Stat 567
3-7-17

1

Recall the corollary to the Compounding identity:

$$P[S_N = 0] = P[N=0],$$

$$P[S_N = k] = \frac{1}{k} E[N] \sum_{j=1}^{k} j\alpha_j P[S_{M-1} = k-j],$$

where $\alpha_j = P[X_1 = j]$ and

$$P[M=n] = \frac{nP[N=n]}{E[N]}$$

2

Application to the Poisson:

Assume $N \sim \text{Poisson}(\lambda)$

$$P[M-1=n] = P[M=n+1]$$

$$= \frac{(n+1)P[N=n+1]}{E[N]}$$

$$= \frac{(n+1)\dfrac{e^{-\lambda}\lambda^{n+1}}{(n+1)!}}{\lambda} = \frac{e^{-\lambda}\lambda^n}{n!}$$

So $M-1 \sim \text{Poisson}(\lambda)$

③

Apply the corollary:

$$P[S_N = 0] = P[N = 0] = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda}$$

$$P[S_N = k] = \frac{1}{k}\lambda \sum_{j=1}^{k} j\alpha_j P[S_{M-1} = k-j]$$

$$= \frac{\lambda}{k}\sum_{j=1}^{k} j\alpha_j P[S_N = k-j]$$

So $P_0 = e^{-\lambda}$, $P_k = \frac{\lambda}{k}\sum_{j=1}^{k} j\alpha_j P_{k-j}$

④

<u>Example</u>: Say $N \sim \text{Poisson}(\lambda = 4)$

Each X_i has a discrete uniform distr.
with values 1, 2, 3, 4

Find $P[S_N = 5]$ $\qquad\qquad$ $\alpha_j = P[X_1 = j]$

$$P_0 = e^{-\lambda} = e^{-4}$$

$$P_1 = \frac{4}{1}(1\alpha_1 P_0) = 4\alpha_1 P_0 = 4\cdot\frac{1}{4}\cdot e^{-4} = e^{-4}$$

$$P_2 = \frac{\cancel{4}}{2}\left(1\cancel{\alpha_1}P_1 + 2\cancel{\alpha_2}P_0\right) = \frac{3}{2}e^{-4}$$

(5)

$$P_3 = \frac{\cancel{4}}{3}\left(1\cancel{\alpha_1}P_2 + 2\cancel{\alpha_2}P_1 + 3\cancel{\alpha_3}P_0\right)$$

$$= \frac{1}{3}\left(\frac{3}{2} + 2 + 3\right)e^{-4} = \frac{13}{6}e^{-4}$$

$$P_4 = \frac{\cancel{4}}{4}\left(1\cancel{\alpha_1}P_3 + 2\cancel{\alpha_2}P_2 + 3\cancel{\alpha_3}P_1 + 4\cancel{\alpha_4}P_0\right)$$

$$= \frac{1}{4}e^{-4}\left(\frac{13}{6} + 2\cdot\frac{3}{2} + 3 + 4\right) = \frac{73}{24}e^{-4}$$

$$P_5 = \frac{\cancel{4}}{5}\left(1\cancel{\alpha_1}P_4 + 2\cancel{\alpha_2}P_3 + 3\cancel{\alpha_3}P_2 + 4\cancel{\alpha_4}P_1 + 5\cancel{\alpha_5}^{\,0}P_0\right)$$

$$= \frac{381}{120}e^{-4} = P[S_N = 5]$$

(6)

Application to the binomial:

Assume $N \sim \text{Bino}(r, p)$

$$P[M-1=n] = P[M = n+1]$$

$$= \frac{(n+1)\,P[N = n+1]}{E[N]}$$

$$= \frac{(n+1)\binom{r}{n+1}p^{n+1}q^{r-n-1}}{rp}$$

$$= \frac{(n+1)}{r} \frac{r!}{(n+1)!\,(r-n-1)!} p^n q^{r-n-1}$$

$$= \frac{(r-1)!}{n!\,(r-1-n)!} p^n q^{r-n-1}$$

So $M-1 \sim \mathrm{Bino}(r-1, p)$

Apply the Corollary:

$$P[S_N=0] = P[N=0] = \binom{r}{0} p^0 q^r = q^r$$

$$P[S_N=k] = \frac{1}{k} r p \sum_{j=1}^{k} j \alpha_j P[S_{M-1}=k-j]$$

Let $P_r(k) = P[S_N=k]$

Then $P_r(0) = q^r$

and $P_r(k) = \frac{1}{k} r p \sum_{j=1}^{k} j \alpha_j P_{r-1}(k-j)$

$$P_r(1) = rp\left(1\,\alpha_1 P_{r-1}(0)\right)$$

$$= rp\,\alpha_1 q^{r-1}$$

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$$P_r(2) = \frac{1}{2} r p \left(1 \alpha_1 P_{r-1}(1) + 2 \alpha_2 P_{r-1}(0) \right)$$

$$= \frac{rp}{2} \left(\alpha_1 (r-1) p \alpha_1 q^{r-2} + 2 \alpha_2 q^{r-1} \right)$$

etc.

Chapter 4 Markov Chains

Defn. A stochastic process $\{X(t), t \in T\}$ is a collection of random variables.

(10)

t often represents time

$X(t)$ is the state of the process at time t

T is the index set of the process

If T is countable, the process is a discrete-time process

If T is an interval, the process is a continuous time process

(11)

The state space is the set of all values that $X(t)$ can take on

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Let $\{X_n, n=0,1,2,\ldots\}$ be a discrete-time stochastic process

Assume that the state space is countable, and label its values $0, 1, 2, \ldots$

(12)

If $X_n = i$, we say that the process is in state i at time n.

Assume that

$$P[X_{n+1}=j \mid X_n=i, X_{n-1}=x_{n-1}, \ldots, X_0=x_0]$$

$$= P[X_{n+1}=j \mid X_n=i] = P_{ij}$$

Then $\{X_n, n=0,1,2,\ldots\}$ is a Markov chain.

1. Three white and three black balls are distributed in two urns in such a way that each contains three balls. We say that the system is in state i, $i = 0, 1, 2, 3$, if the first urn contains i white balls. At each step, we draw one ball from each urn and place the ball drawn from the first urn into the second, and conversely with the ball from the second urn. Let X_n denote the state of the system after the nth step. Explain why $\{X_n, n = 0, 1, 2, \ldots\}$ is a Markov chain and calculate its transition probability matrix.

2. Suppose that whether or not it rains today depends on previous weather conditions through the last three days. Show how this system may be analyzed by using a Markov chain. How many states are needed?

3. In Exercise 2, suppose that if it has rained for the past three days, then it will rain today with probability 0.8; if it did not rain for any of the past three days, then it will rain today with probability 0.2; and in any other case the weather today will, with probability 0.6, be the same as the weather yesterday. Determine **P** for this Markov chain.

5. A Markov chain $\{X_n, n \geqslant 0\}$ with states 0, 1, 2, has the transition probability matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

If $P\{X_0 = 0\} = P\{X_0 = 1\} = \frac{1}{4}$, find $E[X_3]$.