Single-server system with finite capacity N

Stat 568
5-30-17

(1)

If an arriving customer sees N people in the system already, they leave

State	rate of leaving		rate of entering
0	$P_0 \lambda$	$=$	$P_1 \mu$
$1 \le n \le N-1$	$P_n(\lambda+\mu)$	$=$	$P_{n-1}\lambda + P_{n+1}\mu$
N	$P_N \mu$	$=$	$P_{N-1}\lambda$

(2)

$$P_1 = \frac{\lambda}{\mu} P_0$$

$$P_1(\lambda+\mu) = P_0\lambda + P_2\mu$$

$$\frac{\lambda}{\mu} P_0(\lambda+\mu) = P_0\lambda + P_2\mu$$

$$\frac{\lambda^2}{\mu} P_0 = P_2\mu \qquad P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

etc

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

Also, $$1 = \sum_{n=0}^{N} P_n = P_0 \sum_{n=0}^{N} \left(\frac{\lambda}{\mu}\right)^n = P_0 \frac{1-\left(\frac{\lambda}{\mu}\right)^{N+1}}{1-\frac{\lambda}{\mu}}$$

③

Note: we no longer require $\frac{\lambda}{\mu} < 1$

$$P_0 = \frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{N+1}} \quad \text{and}$$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{N+1}}$$

L = expected # people in the system

$$= \sum_{n=0}^{N} n P_n$$

④

$$= \sum_{n=0}^{N} n \left(\frac{\lambda}{\mu}\right)^n \frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{N+1}}$$

Let $q = \frac{\lambda}{\mu}$

$p = \left(1 - \frac{\lambda}{\mu}\right)$

$$= \frac{q}{1 - q^{N+1}} \underbrace{\sum_{n=1}^{N} n q^{n-1} p}_{\star}$$

$$\star = \sum_{n=1}^{\infty} n q^{n-1} p - \sum_{n=N+1}^{\infty} n q^{n-1} p$$

① ②

① $= \frac{1}{p}$ (expected value of a geometric r.v.)

(2) let $x = n - N$

$$\sum_{x=1}^{\infty} (x+N) q^{x+N-1} p = q^N \sum_{x=1}^{\infty} (x+N) q^{x-1} p$$

$$= q^N E[X+N] = q^N\left(\frac{1}{p} + N\right)$$

$$\star = (1) - (2) = \frac{1}{p} - q^N\left(\frac{1}{p} + N\right)$$

$$L = \frac{q}{1-q^{N+1}}\left[\frac{1}{p} - q^N\left(\frac{1}{p} + N\right)\right]$$

$$L = \frac{\lambda}{\left[1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right](\mu-\lambda)}\left[1 + N\left(\frac{\lambda}{\mu}\right)^{N+1} - (N+1)\left(\frac{\lambda}{\mu}\right)^{N}\right] \quad (6)$$

Finally, W = average time in system

$$= \frac{L}{\lambda_a}, \text{ where}$$

$$\lambda_a = \begin{cases} \lambda & \text{if all arrivals are treated as "customers"} \\ \lambda(1-P_N) & \text{if only those entering the system are treated as customers} \end{cases}$$

(7)

<u>Example:</u>

Suppose that it costs $\$c\mu$ to provide service at rate μ.

We receive $\$A$ per customer served

Capacity is N.

How would you determine the value of μ that maximizes your profit?

λ is known.

(8)

Customers arrive at rate λ

Customer enter the system at rate $\lambda(1-P_N)$

We take in $\$\,\lambda(1-P_N)A$ per hour.

Profit per hour $= \lambda(1-P_N)A - c\mu$

$P =$

$$P = \lambda A\left[1 - \frac{\left(\frac{\lambda}{\mu}\right)^N\left(1-\frac{\lambda}{\mu}\right)}{1-\left(\frac{\lambda}{\mu}\right)^{N+1}}\right] - c\mu$$

(9)

$$= \frac{\lambda A\left[\mu^{N+1} - \lambda\mu\right]}{\mu^{N+1} - \lambda^{N+1}} - c\mu$$

Now take $\frac{dP}{d\mu}$ and set it equal to 0.

Find the expected number # of lost customers L_N

"idle" state is 0 in system

"busy" " " ≥ 1 " "

Suppose a busy period has just started

(10)

$$\text{Let } I = \begin{cases} 0 & \text{service completes before next arrival} \\ 1 & \text{arrive occurs before service is completed} \end{cases}$$

So $I \sim \text{Bernoulli}\left(p = \frac{\lambda}{\lambda+\mu}\right)$

$$E[I] = p = \frac{\lambda}{\lambda+\mu}$$

$$V[I] = pq = \frac{\lambda}{\lambda+\mu} \cdot \frac{\mu}{\lambda+\mu}$$

Note: If $I=0$, then no customers will be lost

$$E[L_N \mid I=0] = 0$$

If $I = 1$ 11

$$E[L_N \mid I = 1] = E[L_{N-1}] + E[L_N]$$

$E[L_N] =$

$$E\big[E[L_N \mid I]\big] = 0 \cdot P(I=0)$$

$$+ \big(E[L_{N-1}] + E[L_N]\big) P(I=1)$$

$$E[L_N] = \big(E[L_{N-1}] + E[L_N]\big) \frac{\lambda}{\lambda+\mu}$$

$$(\lambda+\mu) E[L_N] = \lambda E[L_{N-1}] + \lambda E[L_N]$$

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$$\mu E[L_N] = \lambda E[L_{N-1}]$$

$$E[L_N] = \frac{\lambda}{\mu} E[L_{N-1}]$$

$$= \ldots = \left(\frac{\lambda}{\mu}\right)^N \underbrace{E[L_0]}_{1}$$

$$E[L_N] = \left(\frac{\lambda}{\mu}\right)^N$$