Stat 571
11/14/13
①

Midterm : 45 points max

$\bar{x} = 32.3 \qquad n = 26$

1	8 9
2	0 1 7 8 9 9
3	0 0 0 1 1 1 2 6 6 7 8
4	0 0 0 1 1 2 3

<u>Principal Components</u> Let $\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_p \end{bmatrix}$ have

covariance Σ, with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p \geq 0$

and orthonormal eigenvectors $\vec{e}_1, \ldots, \vec{e}_p$. ②

Let $Y_1 = l_{11} X_1 + \ldots + l_{p1} X_p$

$= \vec{l}_1{}' \vec{X}$

Let $Y_2 = l_{12} X_1 + \ldots + l_{p2} X_p$

$= \vec{l}_2{}' \vec{X}$, etc.

$Y_p = \vec{l}_p{}' \vec{X}$

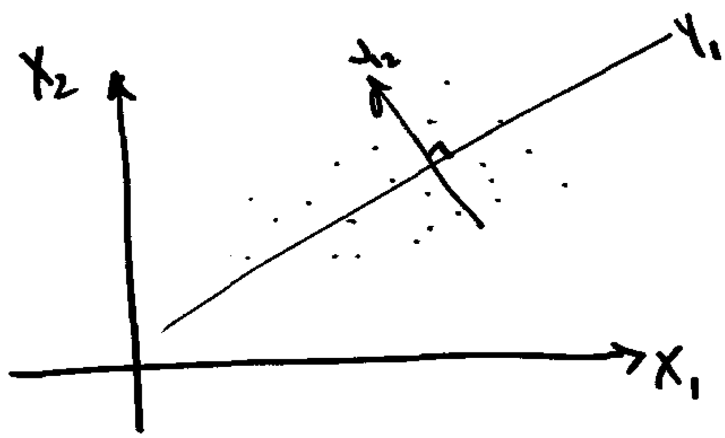
③

Defn: The 1^{st} principal component is the linear combination Y_1 that maximizes $V(Y_1)$, subject to $\vec{l}_1{}'\vec{l}_1 = 1$.

The i^{th} principal component is the linear combination Y_i that maximizes $V(Y_i)$, subject to $\vec{l}_i{}'\vec{l}_i = 1$ and

$Cov(Y_i, Y_k) = 0 \quad \forall\, k < i$

④



Recall the maximization lemma:

$$\max_{\vec{x} \neq \vec{0}} \frac{\vec{x}'B\vec{x}}{\vec{x}'\vec{x}} = \lambda_1 \text{ , attained at } \vec{e}_1$$

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$$\max_{\vec{x} \perp \vec{e}_1, \dots, \vec{e}_k} \frac{\vec{x}' B \vec{x}}{\vec{x}' \vec{x}} = \lambda_{k+1}, \text{ attained at } \vec{e}_{k+1}$$

We want to maximize $V(Y_1)$

$$V(Y_1) = V(\vec{l}_1' X) = \vec{l}_1' \Sigma \vec{l}_1$$

$$= \frac{\vec{l}_1' \Sigma \vec{l}_1}{\vec{l}_1' \vec{l}_1}$$

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So, using the lemma, $\vec{l}_1 = \vec{e}_1$,

and $V(Y_1) = \lambda_1$

Result: The 1^{st} prin. comp. is $Y_1 = \vec{e}_1' \vec{X}$

How about the i^{th} component?

We need to maximize $V(Y_i) = \dfrac{\vec{l}_i' \Sigma \vec{l}_i}{\vec{l}_i' \vec{l}_i}$,

subject to $Cov(Y_i, Y_k) = 0 \ \forall \ k < i$

$$Cov(Y_i, Y_k) = Cov(\vec{l}_i{}'\vec{X}, \vec{l}_k{}'\vec{X})$$

$$= \vec{l}_i{}' \Sigma \vec{l}_k$$

Use induction. Know $\vec{l}_1 = \vec{e}_1$

Assume $\vec{l}_k = \vec{e}_k$ for $k = 1, \dots, i-1$

$$= \vec{l}_i{}' \Sigma \vec{e}_k$$

$$= \vec{l}_i{}' \lambda_k \vec{e}_k = \lambda_k \vec{l}_i{}' \vec{e}_k$$

That is, $Cov(Y_i, Y_k) = 0$ iff $\vec{l}_i \perp \vec{e}_k$

Then apply the 2nd part of maximization lemma, 8

and $\vec{l}_i = \vec{e}_i$.

$$\boxed{\text{Result: The } i^{th} \text{ prin. comp. is } Y_i = \vec{e}_i{}'\vec{X}, \text{ and } V(Y_i) = \lambda_i}$$

<u>Result</u>: $\sum_{i=1}^{p} V(Y_i) = \sum_{i=1}^{p} V(X_i)$

Proof: $\sum_{i=1}^{p} V(X_i) = \operatorname{tr} \Sigma$
$$= \operatorname{tr}(P \Lambda P')$$
$$= \operatorname{tr}(\Lambda P' P)$$
$$= \operatorname{tr} \Lambda = \sum_{i=1}^{p} \lambda_i = \sum_{i=1}^{p} V(Y_i)$$

The proportion of total variance explained by the i^{th} P.C. is $\dfrac{\lambda_i}{\sum_{i=1}^{p} \lambda_i}$

The cumulative proportion explained by $Y_1, \dots, Y_i$ is $\dfrac{\lambda_1 + \dots + \lambda_i}{\lambda_1 + \dots + \lambda_p}$.

Find the correlation between Y_i and X_k

$$Cov(Y_i, X_k) = Cov(\vec{e}_i{}' \vec{X}, \vec{u}_k{}' \vec{X}),$$

where $\vec{u}_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ← index k

$$= \vec{e}_i{}' \Sigma \vec{u}_k$$
$$= \vec{u}_k{}' \Sigma \vec{e}_i = \vec{u}_k{}' \lambda_i \vec{e}_i$$
$$= \lambda_i e_{ki}$$

↑ k^{th} entry in i^{th} eigenvector

$$\text{Corr}(Y_i, X_k) = \frac{\text{Cov}(Y_i, X_k)}{\sqrt{V(Y_i)\,V(X_k)}}$$

$$= \frac{\lambda_i e_{ki}}{\sqrt{\lambda_i \;\sigma_{kk}}} = \frac{e_{ki}\sqrt{\lambda_i}}{\sigma_k}$$

The <u>structure</u> <u>matrix</u> is the $p \times p$ matrix of these correlations.

The <u>pattern</u> <u>matrix</u> is the $p \times p$ matrix of coefficients

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Example

let $S = \begin{bmatrix} 256 & 64 & 0 \\ 64 & 100 & 0 \\ 0 & 0 & 25 \end{bmatrix}$ $\quad tr(S) = 381$

$\lambda_1 = 279$	$\lambda_2 = 77$	$\lambda_3 = 25$
$\vec{e}_1 = \begin{bmatrix} .94 \\ .34 \\ 0 \end{bmatrix}$	$\vec{e}_2 = \begin{bmatrix} .34 \\ -.94 \\ 0 \end{bmatrix}$	$\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Then $Y_1 = .94\,X_1 + .34\,X_2$ $\quad V(Y_1) = 279$

$Y_2 = .34\,X_1 - .94\,X_2$ $\quad V(Y_2) = 77$

$Y_3 = X_3$ $\quad V(Y_3) = 25$

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$$\rho_{Y_i, X_k} = \frac{e_{ki}\sqrt{\lambda_i}}{\sigma_k}$$

$\frac{.94\sqrt{2.79}}{16}$

<u>Structure:</u>

	X_1	X_2	X_3
Y_1	.98	.57	0
Y_2	.19	-.82	0
Y_3	0	0	1