Stat 571

10-3-13

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Note: If $A = P \Lambda P^{-1}$

Then $A^{-1} = P \Lambda^{-1} P^{-1}$

<u>Quadratic forms</u>

If A is symmetric, then

$$\underset{1 \times n}{\vec{x}'}\ \underset{n \times n}{A}\ \underset{n \times 1}{\vec{x}}$$ is a quadratic form

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If $\vec{x}' A \vec{x} \geq 0 \quad \forall \vec{x}$,

then we refer to A and

$\vec{x}' A \vec{x}$ as <u>non-negative definite</u>.

If $\vec{x}' A \vec{x} > 0 \quad \forall \vec{x} \neq \vec{0}$,

then A and $\vec{x}' A \vec{x}$ are

<u>positive definite</u>

③

Defn: A random vector (matrix) is a vector (matrix) whose elements are random variables.

Defn: Let $\vec{X}$ be a random vector,

i.e. $\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$.

Then $E(\vec{X}) = \begin{bmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_n) \end{bmatrix}$.

Similar defn for random matrices.

④

Properties:

$$E[c\vec{X}] = E\begin{bmatrix} cX_1 \\ \vdots \\ cX_n \end{bmatrix} = \begin{bmatrix} cE(X_1) \\ \vdots \\ cE(X_n) \end{bmatrix} = cE[\vec{X}]$$

$$E[\vec{X} + \vec{Y}] = E\begin{bmatrix} X_1 + Y_1 \\ \vdots \\ X_n + Y_n \end{bmatrix} = \begin{bmatrix} E(X_1) + E(Y_1) \\ \vdots \\ E(X_n) + E(Y_n) \end{bmatrix}$$

$$= E[\vec{X}] + E[\vec{Y}]$$

⑤

Let A be a matrix of constant.

Consider $E[A\vec{X}]$

$$= E\left[\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}\right]$$

$$= E\begin{bmatrix} \sum_1^n a_{1i} X_i \\ \vdots \\ \sum_1^n a_{ni} X_i \end{bmatrix} = \begin{bmatrix} \sum a_{1i} E(X_i) \\ \vdots \\ \sum a_{ni} E(X_i) \end{bmatrix} = A E(\vec{X})$$

⑥

Similarly, $E[\vec{X} B] = E[\vec{X}] B$

if B is a matrix of constants.

Recall, for random variables

$$\mu = E(X)$$

$$\sigma^2 = V(X) = Var(X)$$

$$= E[(X-\mu)^2]$$

$$\sigma_{XY} = Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$

For random vectors, (7)

$$\vec{\mu} = E[\vec{X}]$$

Consider $E\left[\underset{n\times 1}{(\vec{X} - \vec{\mu})} \underset{1\times n}{(\vec{X} - \vec{\mu})'} \right]$

$$= E\left[\begin{bmatrix} X_1 - \mu_1 \\ \vdots \\ X_n - \mu_n \end{bmatrix} [X_1 - \mu_1, \ldots, X_n - \mu_n] \right]$$

(8)

$$= E \begin{bmatrix} (X_1 - \mu_1)^2 & & (X_i - \mu_i)(X_j - \mu_j) \\ & \ddots & \\ & & (X_n - \mu_n)^2 \end{bmatrix}$$

$$= \underset{n\times n}{\begin{bmatrix} \sigma_1^2 & & \sigma_{ij} \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix}} = \Sigma =$$

variance-covariance matrix of $\vec{X}$.

Defn: The <u>standard deviation matrix</u> of $\vec{X}$

is $$V^{\frac{1}{2}} = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}$$

Defn: The <u>correlation matrix</u> of $\vec{X}$

is $$\rho = \begin{bmatrix} 1 & & \frac{\sigma_{ij}}{\sigma_i \sigma_j} \\ & \ddots & \\ & & 1 \end{bmatrix}$$

<u>Fact</u>: $$\rho_{n \times n} = (V^{\frac{1}{2}})^{-1} \Sigma (V^{\frac{1}{2}})^{-1}$$

Some properties:

Consider a linear combination of random variables

$$c_1 X_1 + c_2 X_2 + \dots + c_n X_n$$

$$= [c_1 \ c_2 \ \cdots \ c_n] \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} = \vec{c}\,' \vec{X}$$

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So $E[\vec{c}\,'\vec{X}] = \vec{c}\,'E[\vec{X}] = \vec{c}\,'\vec{\mu}$

But $V[\vec{c}\,'\vec{X}] = V[c_1X_1 + \dots + c_nX_n]$

$$= c_1^2 V[X_1] + \dots + c_n^2 V[X_n] + \sum_{i \neq j}\sum c_i c_j \mathrm{Cov}(X_i, X_j)$$

Also $\mathrm{Cov}(\vec{c}\,'\vec{X}) = E\left[(\vec{c}\,'\vec{X} - E(\vec{c}\,'\vec{X}))(\vec{c}\,'\vec{X} - E(\vec{c}\,'\vec{X}))'\right]$

$$= E\left[(\vec{c}\,'\vec{X} - \vec{c}\,'\vec{\mu})(\vec{c}\,'\vec{X} - \vec{c}\,'\vec{\mu})'\right]$$

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$$= E\left[\vec{c}\,'(\vec{X} - \vec{\mu})\,(\vec{c}\,'(\vec{X} - \vec{\mu}))'\right]$$

$$= E\left[\vec{c}\,'(\vec{X} - \vec{\mu})(\vec{X} - \vec{\mu})'\vec{c}\right]$$

$$= \vec{c}\,' \Sigma \vec{c}$$

Fact: If C is a matrix of constants,

then $\mathrm{Cov}(C\vec{X}) = C \Sigma C'$

HW#1 p.103 2, 8, 9, 27, 30 due 10-10