Normal moment generating function (continued)

Stat 567
2-7-17
1

$$\phi(t) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\left(x^2 - 2(\mu+\sigma^2 t)x + (\mu+\sigma^2 t)^2 - (\mu+\sigma^2 t)^2 + \mu^2\right)} dx$$

$$= e^{-\frac{1}{2\sigma^2}\left(-(\mu+\sigma^2 t)^2 + \mu^2\right)} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\left(x - (\mu+\sigma^2 t)\right)^2} dx}_{1}$$

$$= e^{-\frac{1}{2\sigma^2}\left(-\cancel{\mu^2} - 2\mu\sigma^2 t - \sigma^4 t^2 + \cancel{\mu^2}\right)}$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}}$$ Check $\phi(0) = 1$ ✓

2

$$\phi'(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}(\mu + \sigma^2 t)$$

$$\phi''(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}(\sigma^2) + e^{\mu t + \frac{1}{2}\sigma^2 t^2}(\mu+\sigma^2 t)^2$$

$$\phi'(0) = \mu = E[X]$$

$$\phi''(0) = \sigma^2 + \mu^2 = E[X^2]$$

$$V[X] = E[X^2] - (E[X])^2 = \sigma^2 + \mu^2 - \mu^2$$
$$= \sigma^2$$

③

For Bernoulli, Binomial, Geometric, Poisson,

Uniform, Exponential, Gamma, Normal,

You need to know 1) pmf or pdf

2) values that X takes on

3) mgf

4) μ & σ^2

Joint distributions

Defn: The joint cdf of 2 random variables X, Y

is $F(a,b) = P(X \le a \cap Y \le b)$

④

Note: $\lim_{b \to \infty} F(a,b) = \lim_{b \to \infty} P(X \le a \cap Y \le b)$

$F(a,\infty) = P(X \le a)$

$= F_X(a)$

Likewise, $F(\infty, b) = F_Y(b)$

Discrete case

Defn: The joint probability mass function is

$$p(x,y) = P(X = x \cap Y = y)$$

$$p_X(x) = P(X=x)$$
$$= \sum_y P(X=x \cap Y=y) \quad \text{[Law of Total Probability]}$$
$$= \sum_y p(x,y)$$

Similarly, $p_Y(y) = \sum_x p(x,y)$

Continuous case

Defn: The joint probability density function is a function $f(x,y)$ such that

$$P(X \in A \cap Y \in B) = \int_B \int_A f(x,y)\,dx\,dy \qquad (6)$$

Note: $F(a,b) = P(X \leq a \cap Y \leq b)$
$$= \int_{-\infty}^{b} \int_{-\infty}^{a} f(x,y)\,dx\,dy$$

$$F_X(a) = F(a,\infty) = \int_{-\infty}^{\infty} \int_{-\infty}^{a} f(x,y)\,dx\,dy$$

$$F_Y(b) = F(\infty,b) = \int_{-\infty}^{b} \int_{-\infty}^{\infty} f(x,y)\,dx\,dy$$

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By the Fund. Thm. of Calculus in 2 dim,

$$\frac{\partial^2 F(x,y)}{\partial y\, \partial x} = f(x,y)$$

<u>Example</u>: Roll 2 dice. Let $X = \min$, $Y = \max$

$p(x,y)$	Y: 1	2	3	4	5	6	$p_X(x)$
X: 1	1/36	2/36	2/36	2/36	2/36	2/36	11/36
2	0	1/36	2/36	2/36	2/36	2/36	9/36
3	0	0	1/36	2/36	2/36	2/36	7/36
4	0	0	0	1/36	2/36	2/36	5/36
5	0	0	0	0	1/36	2/36	3/36
6	0	0	0	0	0	1/36	1/36

(8)

<u>Example</u>: Let $f(x,y) = c(x^2+y)$, $0<x<1$, $0<y<1$

Find c. $\displaystyle\int_0^1\int_0^1 c(x^2+y)\,dx\,dy \overset{\text{set}}{=} 1$

$$= c\int_0^1 \left(\frac{x^3}{3} + yx\right)\Big|_{x=0}^{1} dy$$

$$= c\int_0^1 \left(\frac{1}{3} + y - 0\right) dy = c\left[\frac{1}{3}y + \frac{y^2}{2}\right]_{y=0}^{1}$$

$$= c\left(\frac{1}{3} + \frac{1}{2} - 0\right) = c \cdot \frac{5}{6} \quad \therefore c = 6/5$$

⑨

Find $f_X(x) = \int_0^1 f(x,y)\,dy$

$$= \int_0^1 \frac{6}{5}(x^2+y)\,dy$$

$$= \frac{6}{5}\left(x^2 y + \frac{y^2}{2}\right)\Big|_{y=0}^{1}$$

$$= \frac{6}{5}\left(x^2+\frac{1}{2}-0\right) = \frac{6}{5}\left(x^2+\frac{1}{2}\right) \quad 0<x<1$$

⑩

Find $f_Y(y) = \int_0^1 \frac{6}{5}(x^2+y)\,dx$

$$= \frac{6}{5}\left(\frac{x^3}{3} + yx\right)\Big|_{x=0}^{1}$$

$$= \frac{6}{5}\left(\frac{1}{3}+y-0\right) = \frac{6}{5}\left(\frac{1}{3}+y\right) \quad 0<y<1$$

<u>Defn.</u> $E[g(X,Y)] = \begin{cases} \sum_x \sum_y g(x,y)\,p(x,y) & \text{discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,f(x,y)\,dx\,dy & \text{continuous} \end{cases}$

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Prove that $E[aX+bY] = aE[X]+bE[Y]$

Proof: (continuous case)

$$E[aX+bY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(ax+by)f(x,y)\,dx\,dy$$

$$= a\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x f(x,y)\,dx\,dy + b\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}y f(x,y)\,dx\,dy$$

$$= aE[X] + bE[Y]$$

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Let $X \sim \text{Bin}(n,p)$

Then $X = X_1 + X_2 + \dots + X_n$,

where each $X_i = \begin{cases} 0 & \text{if failure} \\ 1 & \text{if success} \end{cases}$ on the i^{th} trial

So $X_i \sim \text{Bern}(p)$

and we know $E[X_i]=p$

$$\therefore E[X] = \sum_{i=1}^{n} E[X_i] = np$$

Independence

(13)

Defn. The random variables X and Y are independent if $F(a,b) = F_X(a)\,F_Y(b)$ $\forall\ a,b$

Note: (continuous case)

Assume X & Y are independent.

Then $f(x,y) = \dfrac{\partial^2}{\partial y\,\partial x} F(x,y)$

$$= \frac{\partial^2}{\partial y\,\partial x} F_X(x)\,F_Y(y)$$

$$= \frac{\partial}{\partial y}\underbrace{\frac{\partial}{\partial x} F_X(x)\,F_Y(y)}$$

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$$= \frac{\partial}{\partial y} F_Y(y)\, f_X(x)$$

$$= f_X(x)\, f_Y(y)$$

Assume $f(x,y) = f_X(x)\, f_Y(y)$ $\forall\ x,y$

Then $F(a,b) = \displaystyle\int_{-\infty}^{b}\int_{-\infty}^{a} f(x,y)\,dx\,dy$

$$= \int_{-\infty}^{b}\int_{-\infty}^{a} f_X(x)\, f_Y(y)\,dx\,dy$$

$$= \int_{-\infty}^{b} \Big[f_Y(y) \underbrace{\int_{-\infty}^{a} f_X(x)\,dx}_{F_X(a)} \Big] dy$$

$$= F_X(a) \underbrace{\int_{-\infty}^{b} f_Y(y)\,dy}_{F_Y(b)}$$

The midterm exam is Thursday Feb 14

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+ calculator