<u>Queueing Theory</u>

Stat 528
5-25-17

(1)

Notation: L = average number of customers in the system

L_Q = average number of customers in the queue

W = average time in system

W_Q = Average time in the queue

$X(t)$ = # customers in the system at time t

(2)

$P_n = \lim_{t \to \infty} P[X(t) = n]$ if the limit exists

P_n is the steady state probability

a_n = proportion of customers that find n people in the system when they arrive

d_n = proportion of customers that see n people in the system when they depart

3

Counterexample showing that p_n, a_n, and d_n are not all the same:

Assume service time $\equiv 1$ minute

Assume interarrivals $\sim$ Unif $(1, 2)$

Every arrival will see 0 people in the system

Every departure will see 0 people in the system

So $a_0 = 1 = d_0$

But $p_0 \neq 1$

4

Proposition: In any system in which customers arrive and depart 1 at a time, $a_n = d_n$.

Proof: An arrival who sees n customers in the system causes a transition from state n to $n+1$.

A departure who sees n customers in the system causes a transition from state $n+1$ to n.

(5)

Take any time interval T.

The # of transitions from n to n+1 and the # from n+1 to n are within 1 of each other.

$$a_n = \frac{\text{rate at which arrivals find } n \text{ customers in system}}{\text{overall arrival rate}}$$

$$d_n = \frac{\text{rate at which departures leave } n \text{ customers in system}}{\text{overall departure rate}}$$

$\therefore a_n = d_n$

(6)

M/M/1 system: Single-server queue with Poisson arrivals & departures.

Let λ and μ be the arrival & departure rates

Average time between arrivals is $\frac{1}{\lambda}$

" " " departures is $\frac{1}{\mu}$

Use a previously proven fact that the rate at which the system enters a state must equal the rate at which the system leaves a state.

⑦

	enter	depart
State 0	$P_1\mu$	$P_0\lambda$
State n	$P_{n-1}\lambda + P_{n+1}\mu$	$P_n(\lambda+\mu)$

Balance equations: $\begin{cases} \lambda P_0 = \mu P_1 \\ (\lambda+\mu) P_n = \lambda P_{n-1} + \mu P_{n+1} \quad n \geq 1 \end{cases}$

Rewrite as: $P_1 = \frac{\lambda}{\mu} P_0$

$$P_{n+1} = \frac{\lambda}{\mu} P_n + \left(P_n - \frac{\lambda}{\mu} P_{n-1}\right)$$

⑧

So $P_2 = \frac{\lambda}{\mu} P_1 + \cancel{\left(P_1 - \frac{\lambda}{\mu} P_0\right)}$

$$= \left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$P_3 = \frac{\lambda}{\mu} P_2 + \cancel{\left(P_2 - \frac{\lambda}{\mu} P_1\right)}$$

$$= \left(\frac{\lambda}{\mu}\right)^3 P_0 \qquad \text{etc}$$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

But $1 = \sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n P_0 = P_0 \frac{1}{1-\frac{\lambda}{\mu}}$

if $\lambda/\mu < 1$

(9)

$\therefore$ Need $\lambda < \mu$ for the steady-state probabilities to exist.

$$P_0 = 1 - \frac{\lambda}{\mu} \quad \text{and} \quad P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

$$L = \sum_{n=0}^{\infty} n P_n = \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

$$= \frac{\lambda}{\mu} \sum_{n=1}^{\infty} n \left(\frac{\lambda}{\mu}\right)^{n-1} \left(1 - \frac{\lambda}{\mu}\right) \qquad \text{Let } p = 1 - \frac{\lambda}{\mu}$$

$$= \frac{\lambda}{\mu} \sum_{n=1}^{\infty} n q^{n-1} p = \frac{\lambda}{\mu} \cdot \frac{1}{p} = \frac{\lambda}{\mu} \frac{1}{1 - \frac{\lambda}{\mu}}$$

(10)

$$L = \frac{\lambda}{\mu - \lambda}$$

$$W = \text{Average time in system} = \frac{\text{Average \# customers in system}}{\text{rate of arrival}}$$

$$= \frac{L}{\lambda}$$

$$W = \frac{1}{\mu - \lambda}$$

$$W_Q = W - \text{Average time being served}$$

$$= \frac{1}{\mu - \lambda} - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu - \lambda)}$$

11

$$L_Q = \lambda W_Q = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

Example: M/M/1 arrivals at 5 per hour $\lambda=5$
departures at 7.5 per hour $\mu=7.5$

$$\frac{\lambda}{\mu} = \frac{5}{7.5} = \frac{2}{3} \qquad P_n = \left(\frac{2}{3}\right)^n \frac{1}{3}$$

$$L = \frac{\lambda}{\mu-\lambda} = \frac{5}{7.5-5} = 2$$

$$W = \frac{1}{\mu-\lambda} = \frac{1}{7.5-5} = .4 \text{ hours} = 24 \text{ minutes}$$

12

$$W_Q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{5}{7.5(2.5)} = \frac{4}{15} \text{ hour}$$

$$= 16 \text{ minutes}$$

$$L_Q = \lambda W_Q = 5 \cdot \frac{4}{15} = \frac{4}{3}$$

Suppose $\lambda=6$ instead of 5

$$\frac{\lambda}{\mu} = \frac{6}{7.5} = \frac{4}{5} \qquad P_n = \left(\frac{4}{5}\right)^n \frac{1}{5}$$

$$L = \frac{\lambda}{\mu-\lambda} = \frac{6}{7.5-6} = 4$$

(13)

$$W = \frac{1}{\mu - \lambda} = \frac{1}{7.5-6} = \frac{2}{3}\text{ hr} = 40 \text{ minutes}$$

$$W_Q = 32 \text{ minutes}$$

$$L_Q = 3.2$$

Stat 568

HW #5

5. Let $U_1, U_2, \ldots$ be independent uniform (0, 1) random variables, and define N by

$$N = \min\{n : U_1 + U_2 + \cdots + U_n > 1\}$$

What is $E[N]$?

6. Consider a renewal process $\{N(t), t \geqslant 0\}$ having a gamma (r, λ) interarrival distribution. That is, the interarrival density is

$$f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{r-1}}{(r-1)!}, \quad x > 0$$

(a) Show that

$$P\{N(t) \geqslant n\} = \sum_{i=nr}^{\infty} \frac{e^{-\lambda t}(\lambda t)^i}{i!}$$

(b) Show that

$$m(t) = \sum_{i=r}^{\infty} \left[\frac{i}{r}\right] \frac{e^{-\lambda t}(\lambda t)^i}{i!}$$

where $[i/r]$ is the largest integer less than or equal to i/r.

Hint: Use the relationship between the gamma (r, λ) distribution and the sum of r independent exponentials with rate λ to define $N(t)$ in terms of a Poisson process with rate λ.

3. The manager of a market can hire either Mary or Alice. Mary, who gives service at an exponential rate of 20 customers per hour, can be hired at a rate of \$3 per hour. Alice, who gives service at an exponential rate of 30 customers per hour, can be hired at a rate of \$C per hour. The manager estimates that, on the average, each customer's time is worth \$1 per hour and should be accounted for in the model. Assume customers arrive at a Poisson rate of 10 per hour

(a) What is the average cost per hour if Mary is hired? If Alice is hired?

(b) Find C if the average cost per hour is the same for Mary and Alice.