Stat 567
2-16-17

Transformations of pairs of random variables
(continuous case)

①

Let $Y_1 = g_1(X_1, X_2)$, $Y_2 = g_2(X_1, X_2)$ Note: the transformation must be invertible

We need the joint distribution of Y_1 and Y_2.

The joint density of Y_1 and Y_2 is

$$h(y_1, y_2) = \frac{f(x_1, x_2)}{|J|} \quad \text{where} \quad J = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix}$$

②

OR $h(y_1, y_2) = f(x_1, x_2) \, |J^*|$

$$\text{where} \quad J^* = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

Example: $X \sim \text{Gamma}(\alpha, \lambda)$
$Y \sim \text{Gamma}(\beta, \lambda)$

and X and Y are independent.

Let $U = X+Y$ and $V = \frac{X}{X+Y}$

Find the joint density of U, V.

③

$$f_X(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}$$

$$f_Y(y) = \frac{\lambda e^{-\lambda y}(\lambda y)^{\beta-1}}{\Gamma(\beta)}$$

$f(x,y) = f_X(x) f_Y(y)$ by independence

$$= \frac{\lambda^2 e^{-\lambda(x+y)}(\lambda x)^{\alpha-1}(\lambda y)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}$$

④

$u = x+y \quad v = \dfrac{x}{x+y}$

$x = u-y \quad v = \dfrac{u-y}{u}$

$uv = u-y$

$\underline{y = u - uv}$

$x = u-(u-uv)$

$\underline{= uv}$

$$J^* = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -uv-u(1-v)$$

$$= -u$$

5

$$h(u,v) = f(x,y)\,|J^*|$$

$$= \frac{\lambda^2 e^{-\lambda(x+y)} (\lambda x)^{\alpha-1} (\lambda y)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}\, u$$

$$= \frac{\lambda^2 e^{-\lambda u} \lambda^{\alpha+\beta-2} (uv)^{\alpha-1} (u-uv)^{\beta-1} u}{\Gamma(\alpha)\Gamma(\beta)}$$

$$= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)}\, e^{-\lambda u} u^{\alpha+\beta-1} v^{\alpha-1} (1-v)^{\beta-1}$$

6

$$= \underbrace{\frac{\lambda e^{-\lambda u} (\lambda u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}}_{\text{Gamma}(\alpha+\beta,\,\lambda)} \cdot \underbrace{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} v^{\alpha-1} (1-v)^{\beta-1}}_{\text{Beta}(\alpha,\beta)}$$

Summary: If X and Y are independent Gamma random variables with parameters (α, λ) and (β, λ), then

1. $X+Y \sim \text{Gamma}(\alpha+\beta, \lambda)$
2. $\frac{X}{X+Y} \sim \text{Beta}(\alpha, \beta)$
3. These 2 are independent

⑦

Using moment-generating functions:

If X and Y are independent, find the moment generating function of their sum.

Let $U = X+Y$.

$$\phi_U(t) = E[e^{tU}] = E[e^{t(X+Y)}]$$
$$= E[e^{tX}e^{tY}] = E[e^{tX}]\,E[e^{tY}] \quad \text{by independence}$$
$$= \phi_X(t)\,\phi_Y(t)$$

⑧

Gamma example: $\left.\begin{array}{l} X \sim \text{Gamma}(\alpha, \lambda) \\ Y \sim \text{Gamma}(\beta, \lambda) \end{array}\right\}$ indep.

$U = X+Y.$ $\quad \phi_U(t) = \phi_X(t)\,\phi_Y(t)$

$$= \left(\frac{\lambda}{\lambda-t}\right)^{\alpha}\left(\frac{\lambda}{\lambda-t}\right)^{\beta}$$
$$= \left(\frac{\lambda}{\lambda-t}\right)^{\alpha+\beta}$$

So $U \sim \text{Gamma}(\alpha+\beta, \lambda)$

9

Binomial example: $\left.\begin{array}{l} X \sim \text{Bino}(n_1, p) \\ Y \sim \text{Bino}(n_2, p) \end{array}\right\}$ Indep.

$$U = X+Y. \quad \phi_U(t) = (pe^t+q)^{n_1}(pe^t+q)^{n_2}$$
$$= (pe^t+q)^{n_1+n_2}$$

$$\sim \text{Bino}(n_1+n_2, p)$$

Poisson example: $\left.\begin{array}{l} X \sim \text{Poisson}(\lambda_1) \\ Y \sim \text{Poisson}(\lambda_2) \end{array}\right\}$ Indep.

10

$$U = X+Y. \quad \phi_U(t) = e^{\lambda_1(e^t-1)} e^{\lambda_2(e^t-1)}$$
$$= e^{(\lambda_1+\lambda_2)(e^t-1)}$$

$$\sim \text{Poisson}(\lambda_1+\lambda_2)$$

Normal example: $\left.\begin{array}{l} X \sim \text{Normal}(\mu_1, \sigma_1^2) \\ Y \sim \text{Normal}(\mu_2, \sigma_2^2) \end{array}\right\}$ Indep.

$$U = X+Y. \quad \phi_U(t) = e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2} e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2}$$

(11)

$$= e^{(\mu_1+\mu_2)t + \frac{1}{2}(\sigma_1^2+\sigma_2^2)t^2}$$

$$\sim \text{Normal}(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2)$$

Let $X_1, \ldots, X_n$ be independent, identically distributed (iid) $\text{Norm}(\mu, \sigma^2)$

Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \dfrac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$

(12)

$$\sum_{i=1}^{n}(X_i-\mu)^2 = \sum_{i=1}^{n}(\underbrace{X_i - \bar{X}} + \underbrace{\bar{X} - \mu})^2$$

$$= \sum_{i=1}^{n}(X_i - \bar{X})^2 + \sum_{i=1}^{n}(\bar{X}-\mu)^2 + 2\sum_{i=1}^{n}(X_i - \bar{x})(\bar{x}-\mu)$$

$$= (n-1)S^2 + n(\bar{X}-\mu)^2 + 2(\bar{X}-\mu)\underbrace{\left(\sum_{i=1}^{n} X_i - n\bar{X}\right)}_{0}$$

So $(n-1)S^2 = \sum_{i=1}^{n}(X_i-\mu)^2 - n(\bar{X}-\mu)^2$

$E((n-1)S^2) = \sum_{i=1}^{n} \text{Var}(X_i) - n\,V(\bar{X})$ Used one of the 3 properties of $\bar{X}$

$$= n\sigma^2 - n\frac{\sigma^2}{n}$$ (13)

$$E[(n-1)S^2] = (n-1)\sigma^2$$

$$\therefore E[S^2] = \sigma^2$$

Defn: Let $Z_1, \ldots, Z_n$ be iid $N(0,1)$

Then $Y = \sum_{i=1}^{n} Z_i^2$ has a chi-squared distribution with n degrees of freedom.

(14)

Find the moment generating function of Y

$$\phi_Y(t) = E[e^{tY}] = E[e^{t\sum Z_i^2}]$$

$$= E\left[\prod_{i=1}^{n} e^{tZ_i^2}\right]$$

$$= \prod_{i=1}^{n} E[e^{tZ_i^2}] \quad \text{by independence}$$

$$E[e^{tZ_i^2}] = \int_{-\infty}^{\infty} e^{tz_i^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_i^2} \, dz_i$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(z_i^2(1-2t)\right)} \, dz_i$$ (15)

$$= \frac{1}{\sqrt{(1-2t)}} \underbrace{\int_{-\infty}^{\infty} \frac{1}{(\sqrt{1-2t})^{-1}\sqrt{2\pi}} e^{-\frac{z_i^2}{2(\sqrt{1-2t})^{-2}}} \, dz_i}_{1}$$

$$\phi_{z_i}(t) = (1-2t)^{-\frac{1}{2}}$$

$$\phi_Y(t) = (1-2t)^{-\frac{n}{2}} = \left(\frac{1}{1-2t}\right)^{\frac{n}{2}}$$

$$= \left(\frac{\frac{1}{2}}{\frac{1}{2}-t}\right)^{\frac{n}{2}}$$ (16)

So $Y \sim \text{Gamma}\left(\alpha = \frac{n}{2}, \lambda = \frac{1}{2}\right)$

$\therefore \ \chi^2_n \sim \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$