Stat 567
3-4-17

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Recall $P_{ij} = P(X_{n+1}=j \mid X_n = i)$

So $P_{ij} \geq 0$ and $\sum_{j=0}^{\infty} P_{ij} = 1 \quad \forall i$

Let P be the matrix of transition probabilities

$$P = \begin{bmatrix} P_{00} & P_{01} & \cdots \\ P_{10} & P_{11} & \cdots \\ \vdots & & \\ \vdots & & \end{bmatrix}$$

Each row sums to 1

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<u>Example</u>: If it rains today, α is the prob of rain tomorrow.

If it doesn't rain today, β is the prob of rain tomorrow.

let 0: rain 1: not rain

$P_{00} = \alpha \qquad P_{10} = \beta$

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix} \end{array}$$

Example: rain yesterday & today ⇒ .7 tomorrow

rain not yest & today ⇒ .5 "

rain yest & not today ⇒ .4 "

rain not yest & not today ⇒ .2 "

State 0: RR → RR$\underline{R}$ or RR$\underline{S}$

1: SR → S$\underline{RR}$ or S$\underline{RS}$

2: RS → RS$\underline{R}$ or RS$\underline{S}$

3: SS → SS$\underline{R}$ or SS$\underline{S}$

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & .7 & 0 & .3 & 0 \\ 1 & .5 & 0 & .5 & 0 \\ 2 & 0 & .4 & 0 & .6 \\ 3 & 0 & .2 & 0 & .8 \end{array}$$

If the state space is $0, \pm 1, \pm 2, \ldots$

and $P_{i,i+1} = p$ and $P_{i,i-1} = 1-p$

then the Markov chain is a <u>random walk</u>

⑤

Example: A gambler bets \$1 on each round, and wins a round with prob. p.
The gambler stops when broke or when \$N are won.

$P_{00} = 1$ and $P_{NN} = 1$ $\Big\}$ These are <u>absorbing</u> <u>states</u>

for $i = 1, \dots, N-1$ $\quad P_{i,i+1} = p$

$P_{i,i-1} = 1-p$

⑥

Defn. $P^n_{ij} = P[X_{n+k} = j \mid X_k = i]$

This <u>n-step</u> <u>transition</u> <u>probability</u>.

Note: $P^1_{ij} = P_{ij}$

$$P^{n+m}_{ij} = P[X_{n+m+k} = j \mid X_k = i]$$
$$= P[X_{n+m} = j \mid X_0 = i]$$

(7)

$$= \sum_{k=0}^{\infty} P[X_{n+m}=j \cap X_n=k \mid X_0=i]$$

$$= \sum_{k=0}^{\infty} P[X_{n+m}=j \mid X_n=k \cap X_0=i] \cdot P[X_n=k \mid X_0=i]$$

$$= \sum_{k=0}^{\infty} P_{kj}^{m} P_{ik}^{n}$$

$$P[A \cap B \mid C]$$

$$= \frac{P[A \cap B \cap C]}{P[C]}$$

$$= \frac{P[A \mid B \cap C]\, P[B \cap C]}{P[C]}$$

$$= P[A \mid B \cap C]\, P[B \mid C]$$

(8)

$$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{kj}^{m} P_{ik}^{n}$$

Chapman-Kolmogorov equation

Let $P^{(n)}$ be the matrix of n-step transition probabilities

$$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^{n} P_{kj}^{m}$$

This is the dot product of row i of $P^{(n)}$ & column j of $P^{(m)}$

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That is, $P^{(n+m)} = P^{(n)} P^{(m)}$

Therefore: $P^{(1)} = P$

$$P^{(2)} = P^{(1+1)} = P^{(1)} P^{(1)}$$
$$= PP = P^2$$

Continuing, by induction we get

$$P^{(n)} = P^n$$

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Ex: 1st rain example

$$P = \begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix}$$

Assume $\alpha = .7$
$\beta = .4$

$$= \begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix}$$

State 0: rain
State 1: no rain

Find the prob. that 4 days from now, given that it rains today.

$$P^4_{00} = (0,0)^{th} \text{ entry of } P^4$$

(11)

$$P^2 = \begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix}\begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix} = \begin{bmatrix} .61 & .39 \\ .52 & .48 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} .61 & .39 \\ .52 & .48 \end{bmatrix}\begin{bmatrix} .61 & .39 \\ .52 & .48 \end{bmatrix} = \begin{bmatrix} .5749 & .4251 \\ .5668 & .4332 \end{bmatrix}$$

$$P^4_{00} = .5749$$

<u>Example</u>: Urn contains 2 balls, red + blue

Select 1 ball at random & replace with one of the same color w/ prob. .8

" " " opposite " " " .2

(12)

Let X_n = # red balls in urn after step n.

$= 0, 1, 2$

If the initial state is red + red, find the prob. that the 4^{th} ball chosen is red.

To do this, we need P^3

0 0	.8	1 0	$(.5)(.2) = .1$
0 1	.2	1 1	$(.5)(.8) + (.5)(.8) = .8$
0 2	0	1 2	$(.5)(.2) = .1$

2 0	0
2 1	.2
2 2	.8

(13)

Spectral decomposition of a square matrix

$$P = S \Lambda S^{-1}$$

S: columns are eigenvectors

Λ: diagonal matrix of eigenvalues

$$P^2 = (S \Lambda S^{-1})(S \Lambda S^{-1})$$
$$= S \Lambda^2 S^{-1} \qquad P^n = S \Lambda^n S^{-1}$$

(14)

$\lambda_1 = 1 \quad \vec{e}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_2 = .8 \quad \vec{e}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_3 = .6 \quad \vec{e}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$$P = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & .8 & 0 \\ 0 & 0 & .6 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} .25 & .5 & .25 \\ .5 & 0 & -.5 \\ .25 & -.5 & .25 \end{bmatrix}}_{S^{-1}}$$

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$$P = \begin{bmatrix} .8 & .2 & 0 \\ .1 & .8 & .1 \\ 0 & .2 & .8 \end{bmatrix}$$

$$P^3 = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} \\ \\ 2 \end{array} & \begin{bmatrix} .56 & .392 & .048 \\ .196 & .608 & .196 \\ .048 & .392 & .56 \end{bmatrix} \end{array}$$

P(draw a red on next step)

$$= (.048)(0) + .392(.5) + .56(1)$$

$$= .756$$

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$$\lim_{n \to \infty} \Lambda^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lim_{n \to \infty} P^n = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} S^{-1}$$

$$= \begin{bmatrix} .25 & .5 & .25 \\ .25 & .5 & .25 \\ .25 & .5 & .25 \end{bmatrix}$$