Maximization Lemmas

Stat 571
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I. Cauchy-Schwarz Lemma

$$(\vec{x}'\vec{y})^2 \le (\vec{x}'\vec{x})(\vec{y}'\vec{y}),$$

with equality iff $\vec{x} = c\vec{y}$ for some c.

Proof. If $\vec{x} = \vec{0}$ or $\vec{y} = \vec{0}$, the result holds.

Suppose $\vec{x} \ne c\vec{y}$ for all c.

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Then, $\forall c$, $\vec{x} - c\vec{y} \ne \vec{0}$

So $(\vec{x} - c\vec{y})'(\vec{x} - c\vec{y}) > 0 \quad \forall c$

$$0 < \vec{x}'\vec{x} - c\vec{x}'\vec{y} - c\vec{y}'\vec{x} + c^2\vec{y}'\vec{y}$$

$$= c^2\vec{y}'\vec{y} - 2c\vec{x}'\vec{y} + \vec{x}'\vec{x}$$

$$= \vec{y}'\vec{y}\left[c^2 - 2c\frac{\vec{x}'\vec{y}}{\vec{y}'\vec{y}} + \frac{\vec{x}'\vec{x}}{\vec{y}'\vec{y}}\right]$$

③

$$= \vec{y}'\vec{y}\left[c^2 - 2c\frac{\vec{x}'\vec{y}}{\vec{y}'\vec{y}} + \left(\frac{\vec{x}'\vec{y}}{\vec{y}'\vec{y}}\right)^2\right] + \vec{x}'\vec{x} - \frac{(\vec{x}'\vec{y})^2}{\vec{y}'\vec{y}}$$

$$0 < \vec{y}'\vec{y}\left[c - \frac{\vec{x}'\vec{y}}{\vec{y}'\vec{y}}\right]^2 + \vec{x}'\vec{x} - \frac{(\vec{x}'\vec{y})^2}{\vec{y}'\vec{y}} \qquad \forall c$$

True $\forall c \Rightarrow$ True for $c = \dfrac{\vec{x}'y}{\vec{y}'\vec{y}}$

④

$$0 < \vec{x}'\vec{x} - \frac{(\vec{x}'\vec{y})^2}{\vec{y}'\vec{y}}$$

$$(\vec{x}'\vec{y})^2 < (\vec{x}'\vec{x})(\vec{y}'\vec{y}) \quad //$$

(Equality portion of proof is trivial)

II. Extended Cauchy-Schwarz Inequality

Let A be symmetric, positive definite.

Then $(\vec{x}'\vec{y})^2 \leq (\vec{x}'A\vec{x})(\vec{y}'A^{-1}\vec{y})$ with equality iff $\vec{x} = cA^{-1}\vec{y}$ for some c.

Proof: If $\vec{x} = \vec{0}$ or $\vec{y} = \vec{0}$, then the result holds.

Let $\vec{w} = A^{1/2}\vec{x}$ and $\vec{z} = A^{-1/2}\vec{y}$

Apply the Cauchy-Schwarz lemma to $\vec{w}$ and $\vec{z}$.

$$(\vec{w}'\vec{z})^2 \le (\vec{w}'\vec{w})(\vec{z}'\vec{z}) \quad \text{with equality iff } \vec{w} = c\vec{z}$$

$$(\vec{x}'A^{1/2}A^{-1/2}\vec{y})^2 \le (\vec{x}'A^{1/2}A^{1/2}\vec{x})(\vec{y}'A^{-1/2}A^{-1/2}\vec{y})$$
$$(\vec{x}'\vec{y})^2 \le (\vec{x}'A\vec{x})(\vec{y}'A^{-1}\vec{y}),$$

with equality iff $A^{1/2}\vec{x} = c\,A^{-1/2}\vec{y}$ (6)

$$\vec{x} = c\,A^{-1}\vec{y}$$

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III. Maximization Lemma

Let A be symmetric, positive definite and $\vec{d}$ be a given vector.

Then $\displaystyle \max_{\vec{x}\neq\vec{0}} \frac{(\vec{x}'d)^2}{\vec{x}'A\vec{x}} = \vec{d}'A^{-1}\vec{d}$, and the maximum is attained when $\vec{x} = cA^{-1}\vec{d}$ for any $c \neq 0$

(7)

<u>Proof</u>: Use the extended C-S lemma

$$(\vec{x}'\vec{d})^2 \le (\vec{x}'A\vec{x})(\vec{d}'A^{-1}\vec{d})$$

$$\frac{(\vec{x}'\vec{d})^2}{\vec{x}'A\vec{x}} \le \vec{d}'A^{-1}\vec{d}$$

Suppose $\vec{x} = cA^{-1}\vec{d}$

Then $\dfrac{(c\vec{d}'A^{-1}\vec{d})^2}{c^2\vec{d}'A^{-1}AA^{-1}\vec{d}} = \dfrac{(\vec{d}'A^{-1}\vec{d})^2}{\vec{d}'A^{-1}\vec{d}} = \vec{d}'A^{-1}\vec{d}$ //

(8)

IV. Maximization of quadratic forms

Let $A_{p\times p}$ be symmetric, positive definite with eigenvalues $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_p > 0$ and eigenvectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_p$.

Then $\displaystyle\max_{\vec{x}\neq\vec{0}} \frac{\vec{x}'A\vec{x}}{\vec{x}'\vec{x}} = \lambda_1$, $\displaystyle\min_{\vec{x}\neq\vec{0}} \frac{\vec{x}'A\vec{x}}{\vec{x}'\vec{x}} = \lambda_p$

and $\displaystyle\max_{\vec{x}\perp\vec{e}_1,\dots,\vec{e}_k} \frac{\vec{x}'A\vec{x}}{\vec{x}'\vec{x}} = \lambda_{k+1}$ for $k=1,\dots,p-1$

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<u>Proof</u>: $A = P \Lambda P'$ (since A was symmetric)

Let $\vec{y} = P'\vec{x}$, so $\vec{x} = P\vec{y}$

$$\frac{\vec{x}'A\vec{x}}{\vec{x}'\vec{x}} = \frac{\vec{y}'P\Lambda P'}{\vec{y}'P'P\vec{y}}$$

$$= \frac{\vec{y}'\Lambda\vec{y}}{\vec{y}'\vec{y}} = \frac{[y_1, \ldots, y_p]\begin{bmatrix}\lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_p\end{bmatrix}\begin{bmatrix}y_1 \\ \vdots \\ y_p\end{bmatrix}}{\vec{y}'\vec{y}}$$

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$$= \frac{\sum_{i=1}^{p} \lambda_i y_i^2}{\sum_{i=1}^{p} y_i^2} \leq \frac{\sum_{i=1}^{p} \lambda_1 y_i^2}{\sum_{i=1}^{p} y_i^2} = \lambda_1$$

The maximum is attained when $\vec{x} = \vec{e}_1$:

$$\frac{\vec{x}'A\vec{x}}{\vec{x}'\vec{x}} = \frac{\vec{e}_1'(A\vec{e}_1)}{\vec{e}_1'\vec{e}_1} = \frac{\vec{e}_1'\lambda_1\vec{e}_1}{\vec{e}_1'\vec{e}_1} = \lambda_1$$

Proof of minimum just replaces all λ_i with λ_p.

(11)

Now let $\vec{x} \perp \vec{e}_1, \vec{e}_2, \dots, \vec{e}_k$

Then $\vec{e}_i{}'\vec{x} = 0$ for $i = 1, \dots, k$

So
$$
\begin{aligned}
0 = \vec{e}_i{}'\vec{x} &= \vec{e}_i{}' P \vec{y} \\
&= \vec{e}_i{}' \left[\vec{e}_1 \middle| \vec{e}_2 \middle| \cdots \middle| \vec{e}_p \right] \vec{y} \\
&= \Big[\, 0 \cdots \underset{\uparrow\, i^{\text{th}} \text{ position}}{1} \; 0 \cdots 0 \,\Big] \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix} \\
&= y_i
\end{aligned}
$$

(12)

That is, $y_i = 0$ for $i = 1, \dots, k$

So
$$
\frac{\vec{x}' A \vec{x}}{\vec{x}' \vec{x}} = \frac{\sum_{i=1}^{p} \lambda_i y_i^2}{\sum_{i=1}^{p} y_i^2} = \frac{\sum_{i=k+1}^{p} \lambda_i y_i^2}{\sum_{i=k+1}^{p} y_i^2}
$$

$$
\leq \frac{\sum_{i=k+1}^{p} \lambda_{k+1} y_i^2}{\sum_{i=k+1}^{p} y_i^2} = \lambda_{k+1}
$$

Equality is achieved when $\vec{x} = \vec{e}_{k+1}$

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