Example from last time, continued:

$$X = \max_{1 \le j \le m} X_j = \text{time needed to collect at least 1 of every type}$$

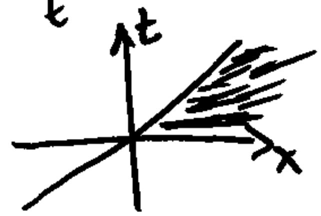
We had $P[X \le t] = \prod_{j=1}^{m} \left(1 - e^{-\lambda p_j t}\right)$

---

show: if $X \ge 0$ then

$$E[X] = \int_0^\infty P[X > t]\,dt = \int_0^\infty \int_t^\infty f(x)\,dx\,dt$$

$$= \int_0^\infty \int_0^x f(x)\,dt\,dx$$



②

$$= \int_0^\infty f(x)\, t \Big|_0^x dx = \int_0^\infty x f(x)\,dx = E[X]$$

So $E[X] = \int_0^\infty \left(1 - \prod_{j=1}^{m} \left(1 - e^{-\lambda p_j t}\right)\right) dt$

Let N = # coupons needed to get a complete set

Let T_i = i^{th} interarrival time of coupons

So $X = \sum_{i=1}^{N} T_i$

③

$$
\begin{aligned}
E[X] &= E\big[E[X \mid N]\big] \\
&= E\Big[E\big[\textstyle\sum_{i=1}^{N} T_i \mid N\big]\Big] \\
&= E\big[N\,E[T_i]\big] = E\big[N\cdot\tfrac{1}{\lambda}\big] \\
&= \tfrac{1}{\lambda}E[N]
\end{aligned}
$$

$$\therefore\ E[N] = \lambda E[X]$$

④

<u>Theorem</u>: For a Poisson process $\{N(t), t\geq 0\}$, the conditional joint density of the arrival times $S_1, S_2, \ldots, S_n$, given that $N(t)=n$ is $\frac{n!}{t^n}$.

<u>Proof</u>: For $s_1 < s_2 < \ldots < s_n < t$, then the event that $S_1 = s_1, S_2 = s_2, \ldots, S_n = s_n, N(t)=n$ is equivalent to the event that

$T_1 = s_1,\ T_2 = s_2 - s_1, \ldots, T_n = s_n - s_{n-1},$ (5)

$T_{n+1} > t - s_n$

Now $f(s_1, \ldots, s_n \mid n) = \dfrac{f(s_1, \ldots, s_n, n)}{P[N(t)=n]}$

$$= \frac{\lambda e^{-\lambda s_1} \cdot \lambda e^{-\lambda(s_2 - s_1)} \cdots \lambda e^{-\lambda(s_n - s_{n-1})} e^{-\lambda(t - s_n)}}{\dfrac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

$$= \frac{\lambda^n e^{-\lambda t}}{e^{\lambda t} \lambda^n t^n} \cdot n! = \frac{n!}{t^n}.$$

(6)

Defn: Let $Y_1, Y_2, \ldots, Y_n$ be iid random variables.

Then $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ are the order statistics

if $Y_{(k)}$ is the k^{th} smallest of $Y_1, \ldots, Y_n$

Proposition: The joint density of the order statistics

is $f_o(c_1, c_2, \ldots, c_n) = n! \prod_{i=1}^{n} f(c_i)$
$c_1 < c_2 < \ldots < c_n$

Proof: $(Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)})$ will equal $(c_1, \ldots, c_n)$

if $(Y_1, Y_2, \ldots, Y_n)$ equals any permutation of $(c_1, \ldots, c_n)$.

(7)

$$\text{So} \quad f_0(c_1,\dots,c_n) = n!\, f(c_1,\dots,c_n) = n! \prod_{i=1}^{n} f(c_i)$$

What if $Y_1,\dots,Y_n \sim$ iid $\text{Unif}(0,t)$

$$f(y_i) = \frac{1}{t}$$

$$\text{Then} \quad f_0(c_1,\dots,c_n) = n! \left(\frac{1}{t}\right)^n = \frac{n!}{t^n}$$

(8)

Our theorem said that the joint density of the arrival times, given $N=n$, is $\frac{n!}{t^n}$. By our proposition, this is also the joint density of the n order statistics when the underlying random variables are iid $U(0,t)$.

Conditioned upon n events, the unordered arrival times behave as independent $U(0,t)$ random variables.

(9)

A generalization

<u>Defn</u>: The counting process $\{N(t), t \geq 0\}$ is a <u>nonhomogeneous Poisson process</u> with <u>intensity function</u> $\lambda(t)$ if

i.) $N(0) = 0$

ii) independent increments

iii) $P[N(t+h) - N(t) = 1] = \lambda(t)h + o(h)$

iv.) $P[N(t+h) - N(t) \geq 2] = o(h)$

(10)

Compound Poisson process

<u>Defn</u>: $\{X(t), t \geq 0\}$ is a compound Poisson process if $X(t) = \sum_{i=1}^{N(t)} Y_i$ where $\{N(t), t \geq 0\}$ is a Poisson process and the Y_i's are iid random variables independent of $N(t)$.

<u>Example</u>: Families move to portland at the rate of 2 per day.

Y_i is the # people in the i^{th} family

and

y	$p(y)$
1	1/6
2	1/3
3	1/3
4	1/6

In 5 days, find the expected # & standard deviation of people moving to portland.

$$\mu_Y = E[Y_i] = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{3} + 3\cdot\frac{1}{3} + 4\cdot\frac{1}{6}$$
$$= 5/2$$

$$E[Y_i^2] = 1^2\cdot\frac{1}{6} + 2^2\cdot\frac{1}{3} + 3^2\cdot\frac{1}{3} + 4^2\cdot\frac{1}{6} = \frac{43}{6}$$

$$\mu_X = E[X] = E(E[X\mid N]) = E\left(E\left[\sum_{i=1}^{N} Y_i \mid N\right]\right)$$
$$= E(N\mu_Y) = \mu_Y E(N) = \mu_Y \lambda t$$
$$= \frac{5}{2}\cdot 2\cdot 5 = 25$$

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$$\sigma_x^2 = \mathrm{Var}[X] = E[V[X|N]] + V[E[X|N]]$$

$$= E\left[V\left[\sum_{i=1}^{N} Y_i \,\middle|\, N\right]\right] + V[N\mu_Y]$$

$$= E[N\sigma_Y^2] + \mu_Y^2 V[N]$$

$$= \sigma_Y^2 E[N] + \mu_Y^2 V[N]$$

$$= (\sigma_Y^2 + \mu_Y^2)\lambda t = E[Y_i^2]\lambda t$$

$$= \frac{43}{6} \cdot 2.5 = 71.\bar{6} \qquad \therefore \sigma_x = 8.47$$