Kolmogorov's forward equations

Stat 568
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① Use the Chapman-Kolmogorov equations from lemma 2:

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s)$$

$$P_{ij}(t+s) - P_{ij}(t) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s) - P_{ij}(t)$$

$$= \sum_{k \neq j} P_{ik}(t) P_{kj}(s) + P_{ij}(t) P_{jj}(s) - P_{ij}(t)$$

$$= \sum_{k \neq j} P_{ik}(t) P_{kj}(s) - (1 - P_{jj}(s)) P_{ij}(t)$$

② Divide by s, take lim as $s \to 0$

$$\frac{d}{dt} P_{ij}(t) = \lim_{s \to 0} \Bigg[\sum_{k \neq j} P_{ik}(t) \underbrace{\frac{P_{kj}(s)}{s}}_{\downarrow\, q_{kj}} - \underbrace{\frac{1 - P_{jj}(s)}{s}}_{\downarrow\, v_j} P_{ij}(t) \Bigg]$$

by lemma 1

$$\frac{d}{dt} P_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t)$$

These are Kolmogorov's Forward Equations

③

Backward: $P_{ij}'(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t)$

Forward: $P_{ij}'(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - \nu_j P_{ij}(t)$

and $q_{ij} = \nu_i P_{ij}$

Apply to a pure birth process $\mu_n = 0$

so $\nu_i = \lambda_i$

Backward: $P_{ij}'(t) = q_{i,i+1} P_{i+1,j}(t) - \lambda_i P_{ij}(t)$

↑ for all other k, $q_{ik} = 0$ since $P_{ik} = 0$

④

$$P_{ij}'(t) = \lambda_i \cdot 1 \cdot P_{i+1,j}(t) - \lambda_i P_{ij}(t)$$

$$= \lambda_i \left(P_{i+1,j}(t) - P_{ij}(t) \right)$$

Forward: $P_{ij}'(t) = q_{j-1,j} P_{i,j-1}(t) - \lambda_j P_{ij}(t)$

$$= \nu_{j-1} P_{j-1,j} P_{i,j-1}(t) - \lambda_j P_{ij}(t)$$

$$= \lambda_{j-1} P_{i,j-1}(t) - \lambda_j P_{ij}(t)$$

Now, let's actually solve the forward equations (5)

$$P_{ij}'(t) = \lambda_{j-1} P_{i,j-1}(t) - \lambda_j P_{ij}(t)$$

If $j < i$, $P_{ij}(t) = 0 \quad \forall t$

If $j = i$, $P_{ii}'(t) = \lambda_{i-1} \cancel{P_{i,i-1}(t)}^{0} - \lambda_i P_{ii}(t)$

$$\frac{P_{ii}'(t)}{P_{ii}(t)} = -\lambda_i$$

$$\ln P_{ii}(t) = -\lambda_i t + c$$

$$P_{ii}(t) = k e^{-\lambda_i t}$$

(6)

When $t=0$, $P_{ii}(0) = 1$

$$1 = k e^0 = k$$

$$\therefore \; P_{ii}(t) = e^{-\lambda_i t}$$

For $j > i$, $P_{ij}'(t) = \lambda_{j-1} P_{i,j-1}(t) - \lambda_j P_{ij}(t)$

$$e^{\lambda_j t}\left[P_{ij}'(t) + \lambda_j P_{ij}(t)\right] = e^{\lambda_j t} \lambda_{j-1} P_{i,j-1}(t)$$

$$\frac{d}{dt}\left[e^{\lambda_j t} P_{ij}(t)\right] = e^{\lambda_j t} \lambda_{j-1} P_{i,j-1}(t)$$

$$= \frac{d}{dt} \int_0^t e^{\lambda_j s} \lambda_{j-1} P_{i,j-1}(s)\, ds$$

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$$e^{\lambda_j t} P_{ij}(t) = \int_0^t e^{\lambda_j s} \lambda_{j-1} P_{i,j-1}(s)\,ds + c$$

When $t=0$, $P_{ij}(0) = 0 = c$

$$P_{ij}(t) = \lambda_{j-1} e^{-\lambda_j t} \int_0^t e^{\lambda_j s} P_{i,j-1}(s)\,ds$$

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Review:

Classification of states as transient, recurrent, positive recurrent, aperiodic, ergodic

Find the stationary probabilities, expected time in transient states

Branching processes

Find expected size + variance of population at n^{th} step, + prob. of die-out. (π_0)

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Counting processes

Distribution of arrival times & interarrival times

Compound Poisson processes

Continuous-time Markov chains

Birth & death processes

Multiserver queues

Expected time from i to j
+ variance

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Yule process

Be able to find popul. size at a given time

Be able to write the K. forward & backward equations for a specific process