The multivariate normal distribution

Stat 571
10-17-13

①

Univariate case, $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$-\infty < x < \infty$

Multivariate case:

$$f(\vec{x}) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})'\Sigma^{-1}(\vec{x}-\vec{\mu})}$$

$-\infty < x_i < \infty$
for $i = 1, \dots, p$

②

Special case:

Bivariate normal ($p=2$)

$$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \vec{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$$

$$\Sigma^{-1} = \frac{1}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \begin{bmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{bmatrix}$$

$$(\vec{x}-\vec{\mu})'\Sigma^{-1}(\vec{x}-\vec{\mu})$$

$$= \left[(x_1-\mu_1), (x_2-\mu_2)\right] \frac{1}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \begin{bmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{bmatrix} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}$$

③

$$= \frac{\sigma_{22}(x_1-\mu_1)^2 - 2\sigma_{12}(x_1-\mu_1)(x_2-\mu_2) + \sigma_{11}(x_2-\mu_2)^2}{\sigma_{11}\sigma_{22} - \sigma_{12}^2}$$

$$= \frac{\dfrac{(x_1-\mu_1)^2}{\sigma_{11}} - 2\sigma_{12}\dfrac{(x_1-\mu_1)}{\sigma_{11}}\dfrac{(x_2-\mu_2)}{\sigma_{22}} + \dfrac{(x_2-\mu_2)^2}{\sigma_{22}}}{1 - \dfrac{\sigma_{12}^2}{\sigma_{11}\sigma_{22}}}$$

$$= \frac{1}{1-\rho^2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]$$

$$= Q$$

④

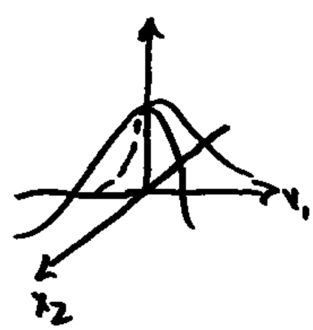
$$f(x_1, x_2) = \frac{1}{2\pi|\Sigma|^{1/2}} e^{-\frac{1}{2}Q}$$

$$|\Sigma|^{1/2} = \sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2}$$

$$= \sqrt{(\sigma_{11}\sigma_{22})\left(1 - \frac{\sigma_{12}^2}{\sigma_{11}\sigma_{22}}\right)}$$

$$= \sigma_1\sigma_2\sqrt{1-\rho^2}$$

$$\boxed{f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2}Q}}$$



⑤

The contours of the graph are found by setting $f(x_1, x_2) = k$,

which is equivalent to setting Q = constant

$$(\vec{x}-\vec{\mu})' \Sigma^{-1} (\vec{x}-\vec{\mu}) = c^2$$

These are ellipses centered at $\vec{\mu}$,

and whose axes are the eigenvectors of Σ, with ½-lengths $c\sqrt{\lambda_1}$ and $c\sqrt{\lambda_2}$,

where λ_1 & λ_2 are the eigenvalues of Σ.

⑥

Special case of the bivariate normal: $\rho = 0$

$$f(x_1, x_2) = \frac{1}{2\pi \sigma_1 \sigma_2} e^{-\frac{1}{2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2} \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2},$$

so X_1 and X_2 are independent.

Back to the $\rho \neq 0$ case:

Find the marginal density of X_1.

(7)

$$f(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2}\frac{1}{1-\rho^2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]} dx_2$$

Let $z_1 = \frac{x_1-\mu_1}{\sigma_1}$, $z_2 = \frac{x_2-\mu_2}{\sigma_2}$

$$\frac{dz_2}{dx_2} = \frac{1}{\sigma_2} \qquad dx_2 = \sigma_2 dz_2$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{1}{1-\rho^2}\left[z_1^2 - 2\rho z_1 z_2 + z_2^2\right]} \sigma_2 dz_2$$

(8)

$$z_2^2 - 2\rho z_1 z_2 + z_1^2$$

$$= z_2^2 - 2\rho z_1 z_2 + \rho^2 z_1^2 + z_1^2 - \rho^2 z_1^2$$

$$= (z_2 - \rho z_1)^2 + z_1^2(1-\rho^2)$$

Now $$f(x_1) = \frac{1}{2\pi\sigma_1\cancel{\sigma_2}\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{1}{1-\rho^2}\left[(z_2-\rho z_1)^2 + z_1^2(1-\rho^2)\right]} \cancel{\sigma_2} dz_2$$

$$= \frac{e^{-\frac{1}{2}z_1^2}}{2\pi\sigma_1\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{1}{1-\rho^2}(z_2-\rho z_1)^2} dz_2$$

⑨

$$= \frac{e^{-\frac{1}{2}z_1^2}\,\cancel{\sqrt{1-\rho^2}}\sqrt{2\pi}}{2\pi\,\sigma_1\cancel{\sqrt{1-\rho^2}}} \underbrace{\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\left(\frac{z_2-\rho z_1}{\sqrt{1-\rho^2}}\right)^2}}{\sqrt{2\pi}\sqrt{1-\rho^2}}\,dz_2}_{1}$$

$$= \frac{e^{-\frac{1}{2}z_1^2}}{\sqrt{2\pi}\;\sigma_1}$$

General result: If $\vec{X} \sim N_p(\vec{\mu}, \Sigma)$,

then $X_i \sim N(\mu_i, \sigma_{ii})$

⑩

<u>Theorem</u>: If $\vec{X} \sim N_p(\vec{\mu}, \Sigma)$,

then $\vec{a}'\vec{X} \sim N(\vec{a}'\vec{\mu}, \vec{a}'\Sigma\vec{a})$

<u>Theorem</u>. If $\vec{a}'\vec{X} \sim N(\vec{a}'\vec{\mu}, \vec{a}'\Sigma\vec{a})$ $\underline{\underline{\forall \vec{a}}}$,

then $\vec{X} \sim N_p(\vec{\mu}, \Sigma)$.

<u>Theorem</u>: If $\vec{X}$ is $N_p(\vec{\mu}, \Sigma)$ and

A is a $q\times p$ matrix of constants, then

$A\vec{X} \sim N_q(A\vec{\mu}, A\Sigma A')$

HW #3 due Oct 24

4. 1,3,4,6