Stat 568
4-25-17

Derivation from last time, continued:

(1)

$$M(t) = \frac{k e^{(\lambda-\mu)t} - \theta}{\lambda-\mu} = \frac{\left[(\lambda-\mu)X_0+\theta\right] e^{(\lambda-\mu)t} - \theta}{\lambda-\mu}$$

$$= X_0 e^{(\lambda-\mu)t} + \frac{\theta}{\lambda-\mu}\left[e^{(\lambda-\mu)t} - 1\right]$$

This works for $\lambda \neq \mu$

If $\lambda=\mu$, then
$$M'(t) = \theta$$
$$M(t) = \theta t + c$$
$$M(0) = X_0 = c$$
$$M(t) = \theta t + X_0$$

(2)

__Example__: M/M/s multiserver queueing system

M: births follow a Poisson process

M: deaths follow a Poisson process

s: # servers

There are s servers.

Customers arrive at rate λ. The wait in a single queue, & then go to the 1st available server.

Each server has departure rate μ.

(3)

Goal: find the $\{\lambda_n\}$ and $\{\mu_n\}$

$\lambda_n \equiv \lambda$

Say $n \le s$ Then n servers will be in use

+ the departure rate will be $n\mu$

because the next departure time will be the minimum of n independent exponentials, each with rate μ.

Say $n > s$ Then s servers will be in use

+ the departure rate will be $s\mu$.

So $\mu_n = \begin{cases} n\mu & n \le s \\ s\mu & n > s \end{cases}$

(4)

Once $\{\lambda_n\}$ and $\{\mu_n\}$ are determined, then

$$P_{01} = 1 \qquad P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i} \qquad P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}$$

$$T_i \sim \exp(\nu_i = \lambda_i + \mu_i)$$

Consider a general birth and death process with rates $\{\lambda_n\}$ and $\{\mu_n\}$ and let

S_i = time it takes to transition from state i to state $i+1$.

Goal: find $E[S_i]$.

(5)

$S_0 \sim \exp(\lambda_0) \qquad E[S_0] = \frac{1}{\lambda_0}$

For $i>0$

Let $I_i = \begin{cases} 1 & \text{if the 1st transition from } i \text{ is to } i+1 \\ 0 & \text{" " " " " " " " } i-1 \end{cases}$

Find:

$E[S_i \mid I_i]$

$E[S_i \mid I_i = 1] = \frac{1}{\lambda_i + \mu_i}$ since if $I_i = 1$, the time to transition to $i+1$ is just the time in i

(6)

$E[S_i \mid I_i = 0] = \frac{1}{\lambda_i + \mu_i} + E[S_{i-1}] + E[S_i]$

Now $E[S_i] = E\left[E[S_i \mid I_i]\right]$

$= E[S_i \mid I_i = 1]\, P[I_i = 1] + E[S_i \mid I_i = 0]\, P[I_i = 0]$

$= \frac{1}{\lambda_i + \mu_i} \cdot \frac{\lambda_i}{\lambda_i + \mu_i} + \left[\frac{1}{\lambda_i + \mu_i} + E[S_{i-1}] + E[S_i]\right] \frac{\mu_i}{\lambda_i + \mu_i}$

$E[S_i] = \frac{1}{\lambda_i + \mu_i} + \frac{\mu_i}{\lambda_i + \mu_i}\left(E[S_{i-1}] + E[S_i]\right)$

$E[S_i]\left[1 - \frac{\mu_i}{\lambda_i + \mu_i}\right] = \frac{1}{\lambda_i + \mu_i} + \frac{\mu_i}{\lambda_i + \mu_i} E[S_{i-1}]$

(7)

$$E[S_i] = \frac{1 + \mu_i E[S_{i-1}]}{\lambda_i}$$

$$E[S_0] = \frac{1}{\lambda_0}$$

$$E[S_1] = \frac{1 + \mu_1 E[S_0]}{\lambda_1} = \frac{1 + \frac{\mu_1}{\lambda_0}}{\lambda_1}$$

and so forth.

How would we get the expected amount of time to go from state k to j, where $k < j$?

$$E[S_k] + E[S_{k+1}] + \dots + E[S_{j-1}]$$

(8)

Expected time from state k to j $\quad j > k$

$$= \sum_{i=k}^{j-1} E[S_i] = \sum_{i=k}^{j-1} \frac{1 - \left(\frac{\mu}{\lambda}\right)^{i+1}}{\lambda - \mu}$$

$$= (j-k)\frac{1}{\lambda-\mu} - \frac{1}{\lambda-\mu} \underbrace{\sum_{i=k}^{j-1}\left(\frac{\mu}{\lambda}\right)^{i+1}}_{\underbrace{\sum_{i=0}^{j-1-k}\left(\frac{\mu}{\lambda}\right)^{i+k+1}}_{\left(\frac{\mu}{\lambda}\right)^{k+1}\sum_{i=0}^{j-1-k}\left(\frac{\mu}{\lambda}\right)^{i}}}$$

Special case: $\lambda_i \equiv \lambda$, $\mu_i \equiv \mu$ $i \geq 1$ (9)

$i \geq 0$ $\mu_0 = 0$

$$E[S_0] = \frac{1}{\lambda_0} = \frac{1}{\lambda}$$

$$E[S_1] = \frac{1 + \frac{\mu}{\lambda}}{\lambda} = \frac{1}{\lambda}\left(1 + \frac{\mu}{\lambda}\right)$$

$$E[S_2] = \frac{1 + \mu E[S_1]}{\lambda} = \frac{1}{\lambda}\left(1 + \mu \frac{1}{\lambda}\left(1 + \frac{\mu}{\lambda}\right)\right)$$

$$= \frac{1}{\lambda}\left(1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2\right)$$

$\vdots$

$$E[S_i] = \frac{1}{\lambda}\left(1 + \frac{\mu}{\lambda} + \dots + \left(\frac{\mu}{\lambda}\right)^i\right) = \frac{1}{\lambda} \frac{1 - \left(\frac{\mu}{\lambda}\right)^{i+1}}{1 - \mu/\lambda}$$

$$= \frac{j-k}{\lambda - \mu} - \frac{1}{\lambda - \mu}\left(\frac{\mu}{\lambda}\right)^{k+1} \frac{1 - \left(\frac{\mu}{\lambda}\right)^{j-k}}{1 - \frac{\mu}{\lambda}}$$ (10)

This works if $\lambda \neq \mu$

If $\lambda = \mu$: $E[S_0] = \frac{1}{\lambda}$

$E[S_1] = \frac{2}{\lambda}$ $E[S_2] = \frac{3}{\lambda}$ $\dots$ $E[S_i] = \frac{i+1}{\lambda}$

The expected time from k to j is

$$\sum_{i=k}^{j-1} \frac{i+1}{\lambda} = \frac{1}{\lambda}\sum_{i=k}^{j-1} i + \frac{1}{\lambda}(j-k) = \frac{1}{\lambda}\left[\frac{(j-1)j}{2} - \frac{(k-1)k}{2} + j - k\right]$$

$$= \frac{1}{2\lambda}\left[j^2 - j + 2j - (k^2 - k + 2k)\right] = \frac{1}{2\lambda}\left[j(j+1) - k(k+1)\right]$$