Stat 267
3-14-17
(1)

<u>Defn</u>: State j is <u>accessible</u> from state i

if $P_{ij}^n > 0$ for some $n \geq 0$

If state i is accessible from state j and $\quad i \leftarrow j$
State j is accessible from state i, then $\quad i \rightarrow j$

State i and state j <u>communicate</u>. $\quad i \leftrightarrow j$

Communication is (1) reflexive $\quad P_{ii}^0 = 1$

(2) symmetric: $i \leftrightarrow j$ then $j \leftrightarrow i$

(3) transitive: $i \leftrightarrow j$ and $j \leftrightarrow k$

$$P_{ij}^n > 0, \ P_{jk}^m > 0 \Rightarrow P_{ij}^n P_{jk}^m > 0$$

$$P_{ik}^{n+m} = \sum_{r=0}^{\infty} P_{ir}^n P_{rk}^m \geq P_{ij}^n P_{jk}^m > 0 \qquad (2)$$

So k is accessible from i.

Similarly, i is accessible from k. $\therefore i \leftrightarrow k$

We have just shown that communication is an
<u>equivalence relation</u>

<u>Defn</u>. 2 states that communicate with each other
are said to be members of the same <u>class</u>.

Note: These classes <u>partition</u> the state space.

③

Defn. A Markov chain is irreducible if there is only 1 class.

Ex.
$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \end{array}$$

$0 \leftrightarrow 1$

$1 \leftrightarrow 2$

There is 1 class, so this chain is irreducible

④

Ex.
$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$0 \leftrightarrow 1$

$\{0,1\}$

$\{2\}$

$\{3\}$

3 classes

Note: 3 is an absorbing state

Defn: f_i is the probability that, starting in state i, the process eventually returns to state i

If $f_i = 1$, then state i is <u>recurrent</u> (5)

If $f_i < 1$, then state i is <u>transient</u>

Note: If a state is recurrent, then the process returns to that state an infinite number of times.

Suppose that state i is transient.
Find the probability that the process is in state i exactly n times, given that it starts in state i.

$$f_i^{n-1}(1-f_i) \sim \text{Geom}(p = f_i) \quad n \geq 1$$

(6)

Let $I_n = \begin{cases} 1 & \text{if } X_n = i \\ 0 & \text{if } X_n \neq i \end{cases}$

So $\sum_{n=0}^{\infty} I_n$ = # times the process is in state i

Find

$$E\left[\sum_{n=0}^{\infty} I_n \,\middle|\, X_0 = i\right] = \sum_{n=0}^{\infty} E[I_n \mid X_0 = i]$$

$$= \sum_{n=0}^{\infty}\Big[0 \cdot P[I_n = 0 \mid X_0 = i] + 1 \cdot P[I_n = 1 \mid X_0 = i]\Big]$$

7

$$= \sum_{n=0}^{\infty} P[I_n = 1 \mid X_0 = i]$$

$$= \sum_{n=0}^{\infty} P[X_n = i \mid X_0 = i]$$

$$= \sum_{n=0}^{\infty} P_{ii}^n = 1 + \sum_{n=1}^{\infty} P_{ii}^n$$

<u>Proposition</u>: State i is recurrent if $\sum_{n=1}^{\infty} P_{ii}^n = \infty$

and transient if $\sum_{n=1}^{\infty} P_{ii}^n < \infty$.

8

<u>Note</u>: Suppose you have a finite state Markov chain. Can all the states be transient? No

States: $0, 1, 2, \ldots, M$

$\uparrow$ if transients

n_1

n_M

Let n_0 be the expected # of times in state 0.

After $n_0 + n_1 + \cdots + n_M$ steps, there would be no place to go.

(9)

<u>Corollary</u> to the proposition:

If state i is recurrent and state i communicates with state j, then state j is recurrent.

<u>Proof</u>: $i \leftrightarrow j$ means $\exists\, k, m$ so that

$$P_{ij}^{k} > 0 \quad \text{and} \quad P_{ji}^{m} > 0$$

$$P_{jj}^{m+n+k} \geq P_{ji}^{m} P_{ii}^{n} P_{ij}^{k}$$

(10)

$$\sum_{n=1}^{\infty} P_{jj}^{m+n+k} \geq \sum_{n=1}^{\infty} P_{ji}^{m} P_{ii}^{n} P_{ij}^{k}$$

$$= \underbrace{P_{ji}^{m}}_{>0} \underbrace{P_{ij}^{k}}_{>0} \underbrace{\sum_{n=1}^{\infty} P_{ii}^{n}}_{\substack{\infty \text{ since } i \\ \text{is recurrent}}}$$

$$= \infty$$

$\therefore$ j is recurrent

11

Note: transience and recurrence are both class properties.

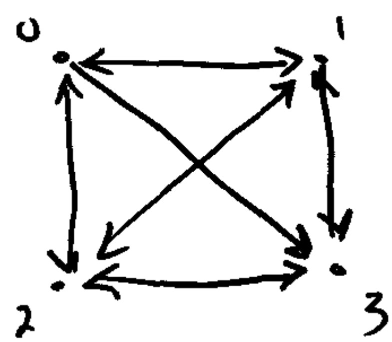
Suppose you have a finite state Markov chain. We saw that at least 1 state must be recurrent. Suppose that this chain is also irreducible. Then all of its states are recurrent.

12

Ex: $$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Irreducible

$\therefore$ All 4 states are recurrent



Ex:

$$P = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 2 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 3 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 4 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} \end{array}$$

$\{0,1\}$ recurrent

$\{2,3\}$ recurrent

$\{4\}$ transient