Limiting probabilities

Stat 528
5-11-17

①

Defn: $P_j = \lim_{t\to\infty} P_{ij}(t)$, assuming that (1) it exists, and (2) it is independent of i

Kolmogorov forward equations

$$P_{ij}'(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t)$$

$$\underset{=0}{\lim_{t\to\infty} P_{ij}'(t)} = \sum_{k\neq j} q_{kj} \underbrace{\lim_{t\to\infty} P_{ik}(t)}_{P_k} - v_j \underbrace{\lim_{t\to\infty} P_{ij}(t)}_{P_j}$$

②

$$0 = \sum_{k\neq j} q_{kj} P_k - v_j P_j$$

$$\therefore \quad v_j P_j = \sum_{k\neq j} q_{kj} P_k$$

$$\text{Also} \quad \sum_j P_j = 1$$

These can be solved to find the P_j's.

Defn: When the P_j's exist, the chain is called <u>ergodic</u>, and the P_j's are called <u>stationary probabilities</u>.

Look in more detail at

(3)

$$v_j P_j = \sum_{k \neq j} q_{kj} P_k$$

Left side: When the process is in state j, v_j is the rate at which it exists that state.

So $v_j P_j$ is the rate at which the process exits state j.

Right side: when the process is in state k, q_{kj} is the rate at which it transitions into state j. $(q_{kj} = v_k P_{kj})$

(4)

So $q_{kj} P_k$ is the rate at which the process transitions from k to j.

Then $\sum_{k \neq j} q_{kj} P_k$ is the rate at which the process enters state j.

$\therefore$ $v_j P_j = \sum_{k \neq j} q_{kj} P_k$ is equivalent to saying that the rate at which the process enters state j equals the rate at which it exits state j.

Example: Birth & death process (5)

	exit rate	entrance rate
$j=0$:	$\lambda_0 P_0$	$\mu_1 P_1$
$j=1$:	$(\lambda_1+\mu_1)P_1$	$\lambda_0 P_0 + \mu_2 P_2$
$j=2$:	$(\lambda_2+\mu_2)P_2$	$\lambda_1 P_1 + \mu_3 P_3$
etc.		

$$q_{01}P_0 + q_{21}P_2$$
$$= v_0 P_{01} P_0 + v_2 P_{21} P_2$$
$$= \lambda_0 \cdot 1 \cdot P_0 + (\lambda_2+\mu_2)\frac{\mu_2}{\lambda_2+\mu_2} P_2$$

$\therefore \lambda_0 P_0 = \mu_1 P_1,$

$$(\lambda_1+\mu_1)P_1 = \mu_1 P_1 + \mu_2 P_2 \Rightarrow \lambda_1 P_1 = \mu_2 P_2$$

(6)

$$(\lambda_2+\mu_2)P_2 = \mu_2 P_2 + \mu_3 P_3$$

$$\Rightarrow \lambda_2 P_2 = \mu_3 P_3 \quad \text{etc.}$$

$$\therefore P_1 = \frac{\lambda_0}{\mu_1} P_0$$

$$P_2 = \frac{\lambda_1}{\mu_2}\frac{\lambda_0}{\mu_1} P_0$$

$$P_3 = \frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2}\frac{\lambda_0}{\mu_1} P_0$$

$$\vdots$$

$$P_n = \frac{\lambda_{n-1}\cdots\lambda_0}{\mu_n\cdots\mu_1} P_0$$

$$\text{Also, } \sum_{n=0}^{\infty} P_n = 1$$ (7)

$$1 = P_0 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1}\cdots\lambda_0}{\mu_n\cdots\mu_1} P_0$$

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1}\cdots\lambda_0}{\mu_n\cdots\mu_1}}$$

Special case: M/M/S example

$$\lambda_n = \lambda \qquad \mu_n = \begin{cases} n\mu & n \le s \\ s\mu & n > s \end{cases}$$

(8)

$$\text{Then } \sum_{n=1}^{\infty} \frac{\lambda_{n-1}\cdots\lambda_0}{\mu_n\cdots\mu_1} = \sum_{n=1}^{s} \frac{\lambda_{n-1}\cdots\lambda_0}{\mu_n\cdots\mu_1} + \sum_{n=s+1}^{\infty} \frac{\lambda_{n-1}\cdots\lambda_0}{\mu_n\cdots\mu_1}$$

$$= \sum_{n=1}^{s} \frac{\lambda^n}{\mu^n n!} + \sum_{n=s+1}^{\infty} \frac{\lambda^n}{\mu^n s^n}$$

This converges if $\frac{\lambda}{\mu s} < 1$

$$\lambda < \mu s$$

How to find the transition probabilities, (9)
in general.

Let $r_{ij} = \begin{cases} q_{ij} & \text{if } i \neq j \\ -\nu_i & \text{if } i = j \end{cases}$

Kolmogorov Backward:

$$P_{ij}'(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t)$$

Kolmogorov Forward:

$$P_{ij}'(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - \nu_j P_{ij}(t)$$

(10)

Back: $P_{ij}'(t) = \sum_k r_{ik} P_{kj}(t)$

For: $P_{ij}'(t) = \sum_k r_{kj} P_{ik}(t)$

Let $R = \{r_{ij}\}$, $P(t) = \{P_{ij}(t)\}$, $P'(t) = \{P_{ij}'(t)\}$

Back: $P'(t) = R\,P(t)$

For: $P'(t) = P(t)\,R$

just as

$$f'(t) = c f(t) \text{ has solution}$$

$$f(t) = f(0) e^{ct},$$

the Kolmogorov equations have solution

$$P(t) = P(0) e^{Rt}$$

where $e^{Rt} = I + tR + \frac{t^2}{2!}R^2 + \ldots$

and $P(0) = \{P_{ij}(0)\} = I$

so $P(t) = e^{Rt}$

Stat 568
HW #4

13. A small barbershop, operated by a single barber, has room for at most two customers. Potential customers arrive at a Poisson rate of three per hour, and the successive service times are independent exponential random variables with mean $\frac{1}{4}$ hour.
   (a) What is the average number of customers in the shop?
   (b) What is the proportion of potential customers that enter the shop?
   (c) If the barber could work twice as fast, how much more business would he do?

20. There are two machines, one of which is used as a spare. A working machine will function for an exponential time with rate λ and will then fail. Upon failure, it is immediately replaced by the other machine if that one is in working order, and it goes to the repair facility. The repair facility consists of a single person who takes an exponential time with rate μ to repair a failed machine. At the repair facility, the newly failed machine enters service if the repairperson is free. If the repairperson is busy, it waits until the other machine is fixed; at that time, the newly repaired machine is put in service and repair begins on the other one. Starting with both machines in working condition, find
   (a) the expected value and
   (b) the variance of the time until both are in the repair facility.
   (c) In the long run, what proportion of time is there a working machine?

21. Suppose that when both machines are down in Exercise 20 a second repairperson is called in to work on the newly failed one. Suppose all repair times remain exponential with rate μ. Now find the proportion of time at least one machine is working, and compare your answer with the one obtained in Exercise 20.

23. A job shop consists of three machines and two repairmen. The amount of time a machine works before breaking down is exponentially distributed with mean 10. If the amount of time it takes a single repairman to fix a machine is exponentially distributed with mean 8, then
   (a) what is the average number of machines not in use?
   (b) what proportion of time are both repairmen busy?