Example 2 from last time:

Stat 568
4-11-17

(1)

$P_0 = .25,\ P_1 = .25,\ P_2 = .5\ ,\ X_0 > 1$

$\mu = 1.25,\ \sigma^2 = \frac{11}{16}$

$E[X_n] = 1.25^n X_0$

$V[X_n] = \frac{11}{16}(1.25)^{n-1}\frac{1-1.25^n}{(-.25)} X_0$

Prob(population dies out) $= \pi_0^{X_0}$

$$\pi_0 = \sum_{j=0}^{\infty} \pi_0^j P_j = 1\cdot P_0 + \pi_0 P_1 + \pi_0^2 P_2$$

(2)

$\pi_0 = \frac{1}{4} + \frac{1}{4}\pi_0 + \frac{1}{2}\pi_0^2$

$2\pi_0^2 - 3\pi_0 + 1 = 0$

$$\pi_0 = \frac{3 \pm \sqrt{9-8}}{4} = \frac{1}{2}, 1$$

Prob(pop dies out) $= \left(\frac{1}{2}\right)^{X_0}$

<u>Defn</u>: $\{N(t), t \geq 0\}$ is a <u>Counting process</u> if $N(t)$ represents the # of events that occur by time t.

(3)

Defn: A counting process has <u>independent increments</u> if the # of events in disjoint time periods are independent.

Defn: A counting process has <u>stationary increments</u> if the # of events in an interval depends only on the length of the interval.

(4)

Defn: A function $f(h)$ is $o(h)$ if

$$\lim_{h \to 0} \frac{f(h)}{h} = 0$$

Example: $f(h) = \frac{1}{2}h$

$$\lim_{h \to 0} \frac{\frac{1}{2}h}{h} = \frac{1}{2}$$ so $\frac{1}{2}h$ is not $o(h)$

$f(h) = \sqrt{h}$

$$\lim_{h \to 0} \frac{\sqrt{h}}{h} = \lim_{h \to 0} \frac{1}{\sqrt{h}} = \infty$$ so $\sqrt{h}$ is not $o(h)$

(5)

$f(h) = h^2$

$$\lim_{h\to 0} \frac{h^2}{h} = \lim_{h \to 0} h = 0 \quad \text{so} \quad h^2 \text{ is } o(h)$$

Note: If $f(h)$ and $g(h)$ are both $o(h)$, then so is $f+g$

If $f(h)$ is $o(h)$, then so is $c\,f(h)$

(6)

Defn: A Poisson process is a counting process such that

i) $N(0) = 0$

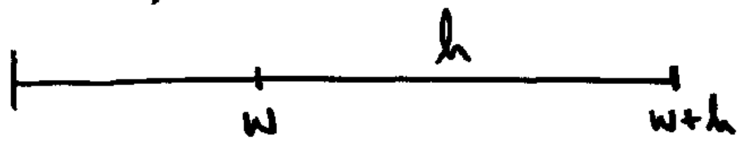
ii) The process has stationary and independent increments

iii) $P[N(h) = 1] = \lambda h + o(h)$

iv) $P[N(h) \geq 2] = o(h)$

7

Let $f(x,h)$ be the probability of seeing exactly x occurrences in an interval of length h.



$$f(0,w+h) = f(0,w)\cdot f(0,h) \quad \text{by indep. incs.}$$

$$f(0,h) = 1-\left[f(1,h) + \sum_{x=2}^{\infty} f(x,h)\right]$$

$$= 1-\left[\lambda h + o(h) + o(h)\right]$$

8

$$f(0,h) = 1-\lambda h + o(h)$$

So
$$f(0,w+h) = f(0,w)\left[1-\lambda h + o(h)\right]$$

$$= f(0,w) + f(0,w)\left[-\lambda h + o(h)\right]$$

$$\lim_{h\to 0} \frac{f(0,w+h)-f(0,w)}{h} = \lim_{h\to 0}\frac{f(0,w)\left[-\lambda h + o(h)\right]}{h}$$

$$\frac{\partial f(0,w)}{\partial w} = -\lambda f(0,w)$$

9

$$\int \frac{d\, f(0,w)}{f(0,w)} = -\lambda \int dw$$

$$\ln f(0,w) = -\lambda w + c$$

$$f(0,w) = e^{-\lambda w + c}$$

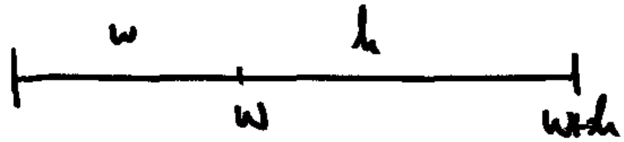
$f(0,0)$ = Prob of seeing 0 events in an interval of length 0 $= 1$

$$1 = e^{c} \quad \therefore \quad c = 0$$

10

$$\therefore f(0,w) = e^{-\lambda w}$$

Suppose $x \geq 1$



$f(x, w+h)$

$$= P[x \text{ in } w \cap 0 \text{ in } h] + P[x-1 \text{ in } w \cap 1 \text{ in } h] + \dots + P[0 \text{ in } w \cap x \text{ in } h]$$

$$= f(x,w)\cdot f(0,h) + f(x-1,w)\, f(1,h) + \sum_{i=2}^{x} f(x-i,w)\, f(i,h)$$

(11)

$$= f(x,w)\left[1-\lambda h + o(h)\right] + f(x-1,w)\left[\lambda h + o(h)\right] + o(h)$$

$$\lim_{h\to 0}\frac{f(x,w+h)-f(x,w)}{h} = \lim_{h\to 0}\frac{f(x,w)\left[-\lambda h + o(h)\right]}{h} + \lim_{h\to 0}\frac{f(x-1,w)\left[\lambda h + o(h)\right] + o(h)}{h}$$

$$\frac{\partial f(x,w)}{\partial w} = -\lambda f(x,w) + \lambda f(x-1,w) \quad \text{for } x \geq 1$$

(12)

for $x=1$:

$$\frac{\partial f(1,w)}{\partial w} = -\lambda f(1,w) + \lambda \underbrace{f(0,w)}_{e^{-\lambda w}}$$

$$e^{\lambda w}\frac{\partial f(1,w)}{\partial w} + \lambda e^{\lambda w} f(1,w) = \lambda$$

$$\frac{d}{dw}\left[e^{\lambda w} f(1,w)\right] = \lambda$$

$$e^{\lambda w} f(1,w) = \lambda w + c$$

(13)

$$f(1,w) = \lambda w e^{-\lambda w} + c e^{-\lambda w}$$

$f(1,0) = \text{Prob}(1$ event in an interval of length $0) = 0$

$$0 = 0 + c \quad \therefore c = 0$$

$$f(1,w) = \lambda w e^{-\lambda w}$$

for $n=2$: $\frac{\partial f(2,w)}{\partial w} = -\lambda f(2,w) + \lambda \underbrace{f(1,w)}_{\lambda w e^{-\lambda w}}$

(14)

$$e^{\lambda w} \frac{\partial f(2,w)}{\partial w} + \lambda e^{\lambda w} f(2,w) = \lambda^2 w$$

$$\frac{d}{dw}\left[e^{\lambda w} f(2,w)\right] = \lambda^2 w$$

$$e^{\lambda w} f(2,w) = \lambda^2 \frac{w^2}{2} + c$$

$$f(2,w) = \frac{\lambda^2 w^2}{2} e^{-\lambda w} + c e^{-\lambda w}$$

$$0 = 0 + c \quad \therefore c = 0$$

15

$$f(0,w) = e^{-\lambda w}$$

$$f(1,w) = \lambda w\, e^{-\lambda w}$$

$$f(2,w) = \frac{\lambda^2 w^2}{2} e^{-\lambda w}$$

$$\vdots$$

$$f(x,w) = e^{-\lambda w} \frac{(\lambda w)^x}{x!}$$