Brownian Motion

Stat 528
6-6-17

(1)

A random walk is a Markov chain where $P_{i,i+1} = p$, $P_{i,i-1} = 1-p$, i is any integer

A symmetric random walk is a random walk with $p = .5$

Suppose that, at each time Δt we take a step of size Δx. Let $X(t)$ be our position at time t.

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Let $X_i = \begin{cases} 1 & \text{if the } i^{\text{th}} \text{ step was to the right} \\ -1 & \text{" " " " " " left} \end{cases}$

So $X_1, X_2, \ldots$ are iid random variables with $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$

$$E[X_i] = 0 \qquad V[X_i] = E[X_i^2] = 1$$

$$X(t) = X_1 \Delta x + X_2 \Delta x + \ldots + X_{[t/\Delta t]} \Delta x$$

$$E[X(t)] = 0$$

③

$$V[X(t)] = (\Delta x)^2 V[X_1] + \dots + (\Delta x)^2 V[X_{[t/\Delta t]}]$$

$$= (\Delta x)^2 [t/\Delta t]$$

Now, let both Δx and Δt go to zero, so that $\frac{(\Delta x)^2}{\Delta t}$ remains constant $= \sigma^2$

Then $V[X(t)] \to \sigma^2 t$

④

Notes: - Since $X(t)$ was the sum of iid random variables, the central limit theorem says that $X(t)$ will have a limiting normal distribution,

i.e. $X(t) \xrightarrow{\mathcal{D}} N(0, \sigma^2 t)$

- Since changes in the value of $X(t)$ in nonoverlapping time intervals are independent,

$\{X(t), t \geq 0\}$ has independent increments, ⑤

i.e. $X(t_1)$, $X(t_2)-X(t_1)$, $X(t_3)-X(t_2)$,

$\ldots$, $X(t_n)-X(t_{n-1})$ are independent

for $0<t_1<t_2<\ldots<t_n$

- $X(t)$ has stationary increments,

  i.e. $X(t+s)-X(t)$ depends only on s

  (not t)

⑥

Defn: A stochastic process $\{X(t), t\geq 0\}$

is a Brownian Motion process if

(i) $X(0)=0$

(ii) $\{X(t), t\geq 0\}$ has stationary and independent increments

(iii) $\forall t$, $X(t)\sim N(0, \sigma^2 t)$

If $\sigma=1$, then $\{X(t), t\geq 0\}$ is a standard Brownian Motion process.

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Note: If $X(t)$ is a Brownian motion process,

then $\frac{X(t)}{\sigma}$ is a standard $\cdots\cdots$

Since $X(t) \sim N(0, t)$, (Assume $\sigma = 1$ without loss of generality)

$$f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}\frac{x^2}{t}}$$

Find the joint density of $X(t_1), X(t_2), \ldots, X(t_n)$

for $0 < t_1 < t_2 < \ldots < t_n$

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The event
$$\begin{cases} X(t_1) = x_1 \\ X(t_2) = x_2 \\ \vdots \\ X(t_n) = x_n \end{cases}$$

is equivalent to the event
$$\begin{cases} X(t_1) = x_1 \\ X(t_2) - X(t_1) = x_2 - x_1 \\ \vdots \\ X(t_n) - X(t_{n-1}) = x_n - x_{n-1} \end{cases}$$

$$X(t_k) - X(t_{k-1}) \sim N(0, t_k - t_{k-1})$$

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Now

$$f(x_1, \dots, x_n) = f_{t_1}(x_1)\, f_{t_2-t_1}(x_2-x_1) \cdots f_{t_n-t_{n-1}}(x_n-x_{n-1})$$

$$= \frac{1}{\sqrt{2\pi t_1}} e^{-\frac{1}{2}\frac{x_1^2}{t_1}} \prod_{k=2}^{n} \frac{1}{\sqrt{2\pi(t_k-t_{k-1})}} e^{-\frac{1}{2}\frac{(x_k-x_{k-1})^2}{t_k-t_{k-1}}}$$

Use this to find the conditional distribution of $X(s)$, given $X(t) = B$, where $s<t$

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$$f_{s|t}(x|B) = \frac{f_{s,t}(x, B)}{f_t(B)}$$

$$= \frac{\frac{1}{\sqrt{2\pi s}} e^{-\frac{1}{2}\frac{x^2}{s}} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{1}{2}\frac{(B-x)^2}{(t-s)}}}{\frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}\frac{B^2}{t}}}$$

$$= K e^{-\frac{1}{2}\left[\frac{x^2}{s} + \frac{(B-x)^2}{t-s}\right]}$$

$\uparrow$

free of x

(11)

$$= K e^{-\frac{1}{2}\left[\frac{(t-s)x^2 + s(B^2 - 2Bx + x^2)}{s(t-s)}\right]}$$

$$= K' e^{-\frac{1}{2}\left[\frac{t x^2 + x(-2sB)}{s(t-s)}\right]}$$

$$= K'' e^{-\frac{1}{2}\left[\frac{t\left[x^2 - \frac{2sB}{t}x + \frac{s^2B^2}{t^2}\right]}{s(t-s)}\right]}$$

$$= K'' e^{-\frac{1}{2}\frac{\left(x - \frac{s}{t}B\right)^2}{\frac{s}{t}(t-s)}}$$

So $X(s) \mid X(t)=B \sim N\left(\frac{s}{t}B, \frac{s}{t}(t-s)\right)$

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Example: Bike race with 2 competitors

Let $Y(t)$ = amount of time (in seconds) that the inside position is ahead when $100t\%$ of the race is completed.

Assume $\{Y(t), 0 \le t \le 1\}$ is a Brownian motion process with variance parameter σ^2.

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If the inside racer is ahead by σ seconds at the midpoint of the race, find the prob that he/she wins.

$$P[Y(1) > 0 \mid Y(\tfrac{1}{2}) = \sigma]$$

$$= P[Y(1) - Y(\tfrac{1}{2}) > -\sigma \mid Y(\tfrac{1}{2}) = \sigma]$$

$$= P[Y(1) - Y(\tfrac{1}{2}) > -\sigma] \quad \text{by independent movements}$$

$$= P[Y(\tfrac{1}{2}) > -\sigma] \quad \text{by stationary movements}$$

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Also $Y(\tfrac{1}{2}) \sim N(0, \sigma^2 \cdot \tfrac{1}{2})$

So $\dfrac{Y(\frac{1}{2}) - 0}{\sigma/\sqrt{2}} \sim N(0,1)$

$$P[Y(\tfrac{1}{2}) > -\sigma] = P\left[Z > \frac{-\sigma}{\sigma/\sqrt{2}}\right]$$

$$= P[Z > -\sqrt{2}] = .9214$$

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If the inside racer wins by σ seconds,
find the probability that he/she was
ahead at the midpoint.

$$P\left[Y(\tfrac{1}{2}) > 0 \,\middle|\, Y(1) = \sigma\right]$$

Need the conditional distribution of $Y(s)$,
given $Y(t) = c$, for $s < t$

Let $X(t) = \dfrac{Y(t)}{\sigma}$. Then $X(t)$ is a stan. B. mo. process

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We showed the cond. dist. of $X(s)$, given $X(t) = B$

$$\text{is } N\left(\frac{s}{t}\underset{\uparrow\ \frac{c}{\sigma}}{B},\ \frac{s}{t}(t-s)\right)$$

$$Y(s) = \sigma X(s)$$

$$\text{so } \quad Y(s) \,|\, Y(t) = c \sim N\left(\frac{s}{t}c,\ \frac{s}{t}(t-s)\sigma^2\right)$$

$$P\left[Y(\tfrac{1}{2}) > 0 \,\middle|\, Y(1) = \sigma\right] = P[W > 0] \text{ where}$$

$$W \sim N\left(\frac{\frac{1}{2}}{1}\sigma,\ \frac{\frac{1}{2}}{1}(1-\tfrac{1}{2})\sigma^2\right)$$

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$$W \sim N\left(\frac{\sigma}{2}, \frac{\sigma^2}{4}\right)$$

$$P[W > 0] = P\left[Z > \frac{0 - \frac{\sigma}{2}}{\frac{\sigma}{2}}\right]$$

$$= P[Z > -1] = .8413$$