Variations on Brownian Motion processes

Stat 568
6-8-17

①

Defn: $\{X(t), t \geq 0\}$ is a Brownian-motion process with drift coefficient μ and variance parameter σ^2 if

i) $X(0) = 0$

ii) $\{X(t), t \geq 0\}$ has stationary & independent increments

iii) $X(t) \sim N(\mu t, \sigma^2 t)$

OR $X(t) = \mu t + \sigma B(t)$, where $B(t)$ is a standard Br-mot process

②

Defn: If $\{Y(t), t \geq 0\}$ is a Brownian motion process with drift coefficient μ and variance parameter σ^2, then

$X(t) = e^{Y(t)}$ is a geometric Brownian motion process.

For geometric Brownian motion, find

$E[X(t) \mid X(u)\ \ 0 \leq u \leq s]$ where $s < t$

(3)

$$E[X(t) \mid X(u), 0 \le u \le s] = E[e^{Y(t)} \mid Y(u), 0 \le u \le s]$$

$$= E[\underbrace{e^{Y(t)-Y(s)+Y(s)}}_{e^{Y(t)-Y(s)}e^{Y(s)}} \mid Y(u), 0 \le u \le s]$$

$$= e^{Y(s)} E[e^{Y(t)-Y(s)} \mid Y(u), 0 \le u \le s]$$

$$= e^{Y(s)} E[e^{Y(t)-Y(s)}]$$ by independent movements

(4)

Recall: The moment-generating function for $W \sim N(\mu, \sigma^2)$ is

$$m_W(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$$\overset{\shortparallel}{E[e^{tW}]}$$

So $E[e^W] = e^{\mu_W + \frac{1}{2}\sigma_W^2}$

⑤

$$Y(t) \sim N(\mu t, \sigma^2 t)$$

$$Y(t) = \mu t + \sigma B(t)$$

$$Y(t) - Y(s) = \mu t + \sigma B(t) - (\mu s + \sigma B(s))$$
$$= \mu(t-s) + \sigma[\underbrace{B(t) - B(s)}_{B(t-s)}]$$

So $Y(t) - Y(s) \sim N(\mu(t-s), \sigma^2(t-s))$

Now $E[e^{Y(t)-Y(s)}] = e^{\mu(t-s) + \frac{1}{2}\sigma^2(t-s)}$

⑥

$$\therefore E[X(t) \mid X(u), 0 \le u \le s]$$
$$= e^{Y(s)} E[e^{Y(t)-Y(s)}]$$
$$= X(s) e^{(t-s)(\mu + \frac{1}{2}\sigma^2)}$$

Why can this be used to model stock prices?

Suppose that the percent changes over time are independent and identically distributed.

(7)

If X_t is the price at time t,

then $Y_t = \frac{X_t}{X_{t-1}}$, $t \geq 1$ are iid.

So $X_t = Y_t X_{t-1} = Y_t Y_{t-1} X_{t-2}$

$= \ldots = Y_t Y_{t-1} \cdots Y_1 X_0$

$$\ln(X_t) = \underbrace{\sum_{i=1}^{t} \ln(Y_i)}_{\text{sum of iid r.v.s}} + \ln(X_0)$$

(8)

$$\sum_{i=1}^{t} \ln(Y_i) \xrightarrow{D} N\Big(t\, \underbrace{E[\ln(Y_i)]}_{\mu},\ t\, V\underbrace{[\ln(Y_i)]}_{\sigma^2}\Big)$$

$$\ln(X_t) \approx N(\ln(X_0) + \mu t,\ \sigma^2 t)$$

$X_t = e^{\ln(X_t)}$ will be approximately a geometric Brownian motion process