Continuous-time Markov chains

Stat 568
4-20-17

①

Defn: $\{X(t), t \geq 0\}$ is a <u>continuous-time</u> <u>Markov</u> <u>chain</u> if $\forall s,t \geq 0$ & integers $i, j, x(u) \geq 0$, $0 \leq u < s$,

$$P[X(t+s) = j \mid X(s) = i,\ X(u) = x(u)]$$
$$= P[X(t+s) = j \mid X(s) = i]$$

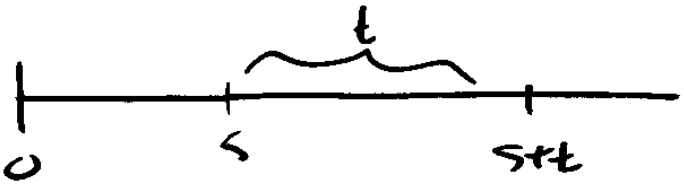
②

In addition, if

$P[X(t+s) = j \mid X(s) = i]$ is independent of s, then the Markov chain has <u>stationary</u> transition probabilities (<u>homogeneous</u>)

Let T_i = time the process stays in state i before making a transition.

$$P[T_i > s+t \mid T_i > s] = P[T_i > t]$$ 3



This is called the "memory-less" property

Suppose that the random variable T has the memoryless property.

Let $\bar{F}(t) = P(T > t)$

The memoryless property says 4

$$\frac{P[T > s+t \cancel{\cap T > s}]}{P[T > s]} = P[T > t]$$

$$P[T > s+t] = P[T > s]\, P[T > t]$$

$$\bar{F}(s+t) = \bar{F}(s)\, \bar{F}(t)$$

Consider $g(s+t) = g(s)g(t)$ for continuous g

Let $x = \frac{m}{n}$, where m & n are integers

(5)

$$g\left(\frac{m}{n}\right) = g\Big(\underbrace{\frac{1}{n} + \dots + \frac{1}{n}}_{m \text{ times}}\Big)$$

$$= g\left(\tfrac{1}{n}\right) \cdots g\left(\tfrac{1}{n}\right) = \left(g\left(\tfrac{1}{n}\right)\right)^m$$

So $$\left(g\left(\tfrac{m}{n}\right)\right)^n = \left(g\left(\tfrac{1}{n}\right)\right)^{mn}$$

$$= \left(\left(g\left(\tfrac{1}{n}\right)\right)^n\right)^m$$

$$= \Big(g\big(\underbrace{\tfrac{1}{n} + \dots + \tfrac{1}{n}}_{n \text{ times}}\big)\Big)^m$$

$$= \left(g(1)\right)^m$$

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And $$g\left(\frac{m}{n}\right) = \left(g(1)\right)^{\frac{m}{n}}$$

$g(x) = \left(g(1)\right)^x$ for any real $x \geq 0$
because x can be sandwiched
between 2 converging sequences
of rational numbers.

Let $\lambda = -\ln g(1)$ $\quad \therefore g(1) = e^{-\lambda}$

$$g(x) = e^{-\lambda x}$$

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$$\therefore \quad \bar{F}(t) = e^{-\lambda t}$$

$$F(t) = 1 - e^{-\lambda t}$$

$$f(t) = \lambda e^{-\lambda t}$$

$\therefore$ In a continuous-time Markov chain, the time spent in state i before transitioning is exponentially distributed.

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Defn. Let $\{X(t), t \geq 0\}$ be a continuous time Markov chain

When $X(t) = n$,

i) new arrivals enter at exponential rate λ_n

ii) departures occur at exponential rate μ_n

Then $X(t)$ is called a birth and death process and $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\mu_n\}_{n=0}^{\infty}$ are the arrival & departure rates

(9)

Let ν_i be the exponential parameter of T_i

From state 0, the process can only go to state 1.

$$\text{So } P_{01} = 1 \qquad \nu_0 = \text{parameter of } T_0 = \lambda_0$$

For $i \geq 1$,

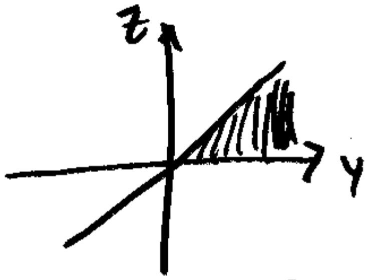
the process can only go from state i to state $i-1$ or $i+1$

To go from i to $i-1$, a death must occur before a birth.

(10)

Let $Y \sim \exp(\lambda)$ (birth) and $Z \sim \exp(\mu)$ (death), indep.

$$P(Z < Y) = \int_0^\infty \int_0^y \lambda e^{-\lambda y} \mu e^{-\mu z} \, dz \, dy$$



$$= \int_0^\infty \lambda e^{-\lambda y} \left(-e^{-\mu z} \right) \Big|_{z=0}^{y} dy$$

$$= \int_0^\infty \lambda e^{-\lambda y} \left(-e^{-\mu y} - (-1) \right) dy$$

$$= \int_0^\infty \lambda \left(e^{-\lambda y} - e^{-(\lambda+\mu) y} \right) dy$$

(11)

$$= \lambda \left[\frac{e^{-\lambda y}}{-\lambda} - \frac{e^{-(\lambda+\mu)y}}{-(\lambda+\mu)} \right]_{y=0}^{\infty}$$

$$= \lambda \left[0 - \left(-\frac{1}{\lambda} + \frac{1}{\lambda+\mu} \right) \right]$$

$$= 1 - \frac{\lambda}{\lambda+\mu} = \frac{\mu}{\lambda+\mu} = P_{i,i-1}$$

So $P_{i,i+1} = \frac{\lambda}{\lambda+\mu}$

(12)

So far: $P_{0,1} = 1$

$$P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}$$

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}$$

T_i is time in state i, so T_i is the minimum of (time until birth, time until death)

(13)

Again $Y \sim \exp(\lambda)$, $Z \sim \exp(\mu)$ indep.

Let $T = \min(Y, Z)$

$$P(T > t) = P(\min(Y,Z) > t)$$
$$= P(Y > t \text{ and } Z > t)$$
$$= P(Y > t)\,P(Z > t)$$
$$= e^{-\lambda t} e^{-\mu t}$$
$$= e^{-(\lambda+\mu)t}$$

So T is $\exp(\lambda+\mu)$ $\quad \therefore \nu_i = \lambda_i + \mu_i$

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<u>Example</u> Linear growth with immigration

Assume $\lambda_n = n\lambda + \theta$

and $\mu_n = n\mu$

Find $E[X(t)]$

let $M(t) = E[X(t)]$

So $M(t+h) = E[X(t+h)]$
$$= E\big[\,E[X(t+h) \mid X(t)]\,\big]$$

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Given $X(t)$,

$$X(t+h) = \begin{cases} X(t)+1 \\ X(t)-1 \\ X(t) \end{cases}$$

Probability:

$(X(t)\lambda+\theta)h + o(h)$

$(X(t)\mu)h + o(h)$

$1-$ other 2

$$E[X(t+h) \mid X(t)] =$$
$$(X(t)+1)\left[(X(t)\lambda+\theta)h + o(h)\right]$$
$$+ (X(t)-1)\left[X(t)\mu h + o(h)\right]$$
$$+ X(t)\left[1-(X(t)\lambda+\theta)h - X(t)\mu h + o(h)\right]$$

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$$= X(t) + \left[(X(t)\lambda+\theta)h - X(t)\mu h + o(h)\right]$$

$$M(t+h) = E(\cdot)$$
$$= M(t) + M(t)(\lambda-\mu)h + \theta h + o(h)$$

$$\frac{M(t+h)-M(t)}{h} = M(t)(\lambda-\mu) + \theta + \frac{o(h)}{h}$$

Take $\lim_{h\to 0}$

$$M'(t) = (\lambda-\mu)M(t) + \theta$$

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Let $g(t) = (\lambda-\mu)M(t) + \theta$

$$g'(t) = (\lambda-\mu)M'(t)$$

So $M'(t) = \dfrac{g'(t)}{\lambda-\mu}$

$$\frac{g'(t)}{\lambda-\mu} = g(t)$$

$$\frac{g'(t)}{g(t)} = \lambda-\mu$$

$$\ln g(t) = (\lambda-\mu)t + c$$

(18)

$$g(t) = ke^{(\lambda-\mu)t}$$

$$ke^{(\lambda-\mu)t} = (\lambda-\mu)M(t) + \theta$$

$$M(0) = E[X(0)] = X(0) = x_0$$

$$k = (\lambda-\mu)x_0 + \theta$$

Finally $M(t) = \dfrac{ke^{(\lambda-\mu)t} - \theta}{\lambda-\mu} = \ldots$

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HW#3

6. Consider a birth and death process with birth rates $\lambda_i = (i+1)\lambda, i \geqslant 0$, and death rates $\mu_i = i\mu, i \geqslant 0$.

(a) Determine the expected time to go from state 0 to state 4.

(b) Determine the expected time to go from state 2 to state 5.

(c) Determine the variances in parts (a) and (b).

9. The birth and death process with parameters $\lambda_n = 0$ and $\mu_n = \mu, n > 0$ is called a pure death process. Find $P_{ij}(t)$.

12. Each individual in a biological population is assumed to give birth at an exponential rate λ, and to die at an exponential rate μ. In addition, there is an exponential rate of increase θ due to immigration. However, immigration is not allowed when the population size is N or larger.

(a) Set this up as a birth and death model.

(b) If $N = 3, 1 = \theta = \lambda, \mu = 2$, determine the proportion of time that immigration is restricted.

17. Each time a machine is repaired it remains up for an exponentially distributed time with rate λ. It then fails, and its failure is either of two types. If it is a type 1 failure, then the time to repair the machine is exponential with rate μ_1; if it is a type 2 failure, then the repair time is exponential with rate μ_2. Each failure is, independently of the time it took the machine to fail, a type 1 failure with probability p and a type 2 failure with probability $1-p$. What proportion of time is the machine down due to a type 1 failure? What proportion of time is it down due to a type 2 failure? What proportion of time is it up?