Limiting probabilities

Stat 568
4-4-17

①

Definition: State i has period d if

$P_{ii}^n = 0$ whenever $d \nmid n$, and d

is the largest integer with this property.

A state whose period is 1 is called aperiodic.

Defn: If state i is recurrent, then it is positive recurrent if, starting in state i, the expected time to return to i is finite.

②

Defn. A positive recurrent aperiodic state is called ergodic.

Facts: ① Period is a class property

② Positive recurrence is a class property

Facts ① & ② imply that ergodic is a class property.

③ In a finite state Markov chain, all recurrent states are positive recurrent.

④ For an irreducible ergodic Markov chain,

$\pi_j = \lim_{n\to\infty} P_{ij}^n$ exists and is independent (3)

of i. Also, π_j is the unique solution

to

$$\star \begin{cases} \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad j \geq 0 \text{ and} \\ \sum_{j=0}^{\infty} \pi_j = 1. \end{cases}$$

Also, the π_j values are called the <u>stationary probabilities</u>

(5) If a Markov chain is irreducible, then there will be a solution to $\star$ iff the Markov chain is positive recurrent.

(4)

Proof of (4): $P[X_{n+1} = j]$

$$= \sum_{i=0}^{\infty} P[X_{n+1} = j \mid X_n = i] P[X_n = i]$$

Take the limit as $n \to \infty$

$$\pi_j = \sum_{i=0}^{\infty} P_{ij} \pi_i$$

(5)

Example: $P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & .5 & .4 & .1 \\ 1 & .3 & .4 & .3 \\ 2 & .2 & .3 & .5 \end{array}$

irreducible
positive recurrent
aperiodic
$\therefore$ ergodic

Use fact (4) $\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}$, $\sum_{j=0}^{\infty} \pi_j = 1$

$$\left.\begin{aligned} \pi_0 &= .5\pi_0 + .3\pi_1 + .2\pi_2 \\ \pi_1 &= .4\pi_0 + .4\pi_1 + .3\pi_2 \\ \pi_2 &= .1\pi_0 + .3\pi_1 + .5\pi_2 \\ \pi_0 &+ \pi_1 + \pi_2 = 1 \end{aligned}\right\}$$

Solve simultaneously

$\pi_0 = \frac{21}{62}$

$\pi_1 = 23/62$

$\pi_2 = 18/62$

Mean time in transient states

(6)

Renumber the states so that $T = \{1, \ldots, t\}$ is the set of transient states.

Let $P_T = \begin{bmatrix} P_{11} & \cdots & P_{1t} \\ \vdots & & \\ P_{t1} & \cdots & P_{tt} \end{bmatrix}$

Let s_{ij} = expected # of time periods in state j, given that the process starts in state i

⑦

Find a formula for s_{ij}, conditioning on the initial transition

Case 1: $i \neq j$ $\quad s_{ij} = \sum_k P_{ik} s_{kj}$ (Law of iterated expectations)

Case 2: $i = j$ $\quad s_{ii} = 1 + \sum_k P_{ik} s_{kj}$

Together, $\quad s_{ij} = \delta_{ij} + \sum_k P_{ik} s_{kj}$, where $\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{o.w.} \end{cases}$

⑧

$$s_{ij} = \delta_{ij} + \sum_{k=1}^{t} P_{ik} s_{kj}$$

because it's impossible to go from a recurrent to a transient state, since recurrence would require an eventual return, make the 2 states communicate

Let $S = \begin{bmatrix} s_{11} & \cdots & s_{1t} \\ \vdots & & \\ s_{t1} & \cdots & s_{tt} \end{bmatrix}$

$$S = I + P_T S$$

$$S - P_T S = I \qquad (I - P_T) S = I$$

$S = (I - P_T)^{-1}$ Fact: this inverse exists