Continuous random variables

Stat 567
1-26-17

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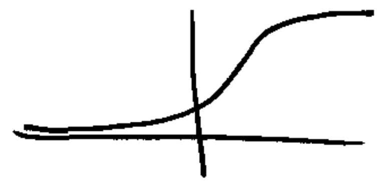
Uniform X takes on any value between α and β.



Recall: $F(b) = P(X \le b)$

$$F(b) - F(a) = P(X \le b) - P(X \le a)$$
$$= P(a < X \le b)$$

Let $f(x) = \frac{d}{dx} F(x)$

By the fundamental theorem of calculus,

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$$\int_a^b f(x)\,dx = F(b) - F(a)$$
$$= P(a < X \le b)$$

Defn: $f(x)$ is the probability density function

Note: In continuous cases,

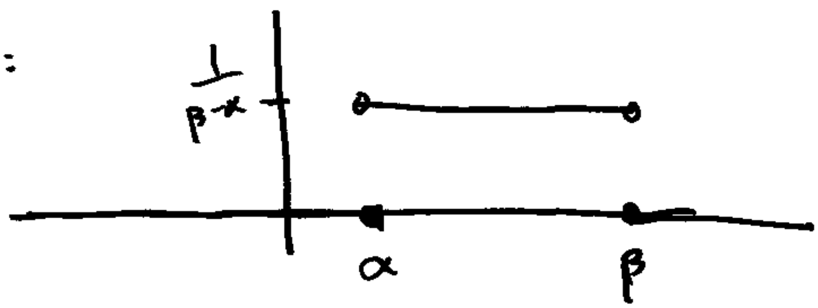
$$\int_a^a f(x)\,dx = 0$$

So $P(X = a)$ will always be 0

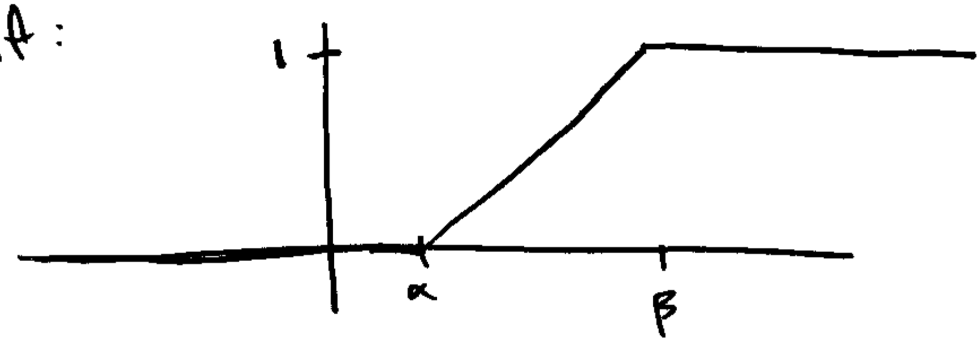
Back to the Uniform distribution:

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & \text{for } \alpha < x < \beta \\ 0 & \text{elsewhere} \end{cases}$$

Density:



Cdf:

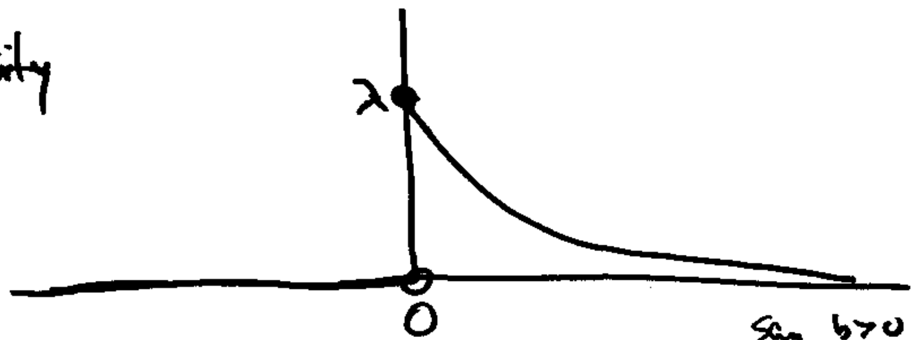


$$F(x) = \begin{cases} 0 & x \le \alpha \\ \dfrac{x-\alpha}{\beta-\alpha} & \alpha < x < \beta \\ 1 & x \ge \beta \end{cases}$$

(5)

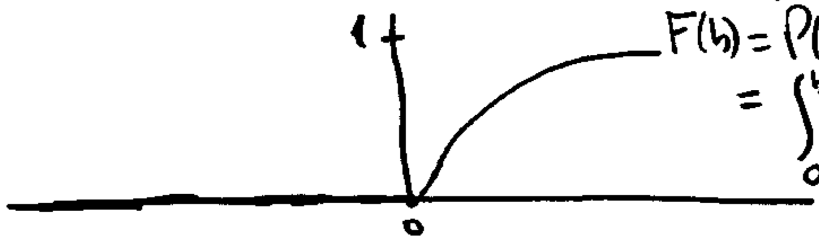
Exponential $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \quad (\lambda > 0) \\ 0 & \text{elsewhere} \end{cases}$

Density



say $b > 0$

cdf



$$F(b) = P(X \leq b) = \int_0^b \lambda e^{-\lambda x}\,dx$$

(6)

$$= -e^{-\lambda x}\Big|_{x=0}^{b}$$

$$= -e^{-\lambda b} - (-1)$$

$$= 1 - e^{-\lambda b}$$

Example: Suppose that lifetimes of a certain electronic component follow an exponential distribution with $\lambda = \frac{1}{3}$

Find the probability that $P(X \leq 3) = F(3)$

$$= 1 - e^{-\frac{1}{3}\cdot 3}$$

$$= 1 - \frac{1}{e}$$

Find $P(1 \le X \le 3) = F(3) - F(1)$

$$= \left(1 - e^{-\frac{1}{3}3}\right) - \left(1 - e^{-\frac{1}{3}\cdot 1}\right)$$

$$= e^{-\frac{1}{3}} - e^{-1}$$

<u>Gamma</u> <u>Defn</u>: The <u>gamma function</u> is

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1}\,dx$$

Properties of the gamma function:

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 1$$

let $\alpha > 1$ (8)

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1}\,dx \qquad \text{let } u = x^{\alpha-1}$$

$$dv = e^{-x}dx$$

$$du = (\alpha-1)x^{\alpha-2}dx \qquad v = -e^{-x}$$

$$= \overset{0}{\cancel{-x^{\alpha-1}e^{-x}\Big|_0^\infty}} + \int_0^\infty (\alpha-1)x^{\alpha-2}e^{-x}\,dx$$

$$\frac{x^{\alpha-1}}{e^x}$$

$$= (\alpha-1)\int_0^\infty x^{\alpha-2}e^{-x}\,dx = (\alpha-1)\Gamma(\alpha-1)$$

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Let α be any positive integer $\quad \Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$

$\Gamma(1) = 1 \qquad \Gamma(2) = 1 \cdot \Gamma(1) = 1$

$\Gamma(3) = 2\,\Gamma(2) = 2$

$\Gamma(4) = 3\,\Gamma(3) = 3 \cdot 2 = 6$

$\Gamma(5) = 4\,\Gamma(4) = 4 \cdot 6 = 24$

In general, $\Gamma(\alpha) = (\alpha-1)!$ if α is a positive integer

<u>Gamma distribution</u> $\quad f(x) = \dfrac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, \quad x \geq 0$

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Note: If $\alpha = 1$, the gamma distribution reduces to the exponential distribution

Check: $\displaystyle\int_0^\infty f(x)\,dx = \int_0^\infty \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}\,dx$

$\displaystyle = \int_0^\infty \frac{e^{-u}u^{\alpha-1}}{\Gamma(\alpha)}\,du \qquad$ Let $u = \lambda x$, $du = \lambda\,dx$

$\displaystyle = \frac{1}{\Gamma(\alpha)}\Gamma(\alpha) = 1$

(11)

$$\text{Cdf} = F(b) = P(X \le b) = \begin{cases} 0 & \text{if } b < 0 \\ \displaystyle\int_0^b \frac{\lambda e^{-\lambda x} (\lambda x)^{\alpha - 1}}{\Gamma(\alpha)}\, dx & \text{if } b \ge 0 \end{cases}$$

Let T be a random variable having a gamma distribution, Assume α is an integer.

Find $$P(T > t) = \int_t^\infty \frac{\lambda e^{-\lambda s} (\lambda s)^{\alpha-1}}{\Gamma(\alpha)}\, ds$$

(12)

$$= -\frac{(\lambda s)^{\alpha-1} e^{-\lambda s}}{\Gamma(\alpha)} \Bigg|_t^\infty + \int_t^\infty \frac{(\alpha-1)(\lambda s)^{\alpha-2} e^{-\lambda s}}{\Gamma(\alpha)}\, ds$$

Let $u = (\lambda s)^{\alpha-1}$

$dv = \dfrac{\lambda e^{-\lambda s}}{\Gamma(\alpha)}\, ds$

$du = (\alpha-1)(\lambda s)^{\alpha-2}\, ds$

$v = -\dfrac{e^{-\lambda s}}{\Gamma(\alpha)}$

$$= \underbrace{\frac{(\lambda t)^{\alpha-1} e^{-\lambda t}}{\cancel{\Gamma(\alpha)}\ (\alpha-1)!}}_{P(X = \alpha - 1) \text{ where } X \sim \text{Poisson}(\lambda t)} + \int_t^\infty \frac{(\alpha-1)(\lambda s)^{\alpha-2} e^{-\lambda s}}{\Gamma(\alpha)}\, ds$$

$$\frac{\mu^{x-1} e^{-\mu}}{(x-1)!}$$

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By continuing this process, we can show

$$P(T > t) = P(X = \alpha-1) + P(X = \alpha-2) + \dots + P(X=0)$$

where $X \sim \text{Poisson}(\lambda t)$

That is, $P(T > t) = P(X \le \alpha - 1)$

$T \sim \text{Gamma}(\alpha, \lambda)$ $\qquad$ $X \sim \text{Poisson}(\lambda t)$

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Example: Suppose that the time until failure of a component follows a gamma distribution with $\lambda = \frac{1}{3}$ and $\alpha = 2$

Find the probability that the time until failure is greater than 6

$$T \sim \text{Gamma}(\lambda = \tfrac{1}{3}, \alpha = 2)$$

$$P(T > 6) = P(X \le \overset{\alpha-1}{1}) \text{ where } X \sim \text{Poisson}\underset{\lambda t}{(2)}$$

$$= p(0) + p(1)$$

$$= e^{-2}\, 2^0/0! + e^{-2}\, 2^1/1! = 3e^{-2}$$

(15)

Normal $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$-\infty < x < \infty$

$-\infty < \mu < \infty, \ \sigma > 0$

Suppose that $X \sim N(\mu, \sigma)$

and $Y = aX + b \quad (a > 0)$

$$G_Y(y) = P(Y \le y) = P(aX + b \le y)$$
$$= P\left(X \le \frac{y-b}{a}\right)$$

(16)

$$= \int_{-\infty}^{\frac{y-b}{a}} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $u = ax + b$

$du = a\,dx$

$$= \int_{-\infty}^{y} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\frac{u-b}{a} - \mu}{\sigma}\right)^2} \frac{du}{a}$$

$$G_Y(y) = \int_{-\infty}^{y} \frac{1}{a\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u-(a\mu+b)}{a\sigma}\right)^2} du$$

So $g(y) = \frac{1}{a\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-(a\mu+b)}{a\sigma}\right)^2} \sim N(a\mu+b, a\sigma)$

4. Suppose a die is rolled twice. What are the possible values that the following random variables can take on?
   (a) The maximum value to appear in the two rolls.
   (b) The minimum value to appear in the two rolls.
   (c) The sum of the two rolls.
   (d) The value of the first roll minus the value of the second roll.
5. If the die in Exercise 4 is assumed fair, calculate the probabilities associated with the random variables in (i)–(iv).

16. An airline knows that 5 percent of the people making reservations on a certain flight will not show up. Consequently, their policy is to sell 52 tickets for a flight that can hold only 50 passengers. What is the probability that there will be a seat available for every passenger who shows up?

32. If you buy a lottery ticket in 50 lotteries, in each of which your chance of winning a prize is $\frac{1}{100}$, what is the (approximate) probability that you will win a prize (a) at least once, (b) exactly once, (c) at least twice?
33. Let X be a random variable with probability density

$$f(x) = \begin{cases} c(1 - x^2), & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

   (a) What is the value of c?
   (b) What is the cumulative distribution function of X?