The Yule process (George Yule, 1871-1951)

Stat 568
5-2-17

Pure birth process with $\lambda_n = n\lambda$ ①

Proposition from last time:

$$P_{ij}(t) = \sum_{k=i}^{j} e^{-\lambda_k t} \prod_{\substack{r=i \\ r\neq k}}^{j} \frac{\lambda_r}{\lambda_r - \lambda_k} - \sum_{k=i}^{j-1} e^{-\lambda_k t} \prod_{\substack{r=i \\ r\neq k}}^{j-1} \frac{\lambda_r}{\lambda_r - \lambda_k}$$

Find $P_{1j}(t)$ ②

$$P_{1j}(t) = \sum_{k=1}^{j} e^{-k\lambda t} \prod_{\substack{r=1 \\ r\neq k}}^{j} \frac{r\lambda}{r\lambda - k\lambda} - \sum_{k=1}^{j-1} e^{-k\lambda t} \prod_{\substack{r=1 \\ r\neq k}}^{j-1} \frac{r\lambda}{r\lambda - k\lambda}$$

$$= \underbrace{e^{-j\lambda t} \prod_{r=1}^{j-1} \frac{r}{r-j}}_{\star} + \sum_{k=1}^{j-1} e^{-k\lambda t} \left(\prod_{\substack{r=1 \\ r\neq k}}^{j} \frac{r}{r-k} - \prod_{\substack{r=1 \\ r\neq k}}^{j-1} \frac{r}{r-k} \right)$$

③

$$\star = \frac{1}{1-j}\cdot\frac{2}{2-j}\cdots\frac{j-1}{-1} = \frac{(j-1)!}{(j-1)!\,(-1)^{j-1}}$$

$$= (-1)^{j-1}$$

$$P_{ij}(t) = e^{-j\lambda t}(-1)^{j-1} + \sum_{k=1}^{j-1} e^{-k\lambda t}\underbrace{\left[\prod_{\substack{r=1\\ r\neq k}}^{j-1}\frac{r}{r-k}\left(\frac{j}{j-k}-1\right)\right]}_{\star\star}$$

$$\star\star = \boxed{\frac{k}{j-k}}\,\frac{1}{1-k}\cdot\frac{2}{2-k}\cdots\frac{j-1}{j-1-k} = \frac{(j-1)!}{(1-k)(2-k)\cdots(-1)(1)\cdots(j-1-k)}$$

skips $r=k$

$j-k$

④

$$= \frac{(j-1)!}{(-1)^{k-1}(k-1)!\,(j-k)!} = (-1)^{k-1}\binom{j-1}{k-1}$$

$$P_{ij}(t) = e^{-j\lambda t}(-1)^{j-1} + \sum_{k=1}^{j-1} e^{-k\lambda t}\left[(-1)^{k-1}\binom{j-1}{k-1}\right]$$

$$= \sum_{k=1}^{j} e^{-k\lambda t}(-1)^{k-1}\binom{j-1}{k-1}$$

$$= e^{-\lambda t}\sum_{k=1}^{j} e^{-(k-1)\lambda t}(-1)^{k-1}\binom{j-1}{k-1}$$

Let $m = k-1$ (5)

$$P_{ij}(t) = e^{-\lambda t} \sum_{m=0}^{j-1} e^{-m\lambda t} (-1)^m \binom{j-1}{m}$$

$$= e^{-\lambda t} \sum_{m=0}^{j-1} \binom{j-1}{m} \left(-e^{-\lambda t}\right)^m 1^{j-1-m}$$

$$P_{ij}(t) = e^{\lambda t} \left(1 - e^{-\lambda t}\right)^{j-1},$$ which is

a geometric distribution with

$$p = e^{-\lambda t},$$

so the expected value is $\frac{1}{p} = e^{\lambda t}$

Assume that you start with i individuals (6)

& assume independent births, then

$P_{ij}(t)$ will follow a Pascal or Negative Binomial distribution, i.e.

$$P_{ij}(t) = \binom{j-1}{i-1} e^{-\lambda i t} \left(1 - e^{-\lambda t}\right)^{j-i}, \quad 1 \le i \le j$$

Next goal: find the limiting probabilities in a general continuous-time Markov chain. (7)

Recall τ_i = time in state i before transitioning

$T_i \sim \text{expon}(\nu_i)$

<u>Defn</u>: let $q_{ij} = \nu_i P_{ij}$

These are the <u>instantaneous</u> <u>transition</u> <u>rates</u>

Note: q_{ij} is the rate at which state i transitions to state j

(8)

Note: Since $\sum_j P_{ij} = 1$,

$$\nu_i = \nu_i \sum_j P_{ij} = \sum_j \nu_i P_{ij} = \sum_j q_{ij}$$

Note: $P_{ij} = \dfrac{q_{ij}}{\nu_i} = \dfrac{q_{ij}}{\sum_j q_{ij}}$

<u>Lemma 1</u>: a) $\displaystyle\lim_{h\to 0} \frac{1-P_{ii}(h)}{h} = \nu_i$

b) $\displaystyle\lim_{h\to 0} \frac{P_{ij}(h)}{h} = q_{ij}$ for $i \neq j$

9

Proof: a) $1 - P_{ii}(h) =$

prob(not in state i at time h | state i at time 0)

= prob(a transition occurred in time h)

$= v_i h + o(h)$

So $\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = \lim_{h \to 0} \frac{v_i h + o(h)}{h} = v_i$

b) $P_{ij}(h) =$ prob(state j at time h | state i at time 0)

= prob(a transition occurred $\cap$ it was into j)

10

Proof: $P[A \cap B | C] = \frac{P[A \cap B \cap C]}{P[C]}$

$$= \frac{P[A \cap B \cap C]}{P[B \cap C]} \frac{P[B \cap C]}{P[C]}$$

$$= P[A | B \cap C] \, P[B | C]$$

$$P_{ij}(t+s) = P[X(t+s) = j \mid X(0) = i]$$

$$= \sum_{k=0}^{\infty} P[X(t+s) = j \cap X(t) = k \mid X(0) = i]$$

$$= (\nu_i h + o(h)) P_{ij}$$ 11

$$\lim_{h \to 0} \frac{P_{ij}(h)}{h} = \lim_{h \to 0} \frac{(\nu_i h + o(h)) P_{ij}}{h}$$

$$= \nu_i P_{ij} = q_{ij}$$

<u>Lemma 2</u>: $\forall\, s \geq 0,\ t \geq 0$,

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s)$$

these are the <u>Chapman-Kolmogorov Equations</u>

12

$$= \sum_{k=0}^{\infty} P[X(t+s)=j \mid X(t)=k \cap X(0)=i] \cdot P[X(t)=k \mid X(0)=i]$$

$$= \sum_{k=0}^{\infty} P[X(t+s)=j \mid X(t)=k]\, P[X(t)=k \mid X(0)=i]$$

$$= \sum_{k=0}^{\infty} P_{kj}(s) P_{ik}(t)$$

<u>Theorem</u>: $\frac{d}{dt} P_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$

Kolmogorov Backward Equations

<u>Proof</u>: By lemma 2,

$$P_{ij}(h+t) - P_{ij}(t) = \sum_{k=0}^{\infty} P_{ik}(h) P_{kj}(t) - P_{ij}(t)$$

$$= \sum_{k \neq i} P_{ik}(h) P_{kj}(t) + P_{ii}(h) P_{ij}(t) - P_{ij}(t)$$

$$= \sum_{k \neq i} P_{ik}(h) P_{kj}(t) - \left(1 - P_{ii}(h)\right) P_{ij}(t)$$