theta)} < c$

(2)

Denominator:

$$L(\theta) = f(\vec{X}_1, \vec{X}_2, \dots, \vec{X}_n)$$

$$= \prod_{i=1}^{n} f_i(\vec{X}_i) \qquad \text{(by independence)}$$

$$= \prod_{i=1}^{n} \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}(\vec{X}_i - \vec{\mu})' \Sigma^{-1} (\vec{X}_i - \vec{\mu})}$$

$$= \frac{1}{(2\pi)^{\frac{np}{2}} |\Sigma|^{n/2}} e^{-\frac{1}{2} \sum_{i=1}^{n} (\vec{X}_i - \vec{\mu})' \Sigma^{-1} (\vec{X}_i - \vec{\mu})}$$

<u>Lemma</u>: $\sum_{i=1}^{n} (\vec{x}_i - \vec{\mu})' \Sigma^{-1} (\vec{x}_i - \vec{\mu})$

$$= (n-1)\, tr[\Sigma^{-1} S] + n(\bar{X} - \vec{\mu})' \Sigma^{-1} (\bar{X} - \vec{\mu})$$

<u>Proof</u>: $(\vec{x}_i - \vec{\mu})' \Sigma^{-1} (\vec{x}_i - \vec{\mu})$

$$= tr\left[(\vec{x}_i - \vec{\mu})' \Sigma^{-1} (\vec{x}_i - \vec{\mu}) \right]$$

$$= tr\left[\Sigma^{-1} (\vec{x}_i - \vec{\mu})(\vec{x}_i - \vec{\mu})' \right]$$

Now $\sum_{i=1}^{n} (\vec{x}_i - \vec{\mu})' \Sigma^{-1} (\vec{x}_i - \vec{\mu}) = tr\left[\Sigma^{-1} \overbrace{\sum_{i=1}^{n} (\vec{x}_i - \vec{\mu})(\vec{x}_i - \vec{\mu})'}^{A} \right]$

4

$$A = \sum_{i=1}^{n} (\underbrace{\vec{x}_i - \bar{x}} + \underbrace{\bar{x} - \vec{\mu}})(\underbrace{\vec{x}_i - \bar{x}} + \underbrace{\bar{x} - \vec{\mu}})'$$

$$= \underbrace{\sum_{i=1}^{n} (\vec{x}_i - \bar{x})(\vec{x}_i - \bar{x})'}_{(n-1) S} + n(\bar{x} - \vec{\mu})(\bar{x} - \vec{\mu})'$$

(cross-products sum to 0)

$$tr(\Sigma^{-1} A) = tr\left(\Sigma^{-1} \left((n-1) S + n(\bar{x} - \vec{\mu})(\bar{x} - \vec{\mu})'\right)\right)$$

$$= (n-1)\, tr(\Sigma^{-1} S) + n\, \cancel{tr}\left((\bar{x} - \vec{\mu})' \Sigma^{-1} (\bar{x} - \vec{\mu})\right)$$

(5)

<u>Lemma</u>: Let B be a $p \times p$ symmetric positive definite matrix and let $b > 0$.

Then
$$\frac{1}{|\Sigma|^b} e^{-\frac{1}{2}tr(\Sigma^{-1}B)} \leq \frac{1}{|B|^b}(2b)^{pb} e^{-pb}$$
$\forall$ pos. def. Σ,

with equality when $\Sigma = \frac{B}{2b}$.

<u>Proof</u>: $tr(\Sigma^{-1}B) = tr(\Sigma^{-1}B^{1/2}B^{1/2}) = tr(B^{1/2}\Sigma^{-1}B^{1/2})$

(6)

Suppose $\vec{y} \neq \vec{0}$. Then $B^{1/2}\vec{y} \neq \vec{0}$

So $(B^{1/2}\vec{y})' \Sigma^{-1} (B^{1/2}\vec{y}) > 0$

$$\vec{y}' B^{1/2} \Sigma^{-1} B^{1/2} \vec{y} > 0$$

Thus $B^{1/2}\Sigma^{-1}B^{1/2}$ is also pos. def.

Also, $tr(B^{1/2}\Sigma^{-1}B^{1/2}) = \sum_{i=1}^{p} \eta_i$, where these are the eigvals of $B^{1/2}\Sigma^{-1}B^{1/2}$

⑦

And $|B^{1/2}\Sigma^{-1}B^{1/2}| = \prod_{i=1}^{p}\eta_i$

$$\shortparallel$$

$$|B^{1/2}|\,|\Sigma^{-1}|\,|B^{1/2}| = \frac{|B|}{|\Sigma|}$$

Now $\dfrac{1}{|\Sigma|^b} e^{-\frac{1}{2}tr(\Sigma^{-1}B)}$

$$= \frac{1}{|B|^b}\left(\frac{|B|}{|\Sigma|}\right)^b e^{-\frac{1}{2}\sum_{i=1}^{p}\eta_i} = \frac{\left(\prod_{i=1}^{p}\eta_i\right)^b}{|B|^b} e^{-\frac{1}{2}\sum_{i=1}^{p}\eta_i}$$

⑧

$$= \frac{1}{|B|^b}\prod_{i=1}^{p}\left(\eta_i^b e^{-\frac{1}{2}\eta_i}\right)$$

If $g(x) = x^b e^{-\frac{1}{2}x}$, find the max

$$g'(x) = x^b\left(e^{-\frac{1}{2}x}\left(-\tfrac{1}{2}\right)\right) + bx^{b-1}e^{-\frac{1}{2}x}$$

$$= \cancel{e^{-\frac{x}{2}}}x^{b-1}\left(-\frac{x}{2}+b\right) \overset{\text{set}}{=} 0$$

$$x=0, \quad x=2b$$

(9)

Thus, $\dfrac{1}{|\Sigma|^b} e^{-\frac{1}{2}\mathrm{tr}(\Sigma^{-1}B)} \leq \dfrac{1}{|B|^b} \prod_{i=1}^{p} \left((2b)^b e^{-\frac{1}{2}2b} \right)$

$$= \frac{1}{|B|^b} (2b)^{pb} e^{-pb}$$

Still need to show equality when $\Sigma = \dfrac{B}{2b}$.

$$|\Sigma|^b = \left|\frac{B}{2b}\right|^b = \left[\frac{1}{(2b)^p}|B|\right]^b = \frac{|B|^b}{(2b)^{pb}}$$

(10)

$$\Sigma^{-1} = 2b\, B^{-1}$$

$$\mathrm{tr}(\Sigma^{-1}B) = \mathrm{tr}(2b\, B^{-1}B) = 2b\,\mathrm{tr}\, I$$
$$= 2bp$$

$$\frac{1}{|\Sigma|^b} e^{-\frac{1}{2}\mathrm{tr}(\Sigma^{-1}B)} = \frac{(2b)^{pb}}{|B|^b} e^{-\frac{1}{2}2bp}$$

//

$$L(\theta) = \frac{1}{(2\pi)^{\frac{np}{2}} |\Sigma|^{\frac{n}{2}}} e^{-\frac{1}{2}\left[(n-1)\,tr(\Sigma^{-1}S) + n(\bar{x}-\vec{\mu})'\Sigma^{-1}(\bar{x}-\vec{\mu})\right]}$$

(11)

Since Σ^{-1} is pos. def.,

$(\bar{x}-\vec{\mu})'\Sigma^{-1}(\bar{x}-\vec{\mu})$ is >0

unless $\vec{\mu} = \bar{x}$.

Let $b = \frac{n}{2}$ and $B = (n-1)S$, and $\vec{\mu} = \bar{x}$

Use the last lemma

(12)

The max occurs when

$$\Sigma = \frac{B}{2b} = \frac{(n-1)S}{2(\frac{n}{2})} = S_n$$

HW #4 chap 5 # 1, 7 due Tues Nov 5