Another Conditional expectation example

Stat 567
3-2-17

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Binomial

$$p_{X|P}(x|p) = \binom{n}{x} p^x q^{n-x}, \quad \text{where } p \sim \text{Unif}(0,1)$$

(prior distribution)

Goal: find $E[p|X=x]$ (Bayesian estimate of p, using squared-error loss)

Joint distribution of X, p

$$f(x,p) = p_{X|p}(x|p) \cdot \underbrace{f_p(p)}_{1} = \binom{n}{x} p^x q^{n-x}$$

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$$p_X(x) = \int_0^1 f(x,p)\,dp$$

$$= \int_0^1 \binom{n}{x} p^x q^{n-x}\,dp$$

$$= \binom{n}{x} \underbrace{\int_0^1 p^x q^{n-x}\,dp}$$

$$= \binom{n}{x} \frac{\Gamma(x+1)\Gamma(n-x+1)}{\Gamma(n+2)} \underbrace{\int_0^1 \frac{\Gamma(n+2)}{\Gamma(x+1)\Gamma(n-x+1)} p^x q^{n-x}\,dp}_{\substack{\text{Beta distribution with} \\ \alpha = x+1,\ \beta = n-x+1}}$$

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$$= \frac{n!}{\cancel{x!(n-x)!}} \frac{\cancel{x!(n-x)!}}{(n+1)!}$$

$$p_X(x) = \frac{1}{n+1} \qquad x = 0, 1, \ldots, n$$

Note: Each of the $n+1$ values that X can take on is equally likely
[X has a "discrete uniform" distribution]

$$f_{p|X}(p|x) = \frac{f(p,x)}{p_X(x)} = \frac{\binom{n}{x} p^x q^{n-x}}{1/_{n+1}}$$

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$$f_{p|X}(p|x) = \underset{\text{(posterior distribution of } p)}{(n+1)\binom{n}{x} p^x q^{n-x}}, \quad x = 0, 1, \ldots, n, \quad 0 < p < 1$$

$$E[p|X=x] = \int_0^1 p\,(n+1)\binom{n}{x} p^x q^{n-x}\, dp$$

$$= (n+1)\binom{n}{x} \int_0^1 p^{x+1} q^{n-x}\, dp$$

$$= (n+1)\binom{n}{x} \frac{\Gamma(x+2)\Gamma(n-x+1)}{\Gamma(n+3)} \underbrace{\int_0^1 \frac{\Gamma(n+3)}{\Gamma(x+2)\Gamma(n-x+1)} p^{x+1} q^{n-x}\, dp}_{1}$$

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$$= (n+1)\frac{n!}{x!\,\cancel{(n-x)!}}\;\frac{(x+1)!\,\cancel{(n-x)!}}{(n+2)!}$$

$$E[p \mid X=x] = \frac{x+1}{n+2}$$

Note: this lies between $\frac{x}{n}$, which would be $\hat{p}$ in a non-Bayesian setting, and $\frac{1}{2}$, which is the mean of the prior distribution.

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The Compounding Identity

Theorem: $X_1, X_2, \ldots$ iid

N independent of the X_i's, non neg. integer

$S_N = \sum_{i=1}^{N} X_i$, $h(S_N)$ is any function of S_N.

Then $E[S_N h(S_N)] = E[N]\,E[X_1 h(S_M)]$,

where M is a new random variable such that $P(M=n) = \dfrac{n\,P(N=n)}{E[N]}$.

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Proof: $E[S_N h(S_N)] = E\left[\sum_{i=1}^{N} X_i h(S_N)\right]$

$$= E\left(E\left[\sum_{i=1}^{N} X_i h(S_N) \mid N\right]\right)$$

$$= \sum_{n=0}^{\infty} E\left[\sum_{i=1}^{N} X_i h(S_N) \mid N\right] P[N=n]$$

$$= \sum_{n=0}^{\infty} E\left[\sum_{i=1}^{n} X_i h(S_n)\right] P[N=n]$$

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$$= \sum_{n=0}^{\infty} \sum_{i=1}^{n} E[X_i h(S_n)] P[N=n]$$

$$= \sum_{n=0}^{\infty} n\, E[X_1 h(S_n)] P[N=n]$$

$$= E[N] \sum_{n=0}^{\infty} E[X_1 h(S_n)] \frac{n P[N=n]}{E[N]}$$

$$= E[N] \sum_{n=0}^{\infty} E[X_1 h(S_n)] P[M=n]$$

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$$= E[N] \sum_{n=0}^{\infty} E[X_1 h(S_n) \mid M=n] \; P[M=n]$$

ok since M is indep. of $X_1, \ldots, X_n$

$$= E[N] \sum_{n=0}^{\infty} E[X_1 h(S_M) \mid M=n] \; P[M=n]$$

$$= E[N] \; E\big[E[X_1 h(S_M) \mid M] \big]$$

$$= E[N] \; E[X_1 h(S_M)]$$

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Corollary: $P[S_N = 0] = P[N=0]$ and

$$P[S_N = k] = \frac{1}{k} E[N] \sum_{j=1}^{k} j \alpha_j P[S_{M-1} = k-j],$$

$k > 0$ and $\alpha_j = P[X_i = j]$ and

$X_1, \ldots, X_n, \ldots$ are integer-valued ≥ 0

Proof. Fix a value of k.

Let $h(S_N) = \begin{cases} 1 & \text{if } S_N = k \\ 0 & \text{otherwise} \end{cases}$

Now $$S_N h(S_N) = \begin{cases} k & \text{if } S_N = k \\ 0 & \text{otherwise} \end{cases}$$

and $$E[S_N h(S_N)] = k P[S_N = k]$$

So the theorem says

$$k P[S_N = k] = E[N]\, E[X_1 h(S_M)]$$

$$= E[N]\, E\big[E[X_1 h(S_M) \mid X_1] \big]$$

$$= E[N] \sum_{j=1}^{k} E[X_1 h(S_M) \mid X_1 = j]\, P[X_1 = j]$$

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$$= E[N] \sum_{j=1}^{k} j\, E[h(S_M) \mid X_1 = j]\, \alpha_j$$

$$= E[N] \sum_{j=1}^{k} j\, P[S_M = k \mid X_1 = j]\, \alpha_j$$

$$P[S_M = k \mid X_1 = j] = P\Big[\sum_{i=1}^{M} X_i = k \;\Big|\; X_1 = j \Big]$$

$$= P\Big[j + \sum_{i=2}^{M} X_i = k \Big]$$

$$= P\left[\sum_{i=2}^{M} X_i = k-j\right]$$

$$= P\left[\sum_{i=1}^{M-1} X_i = k-j\right]$$

$$= P\left[S_{M-1} = k-j\right]$$

$$\therefore \quad k\,P[S_N = k] = E[N] \sum_{j=1}^{k} j\alpha_j\, P[S_{M-1} = k-j]$$