From last time: Stat 567

2-2-17

①

$$E[g(X)] = \begin{cases} \sum_{\text{all } x} g(x)p(x) & \text{discrete} \\ \int_{-\infty}^{\infty} g(x) f(x)\, dx & \text{continuous} \end{cases}$$

and $\text{Var}[X] = E[X^2] - (E[X])^2$

Find the variance for a Poisson r.v.

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad x = 0, 1, \ldots$$

Know $E[X] = \lambda$

②

$$E[X^2] = \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x}{(x-1)!} \qquad \text{Let } y = x-1$$

$$= \sum_{y=0}^{\infty} (y+1) \frac{e^{-\lambda}\lambda^{y+1}}{y!}$$

$$= \lambda\left[\sum_{y=0}^{\infty} y \frac{e^{-\lambda}\lambda^y}{y!} + \sum_{y=0} \frac{e^{-\lambda}\lambda^y}{y!}\right]$$

③

$$= \lambda\Big[\underbrace{E[Y]}_{\lambda} + 1\Big]$$

$$E[X^2] = \lambda^2 + \lambda$$

$$V[X] = E[X^2] - (E[X])^2 = \lambda^2 + \lambda - \lambda^2$$
$$= \lambda$$

$$\therefore \quad \mu = \lambda, \quad \sigma^2 = \lambda \quad \text{for the Poisson r.v.}$$

④

Find the variance for a Uniform r.v.

$$f(x) = \frac{1}{\beta - \alpha}, \qquad \alpha < X < \beta$$

$$E[X] = \frac{\alpha+\beta}{2}$$

$$E[X^2] = \int_\alpha^\beta x^2 \frac{1}{\beta-\alpha}\,dx = \frac{1}{\beta-\alpha}\frac{X^3}{3}\Bigg|_\alpha^\beta$$

$$= \frac{\beta^3 - \alpha^3}{3(\beta-\alpha)} = \frac{\cancel{(\beta-\alpha)}(\beta^2 + \alpha\beta + \alpha^2)}{3\cancel{(\beta-\alpha)}}$$

⑤

$$V[X] = \frac{\beta^2 + \alpha\beta + \alpha^2}{3} - \left(\frac{\alpha+\beta}{2}\right)^2$$

$$= \frac{4\beta^2 + 4\alpha\beta + 4\alpha^2 - [3\alpha^2 + 6\alpha\beta + 3\beta^2]}{12}$$

$$= \frac{\beta^2 - 2\alpha\beta + \alpha^2}{12} = \frac{(\beta-\alpha)^2}{12}$$

$\mu = \frac{\alpha+\beta}{2}$

$\sigma^2 = \frac{(\beta-\alpha)^2}{12}$ for the uniform r.v.

⑥

Skip to sec. 2.6

<u>Defn</u>: The moment generating function $\phi(t)$ for a random variable X is $E[e^{tX}]$.

Note: $E[e^{tX}] = \begin{cases} \sum_{\text{all } x} e^{tx} p(x) & \text{disc.} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx & \text{cont.} \end{cases}$

(7)

Properties: $\phi(0) = E[e^{0X}] = E[1]$
$= 1$

$\phi'(t) = \frac{d}{dt} E[e^{tX}]$

$= E[\frac{d}{dt} e^{tX}] = E[e^{tX} X]$

$\phi'(0) = E[X]$

$\phi''(t) = \frac{d}{dt} E[e^{tX} X] = E[e^{tX} X^2]$

$\phi''(0) = E[X^2]$

(8)

In general, $\phi^{(n)}(0) = E[X^n]$

Bernoulli: $X = \begin{cases} 0 & q \\ 1 & p \end{cases}$

$E[e^{tX}] = \sum_{\text{all } x} e^{tx} p(x) = 1 \cdot q + e^t p$

$\phi(t) = pe^t + q$ $\quad\quad \phi(0) = 1$ ✓

$\phi'(t) = pe^t$ $\quad\quad \phi'(0) = p = \mu$

$\phi''(t) = pe^t$ $\quad\quad \phi''(0) = p = E[X^2]$

(9)

$$\sigma^2 = p - p^2 = p(1-p) = pq$$

Binomial $\quad p(x) = \binom{n}{x} p^x q^{n-x} \quad x = 0, 1, \ldots, n$

$$\phi(t) = E[e^{tX}] = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x}$$

$$= (pe^t + q)^n \sum_{x=0}^{n} \binom{n}{x} \left(\frac{pe^t}{pe^t + q}\right)^x \left(\frac{q}{pe^t + q}\right)^{n-x}$$

(10)

$$\phi(t) = (pe^t + q)^n \qquad \phi(0) = 1 \checkmark$$

$$\phi'(t) = n(pe^t + q)^{n-1} pe^t \qquad \phi'(0) = np = \mu$$

$$\phi''(t) = np\left[(pe^t + q)^{n-1} e^t + (n-1)(pe^t + q)^{n-2} pe^t e^t\right]$$

$$\phi''(0) = np\left[1 + (n-1)p\right]$$

$$= np + n(n-1)p^2 = E[X^2]$$

$$\sigma^2 = np + n(n-1)p^2 - n^2p^2$$

$$= np + \cancel{n^2p^2} - np^2 - \cancel{n^2p^2} = np(1-p) = npq$$

Poisson $\quad p(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad x = 0, 1, \ldots$ (11)

$$\phi(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t} \underbrace{\sum_{x=0}^{\infty} \frac{e^{-\lambda e^t}(\lambda e^t)^x}{x!}}_{1}$$

$$\phi(t) = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$$ (12)

$$\phi(0) = 1 \quad \checkmark$$

Uniform

$$\phi(t) = E[e^{tX}] = \int_{\alpha}^{\beta} e^{tx} \frac{1}{\beta - \alpha} \, dx$$

$$= \frac{1}{\beta - \alpha} \left. \frac{e^{tx}}{t} \right|_{x=\alpha}^{\beta} = \frac{e^{\beta t} - e^{\alpha t}}{t(\beta - \alpha)}$$

(13)

$\phi(0)$ does not exist

But $\lim_{t \to 0} \phi(t) = \lim_{t \to 0} \frac{\beta e^{\beta t} - \alpha e^{\alpha t}}{(\beta - \alpha)} = 1$

Exponential $\quad f(x) = \lambda e^{-\lambda x} \quad x > 0$

$$\phi(t) = E[e^{tX}] = \int_0^\infty e^{tx} \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^\infty e^{-(\lambda - t)x}\, dx$$

$$= \lambda \left. \frac{e^{-(\lambda-t)x}}{-(\lambda - t)} \right|_0^\infty$$

(14)

$$= \lambda \left[0 + \frac{1}{\lambda - t} \right] \quad \text{for } \lambda - t > 0$$

$$t < \lambda$$

$$= \frac{\lambda}{\lambda - t} \qquad \phi(0) = 1 \ \checkmark$$

$$\phi'(t) = \lambda(-1)(\lambda - t)^{-2}(-1) = \lambda(\lambda - t)^{-2}$$

$$\phi''(t) = \lambda(-2)(\lambda - t)^{-3}(-1) = 2\lambda(\lambda - t)^{-3}$$

$$\phi'(0) = \lambda \lambda^{-2} = \frac{1}{\lambda} = \mu \qquad \sigma^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$\phi''(0) = 2\lambda \lambda^{-3} = \frac{2}{\lambda^2} \qquad = \frac{1}{\lambda^2}$$

(15)

Gamma $f(x) = \dfrac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}$ $x > 0$

$$\phi(t) = E[e^{tX}] = \int_0^\infty e^{tx} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} dx$$

$$= \int_0^\infty \frac{\lambda e^{-(\lambda-t)x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} dx$$

$$= \frac{\lambda \lambda^{\alpha-1}}{(\lambda-t)(\lambda-t)^{\alpha-1}} \underbrace{\int_0^\infty \frac{(\lambda-t) e^{-(\lambda-t)x}\left((\lambda-t)x\right)^{\alpha-1}}{\Gamma(\alpha)} dx}_{1}$$

(16)

$$= \left(\frac{\lambda}{\lambda-t}\right)^\alpha \qquad \phi(0) = 1 \checkmark$$

$$\phi'(t) = \lambda^\alpha(-\alpha)(\lambda-t)^{-\alpha-1}(-1)$$

$$= \alpha\lambda^\alpha(\lambda-t)^{-\alpha-1}$$

$$\phi''(t) = \alpha\lambda^\alpha(-\alpha-1)(\lambda-t)^{-\alpha-2}(-1)$$

$$\phi'(0) = \alpha\lambda^\alpha\lambda^{-\alpha-1} = \frac{\alpha}{\lambda} = \mu$$

$$\phi''(0) = \alpha\lambda^\alpha(-\alpha-1)\lambda^{-\alpha-2}(-1) = \frac{\alpha(\alpha+1)}{\lambda^2} = E[X^2]$$

(17)

$$\sigma^2 = \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2}$$

Normal $\quad f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$$\phi(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{tx - \frac{1}{2\sigma^2}(x^2 - 2\mu x + \mu^2)} dx$$

(18)

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\left(-2\sigma^2 tx + x^2 - 2\mu x + \mu^2\right)} dx$$

$$x^2 - 2(\mu + \sigma^2 t)x + \mu^2$$

$$x^2 - 2(\mu+\sigma^2 t)x + (\mu+\sigma^2 t)^2 - (\mu+\sigma^2 t)^2 + \mu^2$$

34. Let the probability density of X be given by

$$f(x) = \begin{cases} c(4x - 2x^2), & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) What is the value of c?

(b) $P\left\{\frac{1}{2} < X < \frac{3}{2}\right\} = ?$

37. Let $X_1, X_2, \ldots, X_n$ be independent random variables, each having a uniform distribution over (0,1). Let $M = \text{maximum}\,(X_1, X_2, \ldots, X_n)$. Show that the distribution function of M, $F_M(\cdot)$, is given by

$$F_M(x) = x^n, \qquad 0 \le x \le 1$$

What is the probability density function of M?

52. (a) Calculate $E[X]$ for the maximum random variable of Exercise 37.
(b) Calculate $E[X]$ for X as in Exercise 33.
(c) Calculate $E[X]$ for X as in Exercise 34.

58. An urn contains $2n$ balls, of which r are red. The balls are randomly removed in n successive pairs. Let X denote the number of pairs in which both balls are red.
(a) Find $E[X]$.
(b) Find $\text{Var}(X)$.

63. Calculate the moment generating function of a geometric random variable.