Stat 528

5-16-17

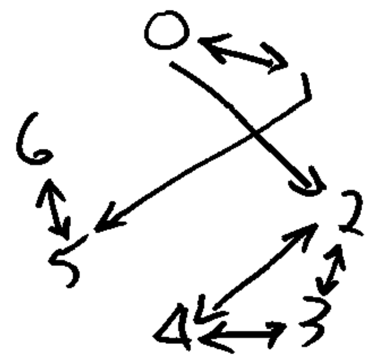
(1)

Midterm results

Max possible = 60 points

1	3
2	9
3	33569
4	22448
5	1146679

1)



a)

$\{0,1\}$ transient

$\{2,3,4\}$ recurrent aperiodic

$\{5,6\}$ recurrent aperiodic

(2)

b) $P_T = \begin{bmatrix} 0 & 2/3 \\ 2/3 & 0 \end{bmatrix}$

$$S = (I - P_T)^{-1} = \begin{bmatrix} 1 & -2/3 \\ -2/3 & 1 \end{bmatrix}^{-1} = \frac{1}{\Delta}\begin{bmatrix} 1 & 2/3 \\ 2/3 & 1 \end{bmatrix}$$

$$\Delta = 1 - 4/9 = 5/9$$

$$= \frac{9}{5}\begin{bmatrix} 1 & 2/3 \\ 2/3 & 1 \end{bmatrix} = \begin{bmatrix} 9/5 & 6/5 \\ 6/5 & 9/5 \end{bmatrix}$$

③

2. $\mu = 0 \cdot \frac{1}{3} + 2 \cdot \frac{2}{3} = \frac{4}{3}$

a) $\pi_0 = \sum_{j=0}^{\infty} \pi_0^j P_j = 1 \cdot \frac{1}{3} + \pi_0^2 \cdot \frac{2}{3}$

$$2\pi_0^2 - 3\pi_0 + 1 = 0$$

$$\pi_0 = \frac{3 \pm \sqrt{9-8}}{4} = \frac{3}{4} \pm \frac{1}{4}$$

$$\therefore \pi_0 = \frac{1}{2}$$

b) $E[X_n] = \mu^n X_0 = \left(\frac{4}{3}\right)^n$

④

$$V[X_n] = \sigma^2 \mu^{n-1} \frac{1-\mu^n}{1-\mu} X_0$$

$$\sigma^2 = 0^2 \cdot \frac{1}{3} + 2^2 \cdot \frac{2}{3} - \mu^2$$

$$= \frac{8}{3} - \left(\frac{4}{3}\right)^2 = 8/9$$

$$V[X_n] = \frac{8}{9} \left(\frac{4}{3}\right)^{n-1} \frac{1-\left(\frac{4}{3}\right)^n}{1-4/3}$$

$$= \frac{8}{3} \left(\frac{4}{3}\right)^{n-1} \left[\left(\frac{4}{3}\right)^n - 1\right]$$

(5)

3. <u>Backward</u> $P_{ij}'(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t)$

$$P_{ij}'(t) = \underbrace{q_{i,i-1}}_{\underbrace{\nu_i}_{i\mu} \underbrace{P_{i,i-1}}_{1}} P_{i-1,j}(t) - \underbrace{\nu_i}_{\mu_i = i\mu} P_{ij}(t)$$

$$P_{ij}'(t) = i\mu \left[P_{i-1,j}(t) - P_{ij}(t) \right]$$

(6)

<u>Forward</u> $P_{ij}'(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - \nu_j P_{ij}(t)$

$$= \underbrace{q_{j+1,j}}_{\underbrace{\underbrace{\nu_{j+1}}_{\mu_{j+1}}}_{(j+1)\mu} \underbrace{P_{j+1,j}}_{1}} P_{i,j+1}(t) - \underbrace{\nu_j}_{\mu_j = j\mu} P_{ij}(t)$$

$$P_{ij}'(t) = (j+1)\mu P_{i,j+1}(t) - j\mu P_{ij}(t)$$

4. $X = \sum_{i=1}^{N} X_i$ $\quad N \sim \text{Poisson}(48)$ (7)

$X_i \sim \text{Unif}(100, 200)$

$E[X_i] = 150$

$V[X_i] = \frac{100^2}{12}$

$$E[X] = E[E[X|N]] = E[N\,E[X_i]] = 150\,E[N]$$

$$= 150 \cdot 48 = 7200$$

$$V[X] = E[V[X|N]] + V[E[X|N]]$$

$$= E[N\,V[X_i]] + V[N\,E[X_i]]$$

(8)

$$= \frac{100^2}{12} E[N] + 150^2 V[N]$$

$$= \frac{100^2}{12} 48 + 150^2 \cdot 48 = 1{,}120{,}000$$

5. $n+1$ need to exit in order for my service to begin. Expected time between exits is $\frac{1}{3}$ minute

$$(n+1)\frac{1}{3} + 1 = \frac{n+4}{3} \text{ minutes}$$

Renewal Processes

(9)

the interarrival times

Defn: Let $\{X_1, X_2, \ldots\}$ be a sequence of iid non-negative random variables, each with cdf F. Then the counting process $\{N(t), t \geq 0\}$ is called a renewal process.

Let $S_1 = X_1$, $S_2 = X_1 + X_2$, … $S_n = \sum_{i=1}^{n} X_i$

Find the distribution of $N(t)$

(10)

Note: $N(t) \geq n \iff S_n \leq t$

The strong law of large numbers said

$$\frac{\sum_{i=1}^{n} X_i}{n} \to \mu \quad \text{as } n \to \infty,$$

with probability 1.

(11)

So $\frac{S_n}{n} \to E[X_i]$ as $n \to \infty$,

with probability 1

Additional assumption about F:

$$F(0) = 0$$

$P[X_i \le 0] = P[X_i = 0]$

This $\Rightarrow \mu > 0$

$$\Rightarrow \lim_{n \to \infty} S_n = \infty$$

(12)

From the note, $N(t) \ge n \Leftrightarrow S_n \le t$

Is it possible for $N(t) = \infty$? No.

If yes, then $N(t) \ge n \ \forall n$

so $S_n \le t \ \forall n$

But $S_n \to \infty$, so for some n,

$S_n > t$ ※

$$P[N(t) = n] = P[N(t) \ge n] - P[N(t) \ge n+1]$$
$$= P[S_n \le t] - P[S_{n+1} \le t]$$

$$= F_n(t) - F_{n+1}(t)$$

where $F_n(t)$ is the cdf of S_n