Stat 571

10-15-13

①

Defn: D = the matrix of <u>deviations</u>

$$= \begin{bmatrix} x_{11}-\bar{x}_1 & \cdots & x_{1p}-\bar{x}_p \\ \vdots & & \vdots \\ x_{n1}-\bar{x}_1 & \cdots & x_{np}-\bar{x}_p \end{bmatrix}$$

$$= X - \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_p \\ \vdots & \vdots & & \vdots \\ \bar{x}_1 & \bar{x}_2 & & \bar{x}_p \end{bmatrix}$$

$$= X - \underset{n\times 1}{\vec{1}}\,\bar{X}'$$

②

Also, write $D = [\vec{d}_1 | \vec{d}_2 | \cdots | \vec{d}_p]$

Consider $\vec{d}_i{}'\vec{d}_j = \sum_{k=1}^{n}(x_{ki}-\bar{x}_i)(x_{kj}-\bar{x}_j)$

$$= (n-1)s_{ij}$$

Also $\vec{d}_i{}'\vec{d}_i = (n-1)s_{ii} = (n-1)s_i^2$

Notice that $\vec{d}_i{}'\vec{d}_j$ would be the $(i,j)^{th}$ entry in $D'D$

So $D'D = (n-1)S$ ③

<u>Theorem</u>: $|S| = 0$ iff the columns of D are linearly dependent.

<u>Proof</u>: ($\Leftarrow$) Suppose $\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_p$ are linearly dependent.

So $\exists\ c_1, \ldots, c_p$ not all 0 such that

$$c_1 \vec{d}_1 + c_2 \vec{d}_2 + \cdots + c_p \vec{d}_p = \vec{0}$$

④

$$[\vec{d}_1 | \vec{d}_2 | \cdots | \vec{d}_p] \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix} = \vec{0}$$

$$D\vec{c} = \vec{0}$$

Then $(n-1)S\vec{c} = (n-1)D'D\vec{c} = \vec{0}$

$$S\vec{c} = \vec{0},$$

which means that the columns of S are linearly dependent, so $|S| = 0$.

($\Rightarrow$) Suppose $|S| = 0$.

Then the columns of S are lin. dep

So $\exists\ \vec{c} \neq \vec{0}$ such that $S\vec{c} = \vec{0}$.

$$(n-1)S\vec{c} = \vec{0}$$

$$D'D\vec{c} = \vec{0}$$

$$\vec{c}\,'D'D\vec{c} = \vec{c}\,'\vec{0} = 0$$

$$\overset{\shortparallel}{(D\vec{c})'D\vec{c}}$$

$$\|D\vec{c}\|^2 = 0 \Rightarrow D\vec{c} = \vec{0}$$

So D has linearly dependent columns.

6

Defn: $|S|$ is the generalized sample variance

$tr(S)$ is the total sample variance

Some useful properties of determinants and traces

① $|AB| = |A|\,|B|$ if A & B are square & have same dimension

② $|A^{-1}| = \dfrac{1}{|A|}$

(3) $|cA| = c^k |A|$ if A is $k \times k$

(4) If A is diagonal, then $|A| = \prod_{i=1}^{k} a_{ii}$

(5) $tr(cA) = c\, tr(A)$

(6) $tr(A+B) = tr(A) + tr(B)$

(7) $tr(AB) = tr(BA)$ if AB and BA are both defined

$A_{n \times p}$ $B_{p \times n}$

(8) $tr(B^{-1}AB) = tr(A)$

(9) $tr(AA') = \sum_i \sum_j a_{ij}^2$

Consider the spectral decomposition of A:

$$A = P \Lambda P^{-1}$$

$$|A| = |P|\ |\Lambda|\ |P^{-1}|$$
$$= |\Lambda| \quad \text{since } |P^{-1}| = \frac{1}{|P|}$$

$$|A| = \prod_{i=1}^{k} \lambda_i$$ ⑨

$$\begin{aligned} tr(A) &= tr(P \Lambda P^{-1}) \\ &= tr(\Lambda) = \sum_{i=1}^{k} \lambda_i \end{aligned}$$

Example $S_1 = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$, $S_2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$, $S_3 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

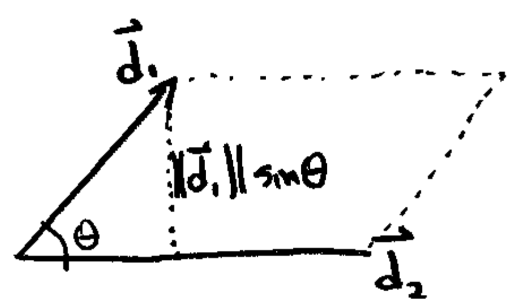
$|S_1| = 9$, $|S_2| = 9$, $|S_3| = 9$

$tr(S_1) = 10$, $tr(S_2) = 10$, $tr(S_3) = 6$

⑩

Consider the 2-dimensional case

D has 2 columns, $\vec{d}_1$ and $\vec{d}_2$



Area of the parallelogram $= \|\vec{d}_2\| \, \|\vec{d}_1\| \sin\theta$

$$= \sqrt{\vec{d}_1' \vec{d}_1} \sqrt{\vec{d}_2' \vec{d}_2} \sqrt{1 - \cos^2\theta}$$

$$= \sqrt{(n-1)s_1^2}\sqrt{(n-1)s_2^2}\sqrt{1-\cos^2\theta}$$ (11)

$$\left[\begin{array}{l} \text{Also } \vec{d}_1{}'\vec{d}_2 = \|\vec{d}_1\|\,\|\vec{d}_2\| \cos\theta \\ \qquad \| \qquad\qquad\qquad \| \\ (n-1)s_{12} \qquad \sqrt{(n-1)s_1^2}\sqrt{(n-1)s_2^2}\cos\theta \\ \cos\theta = \dfrac{s_{12}}{s_1 s_2} = r_{12} \end{array}\right]$$

$$\text{Area of } /\!/\text{-gram} = (n-1)s_1 s_2\sqrt{1-r_{12}^2}$$

$$= (n-1)\sqrt{s_1^2 s_2^2(1-r_{12}^2)}$$

$$= (n-1)\sqrt{s_{11}s_{22} - \frac{s_{12}^2}{s_{11}s_{22}}(s_{11}s_{22})}$$ (12)

$$= (n-1)\sqrt{s_{11}s_{22} - s_{12}^2}$$

$$= (n-1)\sqrt{|S|}$$

Recap: In the 2-dimensional case,

$$\text{Area} = (n-1)\sqrt{|S|}$$

or $$\text{Area}^2 = (n-1)^2|S|$$

<u>Fact</u>: In $\mathbb{R}^p$, $\text{Volume}^2 = (n-1)^p|S| = |(n-1)S|$

(13)

<u>Theorem</u>: If $n \leq p$, then $|S| = 0$.

<u>Proof</u>: Notice that each column of D sums to 0.
So the rows of D sum to a $\vec{0}'$ (row vector)

So the rows of D are linearly dependent

So the rank of D is $< n$.

Now, if $n \leq p$, then $\text{rank}(D) < n \leq p$
$\Rightarrow \text{rank}(D) < p$.

(14)

So the columns of D are linearly dependent.

So, by our theorem, $|S| = 0$.