More on the multivariate normal distr.

Stat 571

10-22-13

①

Theorem: Suppose $\vec{X} \sim N_p(\vec{\mu}, \Sigma)$

and $\vec{X} = \left[\begin{array}{c} \vec{X}_1 \\ \hline \vec{X}_2 \end{array}\right]_{p\times 1}$, $\vec{X}_1$: $q\times 1$, then

$$\vec{X}_1 \sim N_q(\vec{\mu}_1, \Sigma_{11}), \text{ where}$$

$$\vec{\mu} = \left[\begin{array}{c} \vec{\mu}_1 \\ \hline \vec{\mu}_2 \end{array}\right] \text{ and } \Sigma = \left[\begin{array}{c|c} \Sigma_{11} & \Sigma_{12} \\ \hline \Sigma_{21} & \Sigma_{22} \end{array}\right].$$

$\vec{\mu}_1$: $q\times 1$, Σ_{11}: $q\times q$

②

Example: $\vec{X} = N_5(\vec{\mu}, \Sigma)$

Find the distribution of (X_1, X_3)

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} \qquad \text{Let } \vec{Y} = \left[\begin{array}{c} X_1 \\ X_3 \\ \hline X_2 \\ X_4 \\ X_5 \end{array}\right]$$

$$\vec{Y} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} = A\vec{X}$$

So $\vec{Y} \sim N_5(A\vec{\mu}, A\Sigma A')$

$$A\vec{\mu} = \begin{bmatrix} \mu_1 \\ \mu_3 \\ \hline \mu_2 \\ \mu_4 \\ \mu_5 \end{bmatrix} \qquad A\Sigma A' = \left[\begin{array}{cc|ccc} \sigma_{11} & \sigma_{13} & \sigma_{12} & \sigma_{14} & \sigma_{15} \\ \sigma_{13} & \sigma_{33} & \sigma_{32} & \sigma_{34} & \sigma_{35} \\ \hline \sigma_{12} & \sigma_{32} & \sigma_{22} & & \\ \sigma_{14} & \sigma_{34} & & \sigma_{44} & \\ \sigma_{15} & \sigma_{35} & & & \sigma_{55} \end{array}\right]$$

So $\begin{bmatrix} X_1 \\ X_3 \end{bmatrix} \sim N_2\left(\begin{pmatrix} \mu_1 \\ \mu_3 \end{pmatrix}, \begin{pmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{13} & \sigma_{33} \end{pmatrix}\right)$

<u>Theorem</u>: let $\vec{X} = \begin{bmatrix} \vec{X}_1 \\ \hline \vec{X}_2 \end{bmatrix} \sim N_p(\vec{\mu}, \Sigma)$.

Then $\vec{X}_1$ and $\vec{X}_2$ are independent iff $\Sigma_{12} = 0$.

④

<u>Proof</u>. ($\Rightarrow$) Assume $\vec{X}_1$ & $\vec{X}_2$ are independent.

$$\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_q \\ \hline X_{q+1} \\ \vdots \\ X_p \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} X_1 \\ \vdots \\ X_q \end{matrix}}\right\} \vec{X}_1 \\ \left.\vphantom{\begin{matrix} X_1 \\ \vdots \\ X_q \end{matrix}}\right\} \vec{X}_2 \end{matrix}$$

Independence $\Rightarrow$ X_i and X_j are independent $\forall i = 1, \dots, q$ and $\forall j = q+1, \dots, p$

$\Rightarrow \sigma_{ij} = 0 \quad \forall i = 1, \dots, q$ and $\forall j = q+1, \dots, p$

$\Rightarrow \Sigma_{12} = 0$.

($\Leftarrow$) Assume $\Sigma_{12} = 0$ ⑤

So $\Sigma = \left[\begin{array}{c|c} \Sigma_{11} & 0 \\ \hline 0 & \Sigma_{22} \end{array}\right]_{p\times p}$

Then $\Sigma^{-1} = \left[\begin{array}{c|c} \Sigma_{11}^{-1} & 0 \\ \hline 0 & \Sigma_{22}^{-1} \end{array}\right]_{p\times p}$

and $(\vec{x}-\vec{\mu})'\Sigma^{-1}(\vec{x}-\vec{\mu}) =$

$$[(\vec{x}_1-\vec{\mu}_1)' \mid (\vec{x}_2-\vec{\mu}_2)'] \left[\begin{array}{c|c} \Sigma_{11}^{-1} & 0 \\ \hline 0 & \Sigma_{22}^{-1} \end{array}\right] \left[\begin{array}{c} \vec{x}_1-\vec{\mu}_1 \\ \hline \vec{x}_2-\vec{\mu}_2 \end{array}\right]$$

⑥

$$= (\vec{x}_1-\vec{\mu}_1)'\Sigma_{11}^{-1}(\vec{x}_1-\vec{\mu}_1) + (\vec{x}_2-\vec{\mu}_2)'\Sigma_{22}^{-1}(\vec{x}_2-\vec{\mu}_2)$$

So $f(\vec{x})$ will factor into $f_1(\vec{x}_1)\cdot f_2(\vec{x}_2)$

$\Rightarrow \vec{x}_1$ & $\vec{x}_2$ are independent.

<u>Theorem</u>: Let $\vec{X} = \left[\begin{array}{c} \vec{X}_1 \\ \hline \vec{X}_2 \end{array}\right]^{q\times 1} \sim N_p(\vec{\mu}, \Sigma)$.

Then the conditional distribution of $\vec{X}_1 | \vec{x}_2$ is

$$N_q(\vec{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\vec{x}_2-\vec{\mu}_2),\ \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$$

⑦

Proof: Let $A = \begin{bmatrix} I_{q\times q} & -\Sigma_{12}\Sigma_{22}^{-1} \\ \hline O_{(p-q)\times q} & I_{(p-q)\times(p-q)} \end{bmatrix}_{p\times p}$

So $A(\vec{x}-\vec{\mu}) = \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ \hline 0 & I \end{bmatrix} \begin{bmatrix} \vec{x}_1 - \vec{\mu}_1 \\ \hline \vec{x}_2 - \vec{\mu}_2 \end{bmatrix}$

$= \begin{bmatrix} (\vec{x}_1 - \vec{\mu}_1) - \Sigma_{12}\Sigma_{22}^{-1}(\vec{x}_2 - \vec{\mu}_2) \\ \hline \vec{x}_2 - \vec{\mu}_2 \end{bmatrix}$

⑧

and $A\Sigma A'$

$= \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ \hline O & I \end{bmatrix} \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \hline \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} I & O \\ \hline -\Sigma_{22}^{-1}\Sigma_{21} & I \end{bmatrix}$

$= \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & O \\ \hline O & \Sigma_{22} \end{bmatrix}$

$\therefore (\vec{x}_1 - \vec{\mu}_1) - \Sigma_{12}\Sigma_{22}^{-1}(\vec{x}_2 - \vec{\mu}_2)$ is independent of $\vec{x}_2 - \vec{\mu}_2$

$E[A(\vec{X}-\vec{\mu})] = \vec{0}$

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Also,

$$(\vec{X}_1-\vec{\mu}_1) - \Sigma_{12}\Sigma_{22}^{-1}(\vec{X}_2-\vec{\mu}_2) \sim N_q\left(\vec{0},\ \Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right)$$

So if $\vec{x}_2$ is an observation of $\vec{X}_2$,

then
$$\vec{X}_1 | \vec{x}_2 \sim N_q\left(\vec{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\vec{x}_2-\vec{\mu}_2),\ \Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right)$$

Special case: bivariate normal $\vec{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

Find the distribution of $X_1|x_2$

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By the theorem,

$$X_1|x_2 \sim N\Big(\underbrace{\mu_1 + \sigma_{12}\frac{1}{\sigma_{22}}(x_2-\mu_2)}_{\mu_1 + \frac{\sigma_{12}}{\sigma_2^2}(x_2-\mu_2)},\ \underbrace{\sigma_{11} - \sigma_{12}\frac{1}{\sigma_{22}}\sigma_{21}}_{\sigma_1^2 - \frac{\sigma_{12}^2}{\sigma_2^2}}\Big)$$

$$\sigma_1^2 - \frac{\sigma_{12}^2}{\sigma_2^2} = \sigma_1^2\left(1 - \frac{\sigma_{12}^2}{\sigma_1^2\sigma_2^2}\right) = \sigma_1^2(1-\rho^2)$$

(11)

Theorem: Let $\vec{X} \sim N_p(\vec{\mu}, \Sigma)$

Then $(\vec{X}-\vec{\mu})' \Sigma^{-1} (\vec{X}-\vec{\mu}) \sim \chi^2_p$

Proof: Write the spectral decomposition of Σ:

$$\Sigma = \sum_{i=1}^{p} \lambda_i \vec{e}_i \vec{e}_i'$$

so

$$\Sigma^{-1} = \sum_{i=1}^{p} \frac{1}{\lambda_i} \vec{e}_i \vec{e}_i'$$

(12)

Then $(\vec{X}-\vec{\mu})' \Sigma^{-1} (\vec{X}-\vec{\mu})$

$$= (\vec{X}-\vec{\mu})' \left[\sum_{i=1}^{p} \frac{1}{\lambda_i} \vec{e}_i \vec{e}_i' \right] (\vec{X}-\vec{\mu})$$

$$= \sum_{i=1}^{p} \frac{1}{\lambda_i} (\vec{X}-\vec{\mu})' \vec{e}_i \vec{e}_i' (\vec{X}-\vec{\mu})$$

Let $Z_i = \dfrac{\vec{e}_i'(\vec{X}-\vec{\mu})}{\sqrt{\lambda_i}}$

$$= \sum_{i=1}^{p} Z_i^2$$

Need to show $Z_1, \dots, Z_p$ are iid $N(0,1)$

(13)

Let $$\vec{Z} = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_p \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{\lambda_1}} \vec{e}_1{}' \\ \vdots \\ \frac{1}{\sqrt{\lambda_p}} \vec{e}_p{}' \end{bmatrix}}_{A} (\vec{X} - \vec{\mu})$$

i.e., $\vec{Z} = A(\vec{X} - \vec{\mu})$

so $\vec{Z} \sim N_p(\vec{0}, A \Sigma A')$

(14)

$$A \Sigma A' = \begin{bmatrix} \frac{1}{\sqrt{\lambda_1}} \vec{e}_1{}' \\ \hline \vdots \\ \hline \frac{1}{\sqrt{\lambda_p}} \vec{e}_p{}' \end{bmatrix} \sum_{i=1}^{p} \lambda_i \vec{e}_i \vec{e}_i{}' \left[\frac{1}{\sqrt{\lambda_1}} \vec{e}_1 \middle| \cdots \middle| \frac{1}{\sqrt{\lambda_p}} \vec{e}_p \right]$$

$$= \begin{bmatrix} \frac{1}{\sqrt{\lambda_1}} \vec{e}_1{}' \\ \hline \vdots \\ \hline \frac{1}{\sqrt{\lambda_p}} \vec{e}_p{}' \end{bmatrix} \left[\sqrt{\lambda_1} \vec{e}_1 \middle| \cdots \middle| \sqrt{\lambda_p} \vec{e}_p \right]$$

$$= \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{p \times p} = I$$

$\therefore \vec{Z} \sim N_p(\vec{0}, I)$

So $Z_1, \ldots, Z_p$ are indp, each $N(0,1)$

$$\therefore \sum_{i=1}^{p} Z_i^2 \sim \chi_p^2$$