Recall the c.d.f.

$$F(b) = P(X \leq b)$$

---

let $a < b$

$$F(b) - F(a) = P(X \leq b) - P(X \leq a)$$
$$= P(a < X \leq b)$$

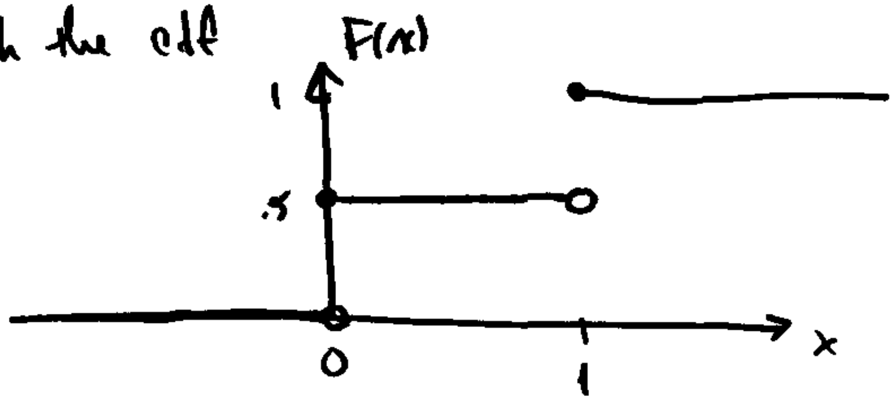
Also $P(X < b) = \lim\limits_{h \to 0^+} P(X \leq b - h)$

<u>Example</u>: Flip a coin $S = \{H, T\}$

Let $X = 1$ for heads, $X = 0$ for tails

②

Graph the cdf



Note that $F(x)$ is always right-continuous

Some discrete random variables:

<u>Bernoulli</u> $X = \begin{cases} 1 & p \\ 0 & 1-p \end{cases}$

(3)

Binomial

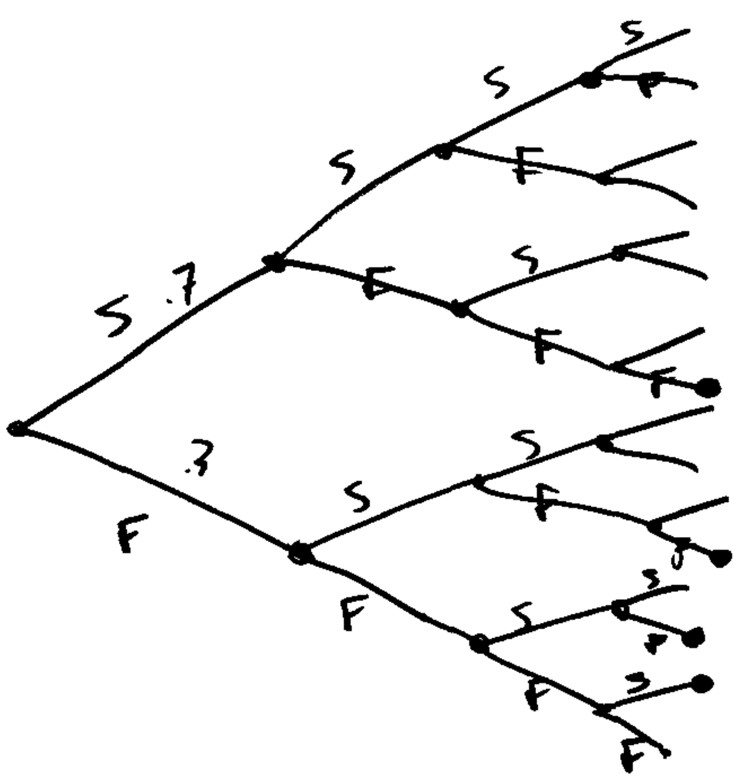
Run a sequence of trials
The trials are independent
Each trial has 2 possible outcomes
(success, failure)
Success prob. remains constant

X = # successes out of n trials

<u>Example</u>: Run 4 trials with $p = .7$

X = # successes

(4)



x	$p(x)$
4	$(.7)^4$
3	$4(.7)^3(.3)$
2	$6(.7)^2(.3)^2$
1	$4(.7)(.3)^3$
0	$(.3)^4$
	1

(5)

$$p(x) = \binom{4}{x}(.7)^x(.3)^{4-x}$$

Pascal's Δ

In general, for n trials and success probability p,

$$\boxed{\begin{gathered} p(x) = \binom{n}{x} p^x (1-p)^{n-x} \\ x = 0, 1, 2, \ldots, n \end{gathered}}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$ Binomial Theorem

(6)

So $$\sum_{x=0}^{n} p(x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= [p + (1-p)]^n = 1$$

<u>Example</u>: Send out 100 invitations to complete a survey. Assume that each individual, independently, completes the survey with probability $p = .1$.

Binomial($n = 100$, $p = .1$)

(7)

Find the prob. that you get exactly 10 responses.

$$p(10) = \binom{100}{10}(.1)^{10}(.9)^{90}$$
$$= .13$$

Find the prob. that you get 10 or fewer responses.

$$P(X \le 10) = F(10)$$
$$= p(0) + p(1) + \dots + p(10) = .5832$$

Find the prob. of getting 20 or more responses.

(8)

$$P(X \ge 20) = p(20) + p(21) + \dots + p(100)$$
$$= 1 - P(X < 20)$$
$$= 1 - P(X \le 19) = 1 - F(19)$$

Geometric

Run a sequence of trials

The trials are independent

Each trial has 2 possible outcomes

The prob. of success, p, is constant

X = trial on which the 1st success occurs

$X = 1, 2, \ldots$

$p(x) = (1-p)^{x-1} p$

Geometric series $a + ar + ar^2 + \ldots$

$$= \frac{a}{1-r} \text{ if } |r| < 1$$

$$\sum_{x=1}^{\infty} (1-p)^{x-1} p = p + p(1-p) + p(1-p)^2 + \ldots$$

$$= \frac{p}{1-(1-p)} = \frac{p}{p} = 1$$

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Example: Survey example: X = trial on which the 1st survey is completed.

Find the prob. that the 5th invitation results in the 1st completed survey.

Geometric$(p=.1)$ $\quad P(X=5) = p(5)$

$$= (1-.1)^4(.1) = .9^4(.1)$$

$$= .06561$$

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<u>Poisson</u> $\quad p(x) = e^{-\lambda} \dfrac{\lambda^x}{x!} \qquad x = 0, 1, 2, \ldots$

$$\sum_{x=0}^{\infty} e^{-\lambda} \frac{\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

$$= e^{-\lambda} \underbrace{\left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots\right)}_{e^{\lambda} \text{ Taylor series}} = 1$$

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Consider a binomial experiment.

Let $n \to \infty$ and $p \to 0$ while $np = \lambda$

$p = \lambda/n$

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$\lim_{n \to \infty} p(x) = \lim_{n \to \infty} \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

(13)

$$= \lim_{n\to\infty} \frac{n!}{x!\,(n-x)!} \frac{\lambda^x}{n^x} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{\overbrace{n(n-1)\cdots(n-x+1)}^{x \text{ terms}}}{n \cdot n \quad \cdots \quad n} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{1\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)}{\left(1-\frac{\lambda}{n}\right)^x} \left(1-\frac{\lambda}{n}\right)^n$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n = \frac{\lambda^x}{x!} e^{-\lambda} \quad \text{because:}$$

(14)

Let $y = \left(1-\frac{\lambda}{n}\right)^n$

$$\ln y = n \ln\left(1-\frac{\lambda}{n}\right)$$

$$= \frac{\ln\left(1-\frac{\lambda}{n}\right)}{\frac{1}{n}}$$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\frac{1}{(1-\lambda/n)} \cdot \lambda n^{-2}}{-n^{-2}} = -\lambda$$

$$\shortparallel$$

$$\ln\left(\lim_{n\to\infty} y\right) \qquad \therefore \lim_{n\to\infty} y = e^{-\lambda}$$

The Poisson experiment looks like

you are observing a continuous time segment and counting the # of a certain type of event, where λ is the expected (or average) number of events.

Example: On the average, your web page gets 10 hits per minute. Find the prob. that a minutes passes with no hits.

Poisson$(\lambda = 10)$ $\quad P(X=0) = p(0) = \dfrac{e^{-10}\ 10^0}{0!}$

$$= e^{-10} = .0000454$$ 16