The Weak Law of Large Numbers (WLLN)

Stat 567
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Let $X_1, X_2, \ldots$ be iid with mean μ and variance σ^2

Then $\lim_{n \to \infty} P[|\bar{X} - \mu| \geq \varepsilon] = 0 \quad \forall \varepsilon > 0$

($\bar{X}$ "converges in probability" to μ)

Pf: Recall Chebyshev's Inequality: $P[|X - \mu| \geq k] \leq \frac{\sigma^2}{k^2}$

Apply this to $\bar{X}$. We know $E[\bar{X}] = \mu$
$V[\bar{X}] = \sigma^2/n$

So $P[|\bar{X} - \mu| \geq \varepsilon] \leq \frac{\sigma^2/n}{\varepsilon^2}$

$$\lim_{n \to \infty} P[|\bar{X} - \mu| \geq \varepsilon] \leq \lim_{n \to \infty} \frac{\sigma^2}{n\varepsilon^2} = 0$$

$$\therefore \lim_{n \to \infty} P[|\bar{X} - \mu| \geq \varepsilon] = 0$$

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The Strong Law of Large Numbers (SLLN)

Let $X_1, X_2, \ldots$ be iid with mean μ and variance σ^2

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Then $P\left[\lim_{n\to\infty} \bar{X}_n = \mu\right] = 1$

($\bar{X}$ "converges almost surely" to μ)

Proof is left for the 600-level course

<u>Chapter 3</u> Recall $P(E|F) = \dfrac{P(E\cap F)}{P(F)}$

<u>Defn</u>: If X & Y are discrete, then

$$p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)}$$ is the conditional probability mass function

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and $$E[X|Y=y] = \sum_{\text{all } x} x\, p_{X|Y}(x|y)$$

If X & Y are continuous, then

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$ is the conditional probability density function

and $$E[X|Y=y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx$$

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Note: $E[X]$ is a real number

$E[X \mid Y=y]$ is a function of y

Example: Let X_1, X_2 be indep Binomial random variables with parameters (n_1, p) and (n_2, p)

Find $E[X_1 \mid \underbrace{X_1+X_2}_{Y} = m]$

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We know that $Y \sim \text{Bino}(n_1+n_2, p)$

$$p_{X_1|Y}(x_1|y) = \frac{p(x_1,y)}{p_Y(y)}$$

$$= \frac{P(X_1=x_1 \cap Y=y)}{P(Y=y)}$$

$$= \frac{P(X_1=x_1 \cap X_1+X_2=y)}{P(Y=y)}$$

$$= \frac{P(X_1=x_1 \cap X_2=y-x_1)}{P(Y=y)}$$

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$$= \frac{P(X_1 = x_1)\, P(X_2 = y - x_1)}{P(Y = y)}$$

$$= \frac{\binom{n_1}{x_1} \cancel{p^{x_1}} \cancel{q^{n_1 - x_1}} \binom{n_2}{y - x_1} \cancel{p^{y - x_1}} \cancel{q^{n_2 - (y - x_1)}}}{\binom{n_1 + n_2}{y} \cancel{p^{y}} \cancel{q^{n_1 + n_2 - y}}}$$

$$p_{X_1|Y}(x_1|y) = \frac{\binom{n_1}{x_1}\binom{n_2}{y - x_1}}{\binom{n_1 + n_2}{y}}$$, which is hypergeometric, with $N = n_1 + n_2$, $R = n_1$, $n = y$

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So $E[X_1 | Y = y]$ = expected value of the hypergeometric distr.

$$= n \frac{R}{N}$$

$$= y \frac{n_1}{n_1 + n_2} = n_1 \frac{y}{n_1 + n_2}$$

Compare to $E[X_1] = n_1 p$

Example: Let X_1 and X_2 be independent Poisson random variables with parameters λ_1 and λ_2 (9)

Find $E[X_1 \mid \underbrace{X_1 + X_2}_{Y} = m]$

$Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$

$$P_{X_1|Y}(x_1|y) = \frac{p(x_1, y)}{P_Y(y)}$$

$$= \frac{P[X_1 = x_1 \cap Y = y]}{P[Y = y]}$$

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$$= \frac{P[X_1 = x_1 \cap X_1 + X_2 = y]}{P[Y = y]}$$

$$= \frac{P[X_1 = x_1 \cap X_2 = y - x_1]}{P[Y = y]}$$

$$= \frac{\dfrac{\cancel{e^{-\lambda_1}}\,\lambda_1^{x_1}}{x_1!} \cdot \dfrac{\cancel{e^{-\lambda_2}}\,\lambda_2^{y-x_1}}{(y-x_1)!}}{\left(\dfrac{\cancel{e^{-(\lambda_1+\lambda_2)}}\,(\lambda_1+\lambda_2)^y}{y!}\right)}$$

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$$= \frac{y!}{x_1!(y-x_1)!} \frac{\lambda_1^{x_1} \lambda_2^{y-x_1}}{(\lambda_1+\lambda_2)^y}$$

$$p_{X_1|Y}(x_1|y) = \binom{y}{x_1} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{x_1} \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{y-x_1}$$

$$\sim \text{Bino}\left(n=y,\ p=\frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

So $E[X_1|Y=y] = np = y\frac{\lambda_1}{\lambda_1+\lambda_2}$

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Example: $f(x,y) = 4y(x-y)e^{-(x+y)}$,

$0<y<x<\infty$

Find $E[X|Y=y]$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_y^{\infty} 4y(x-y)e^{-(x+y)}\,dx$$

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Let $w = x - y$
$dw = dx$

$$f_Y(y) = \int_0^\infty 4y\,w\,e^{-(w+2y)}\,dw$$

$$= 4y e^{-2y} \underbrace{\int_0^\infty w e^{-w}\,dw}_{E[W] \text{ if } W \sim Exp(\lambda=1) \;=\; 1}$$

$$f_Y(y) = 4y e^{-2y} \qquad y > 0$$

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$$f_{X|Y}(x|y) = \frac{\cancel{4y}(x-y)e^{-(x+y)}}{\cancel{4y}e^{-2y}} \qquad 0 < y < x < \infty$$

$$= (x-y)e^{-(x-y)}$$

$$E[X \mid Y=y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\,dx$$

$$= \int_y^\infty x(x-y)e^{-(x-y)}\,dx$$

Let $w = (x-y)$
$dw = dx$

$$= \int_0^\infty (w+y)\, w e^{-w}\, dw$$

$$= \underbrace{\int_0^\infty w^2 e^{-w}\, dw}_{E[W^2] \text{ where}} + y \underbrace{\int_0^\infty w e^{-w}\, dw}_{1}$$

$W \sim Exp(\lambda = 1)$

$$\sigma_W^2 = \lambda^2 = 1$$

$$\sigma_W^2 = E[W^2] - (E[W])^2$$

$$1 = E[W^2] - 1^2$$

$$E[W^2] = 2$$

$$E[X \mid Y=y] = 2+y$$

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Defn: $E[X \mid Y]$ is a new random variable, and it is a function of Y

Find $E[E[X \mid Y]]$ Do for continuous case

$$= \int_{-\infty}^{\infty} E[X \mid Y=y]\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx \right] f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \, \frac{f(x,y)}{\cancel{f_Y(y)}} \cancel{f_Y(y)} \, dy\, dx$$

$$= \int_{-\infty}^{\infty} x \left[\int_{-\infty}^{\infty} f(x,y)\, dy \right] dx$$

$$= \int_{-\infty}^{\infty} x \, f_X(x)\, dx = E[X]$$

$$\therefore \quad E\left[E[X|Y] \right] = E[X]$$

"Law of Iterated Expectations"