Stat 571

Extend the Pearson/Spearman model: 11-21-13

①

$$X_i = \mu_i + l_{i1} F_1 + l_{i2} F_2 + \dots + l_{im} F_m + \varepsilon_i$$

$i = 1, \dots, p$

$$\vec{X}_{p\times1} = \vec{\mu}_{p\times1} + \underset{p\times m}{L} \vec{F}_{m\times1} + \vec{\varepsilon}_{p\times1}$$

$\vec{F}$ and $\vec{\varepsilon}$ are independent

$$E[\vec{F}_{m\times1}] = \vec{0}_{m\times1}, \quad Cov(\vec{F}) = I_{m\times m}$$

②

$$E[\vec{\varepsilon}] = \vec{0}_{p\times1}, \quad Cov(\vec{\varepsilon}) = \Psi = \begin{bmatrix} \psi_1 & & 0 \\ & \ddots & \\ 0 & & \psi_p \end{bmatrix}$$

This is the <u>orthogonal</u> <u>factor</u> <u>model</u>

L is the <u>loading</u> <u>matrix</u>

l_{ij} is the <u>loading</u> of X_i on F_j

F_j is the j^{th} <u>common</u> <u>factor</u>

ε_i is the i^{th} <u>specific</u> <u>factor</u>

③

Assume the orthogonal factor model, and find $Cov(\vec{X})$.

$$\vec{X} = \vec{\mu} + L\vec{F} + \vec{\varepsilon}$$

$$Cov(\vec{X}) = Cov(L\vec{F}) + Cov(\vec{\varepsilon})$$

$$= L \underbrace{Cov(\vec{F})}_{I} L' + \Psi$$

$$Cov(\vec{X}) = LL' + \Psi$$

$$V(X_i) = \underbrace{l_{i1}^2 + l_{i2}^2 + \dots + l_{im}^2}_{i^{th}\text{ commonality}} + \underbrace{\psi_i}_{i^{th}\text{ specific variance}}$$

④

$$Cov(X_i, X_j) = l_{i1} l_{j1} + l_{i2} l_{j2} + \dots + l_{im} l_{jm}$$

$$Cov(\vec{X}, \vec{F}) = Cov(\vec{\mu} + L\vec{F} + \vec{\varepsilon}, \vec{F})$$

$$= Cov(L\vec{F}, \vec{F})$$

$$= L \underbrace{Cov(\vec{F}, \vec{F})}_{I}$$

$$= L$$

(5)

$Cov(\vec{X})$ contains p variances & $\binom{p}{2} = \frac{p(p-1)}{2}$ covariances,

for a total of $p + \frac{p^2}{2} - \frac{p}{2} = \frac{p(p+1)}{2}$ entries

We are attempting to explain those by $\underbrace{L}_{pm}$ and $\underbrace{\Psi}_{p}$,

which have $p(m+1)$ entries

(6)

Suppose that we can find a solution.

Let $T_{m \times m}$ be any orthogonal matrix $(T' = T^{-1})$

$$L^* = LT, \quad \vec{F}^* = T'\vec{F}$$

Then $\vec{X} = \vec{\mu} + L\vec{F} + \vec{\varepsilon}$

$$= \vec{\mu} + LTT'\vec{F} + \vec{\varepsilon}$$

$$= \vec{\mu} + L^*\vec{F}^* + \vec{\varepsilon}$$

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Check assumptions: $E[\vec{F}^*] = E[T'\vec{F}]$

$$= T' E[\vec{F}] = \vec{0}$$

$$Cov(\vec{F}^*) = Cov(T'\vec{F}) = T' Cov(\vec{F}) T$$

$$= T'T = I$$

Also $Cov(\vec{X}) = LL' + \Psi$

$$= (L^* T')(T L^{*\prime}) + \Psi$$

$$= L^* L^{*\prime} + \Psi$$

So the 2 solutions have the same communalities & specific variances

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How do we find the factors?

Extraction step

We are searching for a decomposition of

$$\Sigma = LL' + \Psi$$

Start with the spectral decomposition:

$$\Sigma = P \Lambda P'$$

$$= P \Lambda^{1/2} \Lambda^{1/2} P'$$

$$= (P\Lambda^{1/2})(P\Lambda^{1/2})'$$

Build the structure matrix: (9)

Recall from principal components:

$$Y_i = \vec{e}_i{}' \vec{X}$$

so

$$\vec{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_p \end{bmatrix} = \begin{bmatrix} \vec{e}_1{}'\vec{X} \\ \vdots \\ \vec{e}_p{}'\vec{X} \end{bmatrix} = \begin{bmatrix} \vec{e}_1{}' \\ \vdots \\ \vec{e}_p{}' \end{bmatrix} \vec{X}$$

$$= P'\vec{X}$$

(10)

$Corr(X_i, Y_j)$ was found by multiplying the j^{th} column of P by $\sqrt{\lambda_j}$ & dividing the i^{th} row by σ_i

So the structure matrix can be written:

$$M = V^{-1/2} P \Lambda^{1/2}$$

$$P = V^{1/2} M \Lambda^{-1/2}$$

$$\vec{Y} = P'\vec{X}, \text{ so } \vec{X} = P\vec{Y} = V^{1/2} M \Lambda^{-1/2} \vec{Y}$$

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let $\vec{Y}^* = \Lambda^{-1/2}\vec{Y}$

$$
\begin{aligned}
\mathrm{Cov}(\vec{Y}^*) &= \mathrm{Cov}(\Lambda^{-1/2}\vec{Y}) \\
&= \Lambda^{-1/2}\,\mathrm{Cov}(\vec{Y})\,\Lambda^{-1/2} \\
&= \Lambda^{-1/2}\,\Lambda\,\Lambda^{-1/2} = I
\end{aligned}
$$

So if we let $\vec{F} = \vec{Y}^*$, we have a solutions

Then $\vec{X} = V^{1/2} M \vec{Y}^*$, so $L = V^{1/2}M = P\Lambda^{1/2}$

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	1st solution F_1	1st solution F_2	2nd solution F_1	2nd solution F_2
1. Gaelic	.583	.429	.369	.594
2. English	.568	.288	.433	.467
3. History	.392	.450	.211	.558
4. Arithmetic	.740	-.273	.789	.001
5. Algebra	.724	-.211	.752	.054
6. Geometry	.595	-.132	.604	.083