Stat 568
5-23-17

Renewal processes

①

Recall: $\{X_1, X_2, \dots\}$ are the interarrival times

$\{N(t), t \geq 0\}$ is the renewal process

$F(x)$ is the cdf of each of the X_i's

$$S_n = \sum_{i=1}^{n} X_i$$

$$N(t) \geq n \iff S_n \leq t$$

$$P[N(t) = n] = P[S_n \leq t] - P[S_{n+1} \leq t] \quad *$$
$$= F_n(t) - F_{n+1}(t)$$

②

<u>Example</u> $P[X_n = i] = p(1-p)^{i-1}$ i is an integer ≥ 1

$S_1 = X_1$ = time until the 1^{st} success

$S_n = \sum_{i=1}^{n} X_i$ = time until the n^{th} success

so S_n has a negative binomial (Pascal) distribution

$$P[S_n = k] = \binom{k-1}{n-1} p^n (1-p)^{k-n} \quad k \geq n$$

③

Using $*$,

$$P[N(t)=n] = F_n(t) - F_{n+1}(t)$$

$$= P[S_n \le t] - P[S_{n+1} \le t]$$

$$= \sum_{k=n}^{[t]} P[S_n = k] - \sum_{k=n+1}^{[t]} P[S_{n+1} = k]$$

$$= \sum_{k=n}^{[t]} \binom{k-1}{n-1} p^n q^{k-n} - \sum_{k=n+1}^{[t]} \binom{k-1}{n} p^{n+1} q^{k-n-1}$$

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④

Also,

$$P[N(t)=n] = \binom{[t]}{n} p^n q^{[t]-n}$$

Another identity:

$$P[N(t)=n] = \int_0^\infty P[N(t)=n \mid S_n = y]\, f_{S_n}(y)\, dy$$

If the n^{th} event occurs at time y and $y > t$, then $N(t) < n$, so the integrand is 0.

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$$P[N(t)=n] = \int_0^t P[N(t)=n \mid S_n=y]\, f_{S_n}(y)\, dy$$

If the n^{th} event occurs at time y and $y \le t$ then $N(t)=n$ only if no more events occurred between times y and t, i.e. the $(n+1)^{st}$ interarrival time is greater than $t-y$

$$P[N(t)=n] = \int_0^t \underbrace{P[X_{n+1} > t-y]}_{1-F(t-y)}\, f_{S_n}(y)\, dy$$

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$$P[N(t)=n] = \int_0^t \bar{F}(t-y) f_{S_n}(y)\, dy \quad **$$

<u>Example</u> Suppose $F(x) = 1 - e^{-\lambda x}$

$$\bar{F}(x) = 1 - F(x) = e^{-\lambda x}$$

$$f(x) = \frac{d}{dx} F(x) = \lambda e^{-\lambda x} \sim Exp(\lambda)$$

So $S_n = \sum_{i=1}^{n} X_i \sim Gamma(n, \lambda)$

Using $**$, $P[N(t)=n] = \int_0^t e^{-\lambda(t-y)} \dfrac{\lambda e^{-\lambda y} (\lambda y)^{n-1}}{(n-1)!}\, dy$

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$$= \frac{e^{-\lambda t}\lambda^n}{(n-1)!} \underbrace{\int_0^t y^{n-1}\,dy}_{\frac{y^n}{n}\Big|_0^t = \frac{t^n}{n}}$$

$$= \frac{e^{-\lambda t}(\lambda t)^n}{n!} \sim \text{Poisson}(\lambda t)$$

<u>Definition</u> $m(t) = E[N(t)]$ is the <u>mean-value function</u> or the <u>renewal function</u>

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Yet another identity:

Suppose that the r.v. Y is non-negative, integer-valued.

Then $\displaystyle\sum_{n=1}^{\infty} P[Y \geq n] = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} P[Y=k]$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^{k} P[Y=k] = \sum_{k=1}^{\infty} k\,P[Y=k]$$

$$= \sum_{k=0}^{\infty} k\,P[Y=k] = E[Y]$$

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Use this:

$$m(t) = E[N(t)] = \sum_{n=1}^{\infty} P[N(t) \geq n]$$

$$= \sum_{n=1}^{\infty} P[S_n \leq t]$$

$$m(t) = \sum_{n=1}^{\infty} F_n(t)$$

Assume that F is continuous

$$m(t) = E[N(t)] = \int_0^{\infty} E[N(t) \mid X_1 = x] f(x) dx$$

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Also, when $x > t$

$$E[N(t) \mid X_1 = x] = 0$$

and when $x < t$, $E[N(t) \mid X_1 = x] = 1 + E[N(t-x)]$

$$\text{So } m(t) = \int_0^t \left(1 + E[N(t-x)]\right) f(x)\, dx$$

$$= \int_0^t f(x) dx + \int_0^t m(t-x) f(x) dx$$

$$\boxed{m(t) = F(t) + \int_0^t m(t-x) f(x) dx}$$ Renewal Equation

(11)

<u>Example</u> Suppose that the interarrival times are Unif(0,1)

Let $0 \le t \le 1$

$$m(t) = t + \int_0^t m(t-x) \cdot 1 \, dx \qquad \text{Let } y = t-x$$

$$= t + \int_t^0 m(y)(-dy)$$

$$m(t) = t + \int_0^t m(y)\,dy \qquad \text{Differentiate w.r.t. } t$$

$$m'(t) = 1 + m(t)$$

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Let $h(t) = 1 + m(t)$

$h'(t) = m'(t)$

$$h'(t) = h(t) \qquad h(t) = Ke^t$$

$$m(t) = Ke^t - 1$$

$$0 = m(0) = K - 1 \quad \therefore K = 1$$

and $m(t) = e^t - 1$