From last time: Stat 571

10-29-13

The denominator was maximized ①

when $\hat{\vec{\mu}} = \bar{x}$ and $\hat{\Sigma} = S_n$

Evaluate the denominator at its max.

$$L(\bar{x}, S_n) = \frac{1}{(2\pi)^{\frac{np}{2}} |S_n|^{n/2}} e^{-\frac{1}{2}(n-1)tr(S_n^{-1}S)}$$

Note $S_n = \frac{(n-1)S}{n}$, $S_n^{-1} = \frac{n}{(n-1)} S^{-1}$

$$S_n^{-1}S = \frac{n}{n-1} S^{-1}S = \frac{n}{n-1} I$$ ②

$$tr(S_n^{-1}S) = \frac{n}{n-1} p$$

$$L(\bar{x}, S_n) = \frac{1}{(2\pi)^{\frac{np}{2}} |S_n|^{n/2}} e^{-\frac{1}{2}(n-1)\frac{n}{n-1}p}$$

$$= \frac{1}{(2\pi)^{\frac{np}{2}} |S_n|^{n/2}} e^{-\frac{np}{2}}$$

Maximize the numerator:

$$L(\theta) = \frac{1}{(2\pi)^{\frac{np}{2}} |\Sigma|^{n/2}} e^{-\frac{1}{2}\sum_{i=1}^{n}(\vec{x}_i-\vec{\mu}_0)'\Sigma^{-1}(\vec{x}_i-\vec{\mu}_0)}$$

$$\sum_{i=1}^{n}(\vec{x}_i-\vec{\mu}_0)'\Sigma^{-1}(\vec{x}_i-\vec{\mu}_0)$$

$$= \sum_{i=1}^{n} tr\left[(\vec{x}_i-\vec{\mu}_0)'\Sigma^{-1}(\vec{x}_i-\vec{\mu}_0)\right]$$

$$= \sum_{i=1}^{n} tr\left[\Sigma^{-1}(\vec{x}_i-\vec{\mu}_0)(\vec{x}_i-\vec{\mu}_0)'\right]$$

$$= tr\Big[\Sigma^{-1} \underbrace{\sum_{i=1}^{n}(\vec{x}_i-\vec{\mu}_0)(\vec{x}_i-\vec{\mu}_0)'}_{B}\Big]$$

Now

$$L(\theta) = \frac{1}{(2\pi)^{\frac{np}{2}} |\Sigma|^{n/2}} e^{-\frac{1}{2}tr(\Sigma^{-1}B)}$$

Let $b = n/2$ + use 2^{nd} lemma from last time.

Result is that $L(\theta)$ is maximized

when $\hat{\Sigma}_0 = \frac{B}{2b}$

$$= \frac{1}{n}\sum_{i=1}^{n}(\vec{x}_i-\mu_0)(\vec{x}_i-\mu_0)'$$

Evaluate the numerator:

$$L(\vec{\mu}_0, \hat{\Sigma}_0) = \frac{1}{(2\pi)^{\frac{np}{2}}|\hat{\Sigma}_0|^{n/2}} e^{-\frac{1}{2}tr(\hat{\Sigma}_0^{-1}B)}$$

But $B = n\hat{\Sigma}_0$, or $\hat{\Sigma}_0 = \frac{B}{n}$

$$\hat{\Sigma}_0^{-1} = nB^{-1}$$

$$\hat{\Sigma}_0^{-1}B = nB^{-1}B = nI$$

$$tr(\hat{\Sigma}_0^{-1}B) = tr(nI) = np$$

$$L(\vec{\mu}_0, \hat{\Sigma}_0) = \frac{1}{(2\pi)^{\frac{np}{2}}|\hat{\Sigma}_0|^{n/2}} e^{-\frac{1}{2}np}$$

$$\Lambda = \frac{\frac{1}{(2\pi)^{np/2}|\hat{\Sigma}_0|^{n/2}} e^{-np/2}}{\frac{1}{(2\pi)^{\frac{np}{2}}|S_n|^{n/2}} e^{-\frac{np}{2}}} = \frac{|S_n|^{n/2}}{|\hat{\Sigma}_0|^{n/2}}$$

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The likelihood ratio test says to reject H_0 when $\Lambda < c$.

Equivalently, reject H_0 when

$$\Lambda^{2/n} = \frac{|\hat{\Sigma}_n|}{|\hat{\Sigma}_0|} < c'$$

Defn: $\Lambda^{2/n}$ is called Wilk's Lambda

Lemma: let $A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$, then

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$$|A| = |A_{11}| \cdot |A_{22} - A_{21} A_{11}^{-1} A_{12}|$$

$$= |A_{22}| \cdot |A_{11} - A_{12} A_{22}^{-1} A_{21}|$$

Proof:

$$\left[\begin{array}{c|c} I & -A_{12}A_{22}^{-1} \\ \hline 0 & I \end{array}\right] \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \left[\begin{array}{c|c} I & 0 \\ \hline -A_{22}^{-1}A_{21} & I \end{array}\right]$$

$$= \left[\begin{array}{c|c} A_{11} - A_{12} A_{22}^{-1} A_{21} & 0 \\ \hline 0 & A_{22} \end{array}\right]$$

⑨

Det. of L.H.S.	Det. of R.H.S.
$= \|A\|$	$= \|A_{11} - A_{12}A_{22}^{-1}A_{21}\| \cdot \|A_{22}\|$

We have proven the 2^{nd} equality. The 1^{st} equality is done similarly.

<u>Theorem</u>: $$\Lambda^{2/n} = \frac{1}{1 + (\bar{X} - \vec{\mu}_0)' S_n^{-1} (\bar{X} - \vec{\mu}_0)}$$

⑩

<u>Proof</u>: Let $$A = \left[\begin{array}{c|c} n S_n & \sqrt{n}(\bar{X} - \vec{\mu}_0) \\ \hline \sqrt{n}(\bar{X} - \vec{\mu}_0)' & -1 \end{array}\right]_{(p+1)\times(p+1)}$$

Compute $|A|$ two ways, using the lemma.

$$|A| = |n S_n| \cdot \left| -1 - \sqrt{n}(\bar{X} - \vec{\mu}_0)' (n S_n)^{-1} \sqrt{n} (\bar{X} - \vec{\mu}_0) \right|$$

$$= n^p |S_n| \left(-1 - (\bar{X} - \vec{\mu}_0)' S_n^{-1} (\bar{X} - \vec{\mu}_0) \right)$$

Also,

$$|A| = |-1| \cdot \left| n S_n - \sqrt{n}(\bar{x}-\vec{\mu}_0)(-1)^{-1}\sqrt{n}(\bar{x}-\vec{\mu}_0)' \right|$$

$$= -n^p \left| \underbrace{S_n + (\bar{x}-\vec{\mu}_0)(\bar{x}-\vec{\mu}_0)'} \right|$$

Using page 4 of Thursday's notes,

$$n\left[S_n + (\bar{x}-\vec{\mu})(\bar{x}-\vec{\mu})'\right] = \sum_{i=1}^{n}(\vec{x}_i-\vec{\mu})(\vec{x}_i-\vec{\mu})'$$

So $S_n + (\bar{x}-\vec{\mu}_0)(\bar{x}-\mu_0)'$

$$= \frac{1}{n}\sum_{i=1}^{n}(\vec{x}_i-\vec{\mu}_0)(\vec{x}_i-\vec{\mu}_0)'$$

$$= \hat{\Sigma}_0$$

Now, Equate the 2 expressions for $|A|$:

$$-n^p|S_n|\left(1 + (\bar{x}-\vec{\mu}_0)'S_n^{-1}(\bar{x}-\vec{\mu}_0)\right)$$

$$= -n^p|\hat{\Sigma}_0|$$

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$$\therefore \frac{|\tilde{S}_n|}{|\hat{\Sigma}_0|} = \frac{1}{1 + (\bar{x} - \vec{\mu}_0)' S_n^{-1} (\bar{x} - \vec{\mu}_0)} \quad //$$

Defn: $T^2 = n(\bar{x} - \vec{\mu}_0)' S^{-1} (\bar{x} - \vec{\mu}_0)$

is Hotelling's T^2.

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How are they related?

$$\tilde{S}_n = \frac{(n-1)\tilde{S}}{n}$$

$$\tilde{S}_n^{-1} = \frac{n}{(n-1)} \tilde{S}^{-1}$$

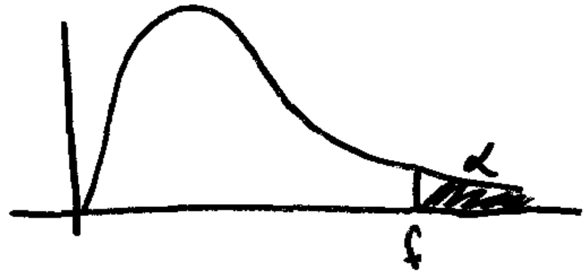
$$\Lambda^{2/n} = \frac{1}{1 + \frac{T^2}{n-1}}$$

$$\Lambda^{2/n} < c' \iff T^2 > c''$$

Write

$$T^2 = \underbrace{\sqrt{n}\,(\bar{x} - \vec{\mu}_0)'}_{[N_p(\vec{0}, \Sigma)]'} \underbrace{\left[\frac{(n-1)S}{(n-1)}\right]^{-1}}_{\left[\frac{\text{Wishart}}{df}\right]^{-1}} \underbrace{\sqrt{n}\,(\bar{x} - \vec{\mu}_0)}_{N_p(\vec{0}, \Sigma)}$$

Hotelling showed that $T^2 \cdot \frac{n-p}{p(n-1)} \sim F_{p,n-p}$



Compute $f \cdot \frac{p(n-1)}{n-p}$

Reject H_0 when

$T^2 > f \cdot \frac{p(n-1)}{n-p}$.