Goal: find $\text{Var}[S_i]$ for the general birth and death process

Stat 568
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①

Recall: $E[S_i|I_i] = \begin{cases} \frac{1}{\lambda_i+\mu_i} & \text{if } I_i = 1 \\ \frac{1}{\lambda_i+\mu_i} + E[S_{i-1}] + E[S_i] & \text{if } I_i = 0 \end{cases}$

$$= \frac{1}{\lambda_i+\mu_i} + (1-I_i)\left(E[S_{i-1}] + E[S_i]\right)$$

$$\text{Var}\left[E[S_i|I_i]\right] = \left(E[S_{i-1}] + E[S_i]\right)^2 \text{Var}(I_i)$$

$\text{Bernoulli}\left(p = \frac{\lambda_i}{\lambda_i+\mu_i}\right)$ $\quad pq$

②

$$V\left[E[S_i|I_i]\right] = \frac{\lambda_i\mu_i}{(\lambda_i+\mu_i)^2}\left(E[S_{i-1}] + E[S_i]\right)^2$$

$$V[S_i | I_i = 1] = V[T_i | I_i = 1] = \frac{1}{(\lambda_i+\mu_i)^2}$$

since $T_i \sim \exp(v_i)$,
$v_i = \lambda_i + \mu_i$

$$V[S_i | I_i = 0] = V\left[T_i + S_{i-1} + S_i\right]$$
$$= V[T_i] + V[S_{i-1}] + V[S_i] \quad \text{by independence}$$
$$= \frac{1}{(\lambda_i+\mu_i)^2} + V[S_{i-1}] + V[S_i]$$

Combining these two ③

$$V[S_i \mid I_i] = \frac{1}{(\lambda_i+\mu_i)^2} + (1-I_i)\left(V[S_{i-1}] + V[S_i]\right)$$

$$E\left[V[S_i \mid I_i]\right] = \frac{1}{(\lambda_i+\mu_i)^2} + \left(1 - \frac{\lambda_i}{\lambda_i+\mu_i}\right)\left(V[S_{i-1}] + V[S_i]\right)$$

$$= \frac{1}{(\lambda_i+\mu_i)^2} + \frac{\mu_i}{\lambda_i+\mu_i}\left(V[S_{i-1}] + V[S_i]\right)$$

$$V[S_i] = V\left[E[S_i \mid I_i]\right] + E\left[V[S_i \mid I_i]\right]$$

④

$$V[S_i] = \frac{\lambda_i\mu_i}{(\lambda_i+\mu_i)^2}\left(E[S_{i-1}] + E[S_i]\right)^2 + \frac{1}{(\lambda_i+\mu_i)^2} + \frac{\mu_i}{\lambda_i+\mu_i}\left(V[S_{i-1}] + V[S_i]\right)$$

$$V[S_i]\underbrace{\left(1 - \frac{\mu_i}{\lambda_i+\mu_i}\right)}_{\frac{\lambda_i}{\lambda_i+\mu_i}} = \frac{\lambda_i\mu_i}{(\lambda_i+\mu_i)^2}\left(E[S_{i-1}] + E[S_i]\right)^2 + \frac{1}{(\lambda_i+\mu_i)^2} + \frac{\mu_i}{\lambda_i+\mu_i}V[S_{i-1}]$$

$$V[S_i] = \frac{1}{\lambda_i(\lambda_i+\mu_i)} + \frac{\mu_i}{\lambda_i}V[S_{i-1}] + \frac{\mu_i}{(\lambda_i+\mu_i)}\left(E[S_{i-1}] + E[S_i]\right)^2$$

Start with $E[S_0] = \frac{1}{\lambda_0}$ and $V[S_0] = \frac{1}{\lambda_0^2}$ (5)

and build.

Defn: Let $P_{ij}(t) = P[X(t+s)=j \mid X(s)=i]$.

These are the transition probabilities for a continuous-time Markov chain.

Pure birth
Proposition: $P_{ij}(t) = \sum_{k=i}^{j} e^{-\lambda_k t} \prod_{\substack{r=i \\ r\neq k}}^{j} \frac{\lambda_r}{\lambda_r - \lambda_k} - \sum_{k=i}^{j-1} e^{-\lambda_k t} \prod_{\substack{r=i \\ r\neq k}}^{j-1} \frac{\lambda_r}{\lambda_r - \lambda_k}$ for $i<j$

Assumption:
$\lambda_i \neq \lambda_j$ for $i \neq j$

Also, $P_{ii}(t) = e^{-\lambda_i t}$ (6)

Proof: Let T_k = time to get from state k to $k+1$

Let $j > i$. Then $\sum_{k=i}^{j-1} T_k$ is the time to get from state i to j

Note: $X(t) < j \iff T_i + T_{i+1} + \ldots + T_{j-1} > t$, assuming $X(0)=i$

So $P[X(t) < j \mid X(0) = i] = P\left[\sum_{k=i}^{j-1} T_k > t\right]$ (7)

Defn: Let T_i, $i = 1, \ldots, n$ be independent exponential random variables with parameters λ_i and $\lambda_i \neq \lambda_j$ if $i \neq j$

Then $T = \sum_{i=1}^{n} T_i$ is a hypoexponential random variable.

(8)

Review: Let X & Y be independent random variables.

Let $U = X + Y$ and $V = Y$

$$X = U - V \text{ and } Y = V$$

$$J = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

$$g(u,v) = f(x,y)\,|J|$$
$$= f_X(x)\, f_Y(y) = f_X(u-v) f_Y(v)$$

$$g_U(u) = \int_{-\infty}^{\infty} f_X(u-v) f_Y(v)\, dv$$ "Convolution formula"

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$$f_{T_1+T_2}(t) = \int_0^t f_{T_1}(s) f_{T_2}(t-s)\, ds$$

$$= \int_0^t \lambda_1 e^{-\lambda_1 s} \lambda_2 e^{-\lambda_2 (t-s)}\, ds$$

$$= \lambda_1 \lambda_2 e^{-\lambda_2 t} \int_0^t e^{-(\lambda_1 - \lambda_2)s}\, ds$$

$$= \lambda_1 \lambda_2 e^{-\lambda_2 t} \left. \frac{e^{-(\lambda_1-\lambda_2)s}}{-(\lambda_1-\lambda_2)} \right|_{s=0}^{t}$$

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$$= \lambda_1 \lambda_2 e^{-\lambda_2 t} \left[\frac{e^{-(\lambda_1-\lambda_2)t}}{-(\lambda_1-\lambda_2)} - \frac{1}{-(\lambda_1-\lambda_2)} \right]$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} e^{-\lambda_2 t} \left(1 - e^{-(\lambda_1-\lambda_2)t} \right)$$

$$= \frac{\lambda_1}{\lambda_1-\lambda_2} \lambda_2 e^{-\lambda_2 t} + \frac{\lambda_2}{\lambda_2-\lambda_1} \lambda_1 e^{-\lambda_1 t}$$

(11)

In general, it can be shown that

$$f_T(t) = \sum_{i=1}^{n} C_{i,n} \lambda_i e^{-\lambda_i t}, \text{ where}$$

$$C_{i,n} = \prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i}$$

So $P[T > t] = \int_t^{\infty} \sum_{i=1}^{n} C_{i,n} \lambda_i e^{-\lambda_i s} ds$

$$= \sum_{i=1}^{n} C_{i,n} e^{-\lambda_i t}$$

(12)

Back to the proposition:

$$P[X(t) < j \mid X(0) = i] = P\left[\sum_{k=i}^{j-1} T_k > t\right]$$

$$= \sum_{k=i}^{j-1} C_{k,j-1} e^{-\lambda_k t}$$

Similarly,

$$P[X(t) < j+1 \mid X(0) = i] = P\left[\sum_{k=i}^{j} T_k > t\right]$$

$$= \sum_{k=i}^{j} C_{k,j} e^{-\lambda_k t}$$

Now

$$P_{ij}(t) = P[X(t) = j \mid X(0) = i]$$

$$= P[\underbrace{X(t) < j+1}_{\le j} \mid X(0) = i] - P[\underbrace{X(t) < j}_{\le j-1} \mid X(0) = i]$$

Finally $P_{ii}(t) = P[T_i > t] = e^{-\lambda_i t}$