Stat 568
4-13-17

In a Poisson process,

① let $T_1$ = time of $1^{st}$ event, ...

$T_n$ = time between the $(n-1)^{st}$ and $n^{th}$ events

$\{T_n,\ n=1,2,\dots\}$ is the sequence of interarrival times

Then $P(T_1 > t) = P[N(t)=0] = e^{-\lambda t}$

$$\underbrace{P(T_1 \le t)}_{F_1(t)} = 1 - e^{-\lambda t}$$

$$f_1(t) = \lambda e^{-\lambda t} \sim \text{exponential}(\lambda)$$

②

And $P(T_2 > t) = \displaystyle\int_0^\infty P(T_2 > t \mid T_1 = s) f_1(s)\, ds$

$$= \int_0^\infty \underbrace{P[0 \text{ events in the interval } (s, s+t)]}_{\substack{P[0 \text{ events in } (0,t)] \\ P[N(t)=0] \\ e^{-\lambda t}}} f_1(s)\, ds$$

$$= e^{-\lambda t} \underbrace{\int_0^\infty f_1(s)\, ds}_{1} = e^{-\lambda t}$$

So $T_n,\ n=1,2,\dots$ are iid exponential$(\lambda)$ $\left[\mu = \frac{1}{\lambda}\right]$

③

Let $S_n$ = waiting time until the $n^{\text{th}}$ event
(arrival time of) - - - -

So $S_n = \sum_{i=1}^{n} T_i$ So $S_n \sim \text{Gamma}(n, \lambda)$

Note: $N(t) \geq n \iff S_n \leq t$

$$F_{S_n}(t) = P[S_n \leq t] = P[N(t) \geq n]$$
$$= \sum_{j=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!}$$

④

And $P[S_n > t] = P[N(t) < n]$
$$= \sum_{j=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^j}{j!}$$

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Consider a Poisson process with rate $\lambda$,
but with two types of events, I and II,
occurring independently with probabilities $p$ and $q = 1-p$.

Define $N_1(t)$ = # of type I events by time $t$
$N_2(t)$ = .. .... II - - - - -

Let $N(t) = N_1(t) + N_2(t)$ (5)

<u>Proposition</u>: $N_1(t)$ and $N_2(t)$ are independent Poisson processes with rates $\lambda p$ and $\lambda q$.

<u>Proof</u>. i) $N_1(0) = 0$ and $N_2(0) = 0$ because $N(0) = 0$

ii) $$P[N_1(t) = k] = \sum_{n=k}^{\infty} \underbrace{P[N_1(t)=k \mid N(t)=n]}_{\text{Binol}(n,p)} \cdot P[N(t)=n]$$

So the stationary & indp. assumptions for $N(t)$ are transferred to $N_1(t)$, and similarly to $N_2(t)$.

(6)

iii) $P[N_1(h) = 1]$

$$= P[N_1(h)=1 \mid N(h)=1] \cdot P[N(h)=1] + P[N_1(h)=1 \mid N(h) \geq 2] \cdot P[N(h) \geq 2]$$

$$= p \cdot [\lambda h + o(h)] + c \cdot o(h)$$

$$= \underline{p\lambda} h + o(h)$$

Similarly, $P[N_2(h) = 1] = \underline{\lambda q} h + o(h)$

⑦

iv) $P[N_1(h) \geq 2] \leq P[N(h) \geq 2] = o(h)$

Likewise for $N_2$.

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<u>Example</u> Sell an item

Assume that arrival of offers follows a Poisson process with rate $\lambda$.

Assume each offer follows a continuous density function $f(y)$.

⑧

You will accept the first offer that exceeds $y_0$.

Your cost is $ct$, where $t$ is the time until sale.

Find the optimal $y_0$

Let Type I indicate an offer $> y_0$

So $p = P[Y > y_0] = 1 - F(y_0) = \bar{F}(y_0)$

Let $R(y_0)$ be your total return

$E[R(y_0)] = E[\text{amount of accepted offer} - ct]$

$$= E[\text{amount of accepted offer}] - \frac{c}{\lambda p}$$ (9)

$$= E[Y \mid Y > y_0] - \frac{c}{\lambda p}$$

$$= \int_{y_0}^{\infty} y \, f_{Y|Y>y_0}(y)\, dy - \frac{c}{\lambda p}$$

$$= \int_{y_0}^{\infty} y \, \frac{f(y)}{\overline{F}(y_0)}\, dy - \frac{c}{\lambda p}$$

$$= \frac{1}{p}\left[ \int_{y_0}^{\infty} y\, f(y)\, dy - \frac{c}{\lambda} \right]$$

Now, find the value of $y_0$ which maximizes this (10)

$$\frac{d}{dy_0}\left[ \frac{\int_{y_0}^{\infty} y f(y)\, dy - \frac{c}{\lambda}}{1 - F(y_0)} \right]$$

$$= \frac{[1-F(y_0)]\left[-y_0 f(y_0)\right] - \left[\int_{y_0}^{\infty} y f(y)\, dy - \frac{c}{\lambda}\right](-f(y_0))}{[1-F(y_0)]^2}$$

$$\stackrel{\text{set}}{=} 0$$

(11)

$$\cancel{f(y_0)}\left[ y_0 \bar{F}(y_0) - \int_{y_0}^{\infty} y f(y)\,dy + \frac{c}{\lambda} \right] = 0$$

$$\int_{y_0}^{\infty} y f(y)\,dy - y_0 \underbrace{\bar{F}(y_0)}_{\int_{y_0}^{\infty} f(y)\,dy} = \frac{c}{\lambda}$$

$$\int_{y_0}^{\infty} (y - y_0) f(y)\,dy = \frac{c}{\lambda}$$

& solve numerically for $y_0$

(12)

Example: "The Coupon problem"

There are $m$ coupons or game pieces with probabilities $p_j$, $j = 1, \ldots, m$. You collect the coupons according to a Poisson process with rate $\lambda$.

Let $N_j(t)$ be the # of type $j$ events by time $t$.

By extending our proposition, the $N_j(t)$'s are independent Poisson processes with rates $\lambda p_j$

(13)

Let $X_j$ be the arrival time of the first type $j$ event.

Let $X = \max_{1 \le j \le m} X_j$ This is the time needed to collect at least 1 of every piece.

The $X_j$'s are independent exponential random variables with parameters $\lambda p_j$

(14)

$$P[X \le t] = P[\max_{1 \le j \le m} X_j \le t]$$

$$= P[X_1 \le t \cap X_2 \le t \cap \ldots \cap X_m \le t]$$

$$= \prod_{j=1}^{m} P(X_j \le t) = \prod_{j=1}^{m} (1 - e^{-\lambda p_j t})$$

Next goal: find $E[X]$

HW #2 follows this page.
due 4/20

Stat 568
HW #2

39. A certain scientific theory supposes that mistakes in cell division occur according to a Poisson process with rate 2.5 per year, and that an individual dies when 196 such mistakes have occurred. Assuming this theory, find
   (a) the mean lifetime of an individual,
   (b) the variance of the lifetime of an individual.

   Also approximate
   (c) the probability that an individual dies before age 67.2,
   (d) the probability that an individual reaches age 90,
   (e) the probability that an individual reaches age 100.

42. Let $\{N(t),\ t \geqslant 0\}$ be a Poisson process with rate $\lambda$. Let $S_n$ denote the time of the $n$th event. Find
   (a) $E[S_4]$,
   (b) $E[S_4|N(1) = 2]$,
   (c) $E[N(4) - N(2)|N(1) = 3]$.

47. Consider a two-server parallel queuing system where customers arrive according to a Poisson process with rate $\lambda$, and where the service times are exponential with rate $\mu$. Moreover, suppose that arrivals finding both servers busy immediately depart without receiving any service (such a customer is said to be lost), whereas those finding at least one free server immediately enter service and then depart when their service is completed.
   (a) If both servers are presently busy, find the expected time until the next customer enters the system.
   (b) Starting empty, find the expected time until both servers are busy.
   (c) Find the expected time between two successive lost customers.

50. The number of hours between successive train arrivals at the station is uniformly distributed on $(0, 1)$. Passengers arrive according to a Poisson process with rate 7 per hour. Suppose a train has just left the station. Let $X$ denote the number of people who get on the next train. Find
   (a) $E[X]$,
   (b) $\text{Var}(X)$.