Stat 567

1-31-17

Last time, we showed:

If $X \sim \text{Norm}(\mu, \sigma)$ and $Y = aX + b$, then ①

$$Y \sim N(a\mu + b, a\sigma).$$

So if $Z = \frac{X-\mu}{\sigma}$, then $Z \sim N(0, 1)$

(standard normal)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

Show that the normal density integrates to 1 ②

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $z = \frac{x-\mu}{\sigma}$

$dz = \frac{1}{\sigma} dx$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$= 2 \underbrace{\int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz}_{A}$$

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$$A^2 = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2} dw$$

$$= \int_0^\infty \int_0^\infty \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2+w^2)} dz\, dw$$

let $z = r\cos\theta$

$w = r\sin\theta$

$dz\,dw = r\,dr\,d\theta$

$$= \int_0^{\frac{\pi}{2}} \int_0^\infty \frac{1}{2\pi} e^{-\frac{1}{2}r^2} r\,dr\,d\theta$$

let $u = \frac{1}{2}r^2$

$du = r\,dr$

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$$= \int_0^{\frac{\pi}{2}} \left[\int_0^\infty \frac{1}{2\pi} e^{-u} du \right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2\pi} \left(-e^{-u}\right)\Big|_0^\infty d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2\pi} (0+1)\, d\theta = \frac{1}{2\pi}\, \theta \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2\pi} \frac{\pi}{2} = \frac{1}{4} = A^2$$

$\therefore A = \frac{1}{2}$

$\therefore 2A = 1$ ✓

Here's a consequence: Recall (5)

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} \, dx$$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-x} x^{-\frac{1}{2}} \, dx$$

Let $x = \frac{1}{2} z^2$

$dx = z \, dz$

$$= \int_0^\infty e^{-\frac{1}{2}z^2} \left(\tfrac{1}{2} z^2\right)^{-\frac{1}{2}} z \, dz$$

$$= \sqrt{2} \int_0^\infty e^{-\frac{1}{2}z^2} \, dz = \sqrt{2}\sqrt{2\pi} \underbrace{\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \, dz}_{A = \frac{1}{2}}$$

$$= \sqrt{\pi}$$

(6)

Defn. For a random variable X,

$$\mu = \underbrace{E[X]}_{\text{"expected value of } X\text{"}} = \begin{cases} \sum_{\text{all } x} x \, p(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x f(x) \, dx & \text{if } X \text{ is continuous} \end{cases}$$

Bernoulli: $X = \begin{cases} 0 & 1-p \\ 1 & p \end{cases}$

$$\mu = E[X] = 0(1-p) + 1(p) = p$$

Binomial: $X = 0, 1, 2, \ldots, n$

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$E[X] = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} x \frac{n!}{x!\,(n-x)!} p^x q^{n-x} \qquad (q = 1-p)$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!\,(n-x)!} p^x q^{n-x} \qquad \text{Let } y = x-1$$

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$$= \sum_{y=0}^{n-1} \frac{n!}{y!\,(n-y-1)!} p^{y+1} q^{n-y-1}$$

Let $m = n-1$

$$= \sum_{y=0}^{m} \frac{(m+1)!}{y!\,(m-y)!} p^{y+1} q^{m-y}$$

$$= (m+1)\, p \underbrace{\sum_{y=0}^{m} \frac{m!}{y!\,(m-y)!} p^y q^{m-y}}_{1} = np$$

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Geometric $X = 1, 2, \ldots \qquad p(x) = (1-p)^{x-1} p$

$$E[X] = \sum_{x=1}^{\infty} x (1-p)^{x-1} p$$

$$= p \sum_{x=1}^{\infty} x q^{x-1} = p \sum_{x=1}^{\infty} \frac{d}{dq} q^x$$

$$= p \frac{d}{dq}\left(\sum_{x=1}^{\infty} q^x\right) = p \frac{d}{dq}\left[q + q^2 + q^3 + \ldots\right]$$

$$= p \frac{d}{dq}\left[q \underbrace{(1 + q + q^2 + \ldots)}_{\frac{1}{1-q}}\right]$$

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$$= p \frac{d}{dq}\left(\frac{q}{1-q}\right) = p \frac{(1-q)1 - q(-1)}{(1-q)^2}$$

$$= p \frac{1}{p^2} = \frac{1}{p}$$

Poisson $X = 0, 1, 2, \ldots \qquad p(x) = \frac{e^{-\lambda} \lambda^x}{x!}$

$$E[X] = \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x!} = \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-1)!}$$

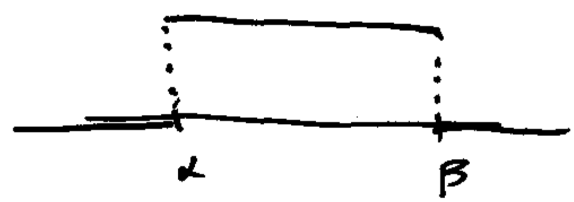
Let $y = x-1$

$$= \sum_{y=0}^{\infty} \frac{e^{-\lambda} \lambda^{y+1}}{y!}$$

$$= \lambda \underbrace{\sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!}}_{1} = \lambda$$ (11)

Uniform $f(x) = \frac{1}{\beta - \alpha}, \quad \alpha < x < \beta$

$$E[X] = \int_\alpha^\beta x \frac{1}{\beta-\alpha} dx$$



$$= \frac{1}{\beta-\alpha} \frac{x^2}{2}\Big|_\alpha^\beta = \frac{1}{\beta-\alpha}\frac{1}{2}(\beta^2-\alpha^2) = \frac{1}{\beta-\alpha}\frac{1}{2}(\beta+\alpha)(\beta-\alpha)$$

$$= \frac{\beta+\alpha}{2}$$ (12)

Exponential $f(x) = \lambda e^{-\lambda x} \quad x > 0$

$$E[X] = \int_0^\infty x \lambda e^{-\lambda x} dx$$

let $u = x \quad dv = \lambda e^{-\lambda x} dx$

$du = dx \quad v = -e^{-\lambda x}$

$$= \cancel{-xe^{-\lambda x}\Big|_0^\infty} + \int_0^\infty e^{-\lambda x} dx$$

(13)

$$= -\frac{1}{\lambda} e^{-\lambda x} \Big|_0^\infty$$

$$= 0 - \left(-\frac{1}{\lambda}\right) = \frac{1}{\lambda}$$

<u>Gamma</u> (wait until the next section)

<u>Normal</u> $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty$

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

(14)

Let $z = \frac{x-\mu}{\sigma}$

$dz = \frac{1}{\sigma} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z + \mu) e^{-\frac{1}{2}z^2} dz$$

$$= \underbrace{\frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-\frac{1}{2}z^2} dz}_{0} + \mu \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz}_{1}$$

$$= \mu$$

(15)

<u>Defn</u>: $$E[g(X)] = \begin{cases} \sum_{\text{all } x} g(x)\,p(x) & \text{discrete} \\ \int_{-\infty}^{\infty} g(x)\, f(x)\,dx & \text{contin.} \end{cases}$$

<u>Note</u>: $$E[aX+b] = \int_{-\infty}^{\infty} (ax+b)\, f(x)\,dx$$

$$= a \underbrace{\int_{-\infty}^{\infty} x f(x)\,dx}_{E[X]} + b \underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{1}$$

$$= a\,E[X] + b$$

The expectation is a <u>linear operator</u>

(16)

<u>Defn</u>. $E[X^n]$ is called the <u>n^{th} moment</u> of X.

<u>Defn</u>. $\mathrm{Var}[X] = E\left[(X-\mu)^2\right]$

<u>Note</u>: $$E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2]$$

$$= E[X^2] - 2\mu \underbrace{E[X]}_{\mu} + \mu^2$$

$$= E[X^2] - \mu^2 = E[X^2] - (E[X])^2$$