<u>Branching Processes</u>

Stat 528
4-6-17

Each individual produces j offspring with (1)
probability P_j, independent of all other individuals.

Assume $0 \le P_j < 1$ and $\sum_{j=0}^{\infty} P_j = 1$

X_0 = # of individuals initially present
(generation 0)

X_n = # of individuals in the n^{th} generation

The state space is $\{0, 1, 2, \dots\}$ (2)

$P_{00} = 1$ So 0 is an absorbing state

<u>If $P_0 > 0$,</u> then

$$P_{i0} = P_0^i > 0$$

(by independence

So all other states must be transient.

If the # of transient states is finite, then
the total # of visits to all transient

States is also finite, so eventually the process would have to visit state 0. (3)

$\therefore$ Either the # of transient states is infinite or the population dies out.

$\therefore$ In the limit as $n \to \infty$, the population either increases without bound or dies out.

Let μ and σ^2 be the mean and variance of the # of offspring per individual.

That is, let Z_i = # offspring for i^{th} individual (4)

$$\mu = E[Z_i] = \sum_{j=0}^{\infty} j P_j = \sum_{j=1}^{\infty} j P_j$$

$$\sigma^2 = Var[Z_i] = E[Z_i^2] - \mu^2$$
$$= \sum_{j=0}^{\infty} j^2 P_j - \mu^2 = \sum_{j=1}^{\infty} j^2 P_j - \mu^2$$

Goal: find $E[X_n]$ and $Var[X_n]$

Note $X_n = \sum_{i=1}^{X_{n-1}} Z_i$. Now, condition on X_{n-1}

$$E[X_n] = E\big[E[X_n \mid X_{n-1}]\big]$$

Recall: this is the Law of Iterated Expectations

$$= E\Big[E\Big[\sum_{i=1}^{X_{n-1}} Z_i \,\Big|\, X_{n-1}\Big]\Big]$$

$$= E[\mu \cdot X_{n-1}] = \mu E[X_{n-1}]$$

So $E[X_1] = \mu E[X_0] = \mu X_0$

6

$$E[X_2] = \mu E[X_1] = \mu \cdot \mu X_0 = \mu^2 X_0$$

$$\vdots$$

$$E[X_n] = \mu^n X_0$$

$$V[X_n] = \underbrace{E\big[V[X_n \mid X_{n-1}]\big]}_{1} + \underbrace{V\big[E[X_n \mid X_{n-1}]\big]}_{2}$$

2 : $V[\mu X_{n-1}] = \mu^2 V[X_{n-1}]$

$$① : E\left[V\left[\sum_{i=1}^{X_{n-1}} Z_i \,\middle|\, X_{n-1}\right]\right]$$

$$= E\left[\sum_{i=1}^{X_{n-1}} V[Z_i]\right] = E[\sigma^2 X_{n-1}]$$

$$= \sigma^2 E[X_{n-1}] = \sigma^2 \mu^{n-1} X_0$$

$$\therefore V[X_n] = \sigma^2 \mu^{n-1} X_0 + \mu^2 V[X_{n-1}]$$

$$= \sigma^2 \mu^{n-1} X_0 + \mu^2\left[\sigma^2 \mu^{n-2} X_0 + \mu^2 V[X_{n-2}]\right]$$

$$= \sigma^2 (\mu^{n-1} + \mu^{n}) X_0 + \mu^4 V[X_{n-2}]$$

$$= \sigma^2 (\mu^{n-1} + \mu^{n}) X_0 + \mu^4\left[\sigma^2 \mu^{n-3} X_0 + \mu^2 V[X_{n-3}]\right]$$

$$= \sigma^2 (\mu^{n-1} + \mu^{n} + \mu^{n+1}) X_0 + \mu^6 V[X_{n-3}]$$

$$\vdots$$

$$= \sigma^2 (\underbrace{\mu^{n-1} + \mu^{n} + \mu^{n+1} + \dots + \mu^{2n-2}}_{n \text{ terms}}) X_0 + \mu^{2n} \underbrace{V[X_0]}_{0}$$

$$\therefore V[X_n] = \sigma^2 (\mu^{n-1})(1 + \mu + \dots + \mu^{n-1}) X_0$$

$$= \sigma^2 \mu^{n-1} \frac{1-\mu^n}{1-\mu} X_0 \quad \text{if } \mu \neq 1$$

(9)

$= n\sigma^2 X_0$ if $\mu = 1$

Let π_0 = prob that the population dies out, given that $X_0 = 1$

That is, $\pi_0 = \lim_{n\to\infty} P[X_n = 0 \mid X_0 = 1]$

Case 1: $\mu < 1$

$E[X_n] = \mu^n$ (since $X_0 = 1$)

(10)

So $\mu^n = E[X_n] = \sum_{j=0}^{\infty} j\, P[X_n = j]$

$= \sum_{j=1}^{\infty} j\, P[X_n = j]$

$\geq \sum_{j=1}^{\infty} 1 \cdot P[X_n = j] = P[X_n \geq 1]$

$= 1 - P[X_n = 0]$

$\mu^n \geq 1 - P[X_n = 0]$

So $\underbrace{\lim_{n\to\infty} \mu^n}_{0} \geq 1 - \lim_{n\to\infty} P[X_n = 0]$ $\therefore \lim_{n\to\infty} P[X_n = 0] = 1$

$\therefore \pi_0 = 1$

Case 2: $\mu = 1$

It can be shown that, again, $\pi_0 = 1$

Case 3: $\mu > 1$

$$\pi_0 = P[\text{population dies out}]$$
$$= \sum_{j=0}^{\infty} P[\text{population dies out} \mid X_1 = j]\, P_j$$
$$\pi_0 = \sum_{j=0}^{\infty} \pi_0^{\,j} P_j$$

by independence

(12)

Example 1: $P_0 = .5$, $P_1 = .25$, $P_2 = .25$, $X_0 = 1$

$$\mu = \sum j P_j = 0(.5) + 1(.25) + 2(.25)$$
$$= .75 \qquad \therefore \pi_0 = 1$$

$$E[X_n] = \mu^n X_0 = (.75)^n$$

$$E[Z_i^2] = \sum j^2 P_j = 0^2(.5) + 1^2(.25) + 2^2(.25)$$
$$= 1.25$$

$$\sigma^2 = 1.25 - \mu^2 = \frac{11}{16} = .6875$$

$$V[X_n] = \sigma^2 \mu^{n-1} \frac{1-\mu^n}{1-\mu} X_0$$

$$= \frac{11}{16}(.75)^{n-1} \frac{1-.75^n}{.25}$$

Example 2 : $P_0 = .25,\ P_1 = .25,\ P_2 = .5$

$$\mu = 0(.25) + 1(.25) + 2(.5) = 1.25$$

$$\sigma^2 = \frac{11}{16} = .6875$$

HW #1 follows this page, due Thur 4/13

Stat 568
HW1
Due 4/13/17

63. For the Markov chain with states 1, 2, 3, 4 whose transition probability matrix **P** is as specified below find f_{i3} and s_{i3} for $i = 1, 2, 3$.

$$\mathbf{P} = \begin{bmatrix} 0.4 & 0.2 & 0.1 & 0.3 \\ 0.1 & 0.5 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.2 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

64. Consider a branching process having $\mu < 1$. Show that if $X_0 = 1$, then the expected number of individuals that ever exist in this population is given by $1/(1-\mu)$. What if $X_0 = n$?

65. In a branching process having $X_0 = 1$ and $\mu > 1$, prove that π_0 is the *smallest* positive number satisfying Equation (4.20).

Hint: Let π be any solution of $\pi = \sum_{j=0}^{\infty} \pi^j P_j$. Show by mathematical induction that $\pi \geqslant P\{X_n = 0\}$ for all n, and let $n \to \infty$. In using the induction argue that

$$P\{X_n = 0\} = \sum_{j=0}^{\infty} (P\{X_{n-1} = 0\})^j P_j$$

66. For a branching process, calculate π_0 when
(a) $P_0 = \frac{1}{4}, P_2 = \frac{3}{4}$.
(b) $P_0 = \frac{1}{4}, P_1 = \frac{1}{2}, P_2 = \frac{1}{4}$.
(c) $P_0 = \frac{1}{6}, P_1 = \frac{1}{2}, P_3 = \frac{1}{3}$.