Stat 567
2-21-17

①

From last time:

$Z_1, \dots, Z_n$ iid $N(0,1)$

$$Y = \sum_{i=1}^{n} Z_i^2$$

We saw that $Y \sim \underbrace{\text{Gamma}\left(\alpha = \frac{1}{2}, \lambda = \frac{1}{2}\right)}_{\chi^2_n}$

<u>Defn</u>: The bivariate normal density function is

$$f(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{\bigstar}$$

②

where

$$\bigstar = -\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]$$

and $\rho = \dfrac{\sigma_{12}}{\sigma_1\sigma_2}$ $\leftarrow \text{Cov}(X_1,X_2)$

Note: $\rho = 0 \iff f(x_1,x_2) = f_{X_1}(x_1)\, f_{X_2}(x_2)$

That is, in the bivariate normal distribution, independence is equivalent to 0 correlation (or 0 covariance)

③

Recall the 3rd property of $\bar{X}$:

$$\mathrm{Cov}(\bar{X}, X_i - \bar{X}) = 0 \quad \forall i$$

If $X_1, \ldots, X_n$ are multivariate normal,

then $\bar{X}, X_i - \bar{X}$ will have a bivariate

normal distribution.

Since $\mathrm{Cov} = 0$, they must be independent.

Since $\bar{X}$ is independent of $X_i - \bar{X}$ $\forall i$,

④

$\bar{X}$ must be indep. of $\dfrac{\sum (X_i - \bar{X})^2}{n-1} = S^2$

$\therefore$ If $X_1, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$,

then $\bar{X}$ is indep of S^2

Recall $(n-1)S^2 = \sum_{i=1}^{n} (X_i - \mu)^2 - n(\bar{X} - \mu)^2$

(Used this to show $E(S^2) = \sigma^2$)

(5)

Now assume $X_1, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$

$$\frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^{n} \left(\frac{X_i-\mu}{\sigma}\right)^2 - \frac{n(\bar{X}-\mu)^2}{\sigma^2}$$

$$\underbrace{\frac{(n-1)S^2}{\sigma^2}}_{?} + \underbrace{\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2}_{\chi^2_1} = \underbrace{\sum_{i=1}^{n} \left(\frac{X_i-\mu}{\sigma}\right)^2}_{\chi^2_n}$$

these are independent

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Since they are indep., the mgf of their sum is the product of their individual mgfs.

$$\phi(t) \left(\frac{1}{1-2t}\right)^{\frac{1}{2}} = \left(\frac{1}{1-2t}\right)^{\frac{n}{2}}$$

$$\therefore \quad \phi(t) = \left(\frac{1}{1-2t}\right)^{\frac{n-1}{2}}$$

$$\therefore \quad \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

(7)

Limit Theorems

Markov's Inequality:

If X is a nonnegative random variable, then

$$\forall a > 0, \quad P(X \geq a) \leq \frac{E[X]}{a}$$

Proof: (continuous case)

$$E[X] = \int_0^\infty x f(x)dx = \int_0^a x f(x)dx + \int_a^\infty x f(x)dx$$

$$\geq \int_a^\infty x f(x)dx \geq \int_a^\infty a f(x)dx$$

$$= a P[X \geq a] \quad //$$

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Chebyshev's Inequality:

$$\forall k > 0, \quad P[|X-\mu| \geq k] \leq \frac{\sigma^2}{k^2}$$

Pf: $(X-\mu)^2$ is non-negative. Apply the Markov inequality with $a = k^2$

$$P[(X-\mu)^2 \geq k^2] \leq \frac{E[(X-\mu)^2]}{k^2}$$

$$P[|X-\mu| \geq k] \leq \frac{\sigma^2}{k^2} \quad //$$

(9)

Note: Chebyshev's Inequality is often stated as:

$$P[|X-\mu| \ge k\sigma] \le \frac{1}{k^2} \quad [\text{let } k = k^*\sigma)$$

<u>Central Limit Theorem</u>

Let $X_1, X_2, \ldots$ be iid with mean μ,
variance σ^2

Then $$\lim_{n\to\infty} P\left[\frac{X_1+\ldots+X_n-n\mu}{\sigma\sqrt{n}} \le a\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-\frac{x^2}{2}}dx$$

(10)

Let $Z_i = \frac{X_i-\mu}{\sigma}$. So $E[Z_i]=0$
and $V(Z_i) = 1$ $(E(Z^2)=1)$

Let $$Y_n = \frac{X_1+X_2+\ldots+X_n-n\mu}{\sigma\sqrt{n}}$$

$$= \frac{(X_1-\mu)+\ldots+(X_n-\mu)}{\sigma\sqrt{n}} = \frac{1}{\sqrt{n}}\left(\frac{X_1-\mu}{\sigma}+\ldots+\frac{X_n-\mu}{\sigma}\right)$$

$$= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_i$$

(11)

$$\phi_{Y_n}(t) = E[e^{tY_n}] = E[e^{t\frac{1}{\sqrt{n}}\Sigma Z_i}]$$

$$= E[\prod_{i=1}^{n} e^{\frac{t}{\sqrt{n}}Z_i}]$$

$$= \prod_{i=1}^{n} E[e^{t/\sqrt{n}\, Z_i}] \quad \text{by independence}$$

$$= \left(E[e^{t/\sqrt{n}\, Z}]\right)^n \quad \text{by i.d.}$$

$$= \left(E[1 + \frac{t}{\sqrt{n}}Z + \frac{t^2 Z^2}{2n} + O(n^{-3/2})]\right)^n$$

(12)

$$\phi_{Y_n}(t) = \left(1 + 0 + \frac{t^2}{2n} + O(n^{-3/2})\right)^n$$

$$\ln \phi_{Y_n}(t) = n\ln\left(1 + \frac{t^2}{2n} + O(n^{-3/2})\right)$$

$$= \frac{\ln(1 + \frac{t^2}{2n} + O(n^{-3/2}))}{\frac{1}{n}}$$

$$\lim_{n\to\infty} \ln \phi_{Y_n}(t) = \lim_{n\to\infty} \frac{\frac{1}{1 + \frac{t^2}{2n} + O(n^{-3/2})}\left[\frac{t^2}{2}\left(\frac{-1}{n^2}\right) + O(n^{-5/2})\right]}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{t^2}{2n}+O(n^{-3/2})}\left[\frac{t^2}{2}+O(n^{-1/2})\right]$$

$$\lim_{n\to\infty} \ln \phi_{Y_n}(t) = \frac{t^2}{2}$$

$\lim_{n\to\infty} \phi_{Y_n}(t) = e^{t^2/2}$, which is the mgf of the $N(0,1)$ distribution

So the distribution of Y_n approaches the $N(0,1)$ distribution as $n \to \infty$

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68. Let $X_1, X_2, \ldots, X_{10}$ be independent Poisson random variables with mean 1.

(a) Use the Markov inequality to get a bound on $P\{X_1 + \cdots + X_{10} \geq 15\}$.

(b) Use the central limit theorem to approximate $P\{X_1 + \cdots + X_{10} \geq 15\}$.

70. Show that

$$\lim_{n \to \infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}$$

Hint: Let X_n be Poisson with mean n. Use the central limit theorem to show that $P\{X_n \leq n\} \to \frac{1}{2}$.

8. An unbiased die is successively rolled. Let X and Y denote, respectively, the number of rolls necessary to obtain a six and a five. Find (a) $E[X]$, (b) $E[X|Y = 1]$, (c) $E[X|Y = 5]$.

14. Let X be uniform over (0, 1). Find $E[X|X < \frac{1}{2}]$.

15. The joint density of X and Y is given by

$$f(x, y) = \frac{e^{-y}}{y}, \quad 0 < x < y, \quad 0 < y < \infty$$

Compute $E[X^2|Y = y]$.