ANCOVA, Continued

$N = an$ Stat 506

$i = 1, \ldots, a$ 4-28-15

$j = 1, \ldots, n$

$$y_{ij} = \mu + \tau_i + \beta(x_{ij} - \bar{x}_{..}) + \varepsilon_{ij}$$

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$$\begin{bmatrix} y_{11} \\ \vdots \\ y_{1n} \\ \hline y_{21} \\ \vdots \\ y_{2n} \\ \hline \vdots \\ \hline y_{a1} \\ \vdots \\ y_{an} \end{bmatrix}_{N \times 1} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & x_{11} - \bar{x}_{..} \\ \vdots & \vdots & & \vdots & & \vdots \\ 1 & 1 & 0 & \cdots & 0 & x_{1n} - \bar{x}_{..} \\ \hline 1 & 0 & 1 & 0 \cdots & 0 & x_{21} - \bar{x}_{..} \\ \vdots & \vdots & \vdots & & & \vdots \\ 1 & 0 & 1 & 0 \cdots & 0 & x_{2n} - \bar{x}_{..} \\ \hline \vdots & & \vdots & & & \vdots \\ \hline 1 & -1 & \cdots & & -1 & x_{a1} - \bar{x}_{..} \\ \vdots & \vdots & & & \vdots & \vdots \\ 1 & -1 & & & -1 & x_{an} - \bar{x}_{..} \end{bmatrix}_{N \times (a+1)} \begin{bmatrix} \mu \\ \tau_1 \\ \vdots \\ \tau_{a-1} \\ \beta \end{bmatrix}_{(a+1) \times 1} + \vec{\varepsilon}_{N \times 1}$$

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Multiple Comparisons in ANCOVA

Suppose that $H_0: \tau_i \equiv 0$ is rejected.

Need to know the properties of $\hat{\tau}_i - \hat{\tau}_j$

(or $\hat{\mu}_i - \hat{\mu}_j$, where $\hat{\mu}_i = \hat{\mu} + \hat{\tau}_i$)

$$y_{ij} = \mu + \tau_i + \beta(x_{ij} - \bar{x}_{..}) + \varepsilon_{ij}$$

$$\bar{y}_{i.} = \mu + \tau_i + \beta(\bar{x}_{i.} - \bar{x}_{..}) + \bar{\varepsilon}_{i.}$$

$$\bar{y}_{..} = \mu + \bar{\varepsilon}_{..}$$

③

$$\hat{\beta} = \frac{E_{xy}}{E_{xx}} = \frac{1}{E_{xx}}\left[\sum_i\sum_j (x_{ij}-\bar{x}_{i\cdot})(y_{ij}-\bar{y}_{i\cdot})\right]$$

$$= \frac{1}{E_{xx}}\left[\sum_i\sum_j (x_{ij}-\bar{x}_{i\cdot})\left[\beta(x_{ij}-\bar{x}_{i\cdot})+\varepsilon_{ij}-\bar{\varepsilon}_{i\cdot}\right]\right]$$

$$= \frac{1}{E_{xx}}\left[\beta E_{xx} + \overbrace{\sum_i\sum_j (x_{ij}-\bar{x}_{i\cdot})(\varepsilon_{ij}-\bar{\varepsilon}_{i\cdot})}^{\sum_i\sum_j (x_{ij}-\bar{x}_{i\cdot})\varepsilon_{ij} \,-\, \sum_i \bar{\varepsilon}_{i\cdot}\left[\cancel{\sum_j (x_{ij}-\bar{x}_{i\cdot})}\right]^{0}}\right]$$

$$E(\hat{\beta}) = \beta + 0 \qquad V(\hat{\beta}) = \frac{1}{E_{xx}^2}\left[\sum_i\sum_j (x_{ij}-\bar{x}_{i\cdot})^2\sigma^2\right]$$

$$= \sigma^2 / E_{xx}$$

④

$$\hat{\mu}_i = \hat{\mu} + \hat{\tau}_i = \bar{y}_{\cdot\cdot} + (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot}) - \hat{\beta}(\bar{x}_{i\cdot} - \bar{x}_{\cdot\cdot})$$

$$= \bar{y}_{i\cdot} - \hat{\beta}(\bar{x}_{i\cdot} - \bar{x}_{\cdot\cdot})$$

$$= \overbrace{\mu + \tau_i + \beta(\bar{x}_{i\cdot} - \bar{x}_{\cdot\cdot}) + \bar{\varepsilon}_{i\cdot}} - \hat{\beta}(\bar{x}_{i\cdot} - \bar{x}_{\cdot\cdot})$$

$$= \mu + \tau_i + (\beta - \hat{\beta})(\bar{x}_{i\cdot} - \bar{x}_{\cdot\cdot}) + \bar{\varepsilon}_{i\cdot}$$

$$E(\hat{\mu}_i) = \mu + \tau_i = \mu_i$$

$$V(\hat{\mu}_i) = (\bar{x}_{i\cdot} - \bar{x}_{\cdot\cdot})^2 V(\hat{\beta}) + V(\bar{\varepsilon}_{i\cdot})$$

$$+ 2\,\mathrm{Cov}\left[(\beta - \hat{\beta})(\bar{x}_{i\cdot} - \bar{x}_{\cdot\cdot}), \bar{\varepsilon}_{i\cdot}\right]$$

⑤

$$= (\bar{x}_{i\cdot} - \bar{x}_{\cdot\cdot})^2 \frac{\sigma^2}{E_{xx}} + \frac{\sigma^2}{n} - 2(\bar{x}_{i\cdot} - \bar{x}_{\cdot\cdot})\underbrace{Cov(\hat{\beta}, \bar{\varepsilon}_{i\cdot})}_{*}$$

$$* = Cov(\hat{\beta}, \bar{\varepsilon}_{i\cdot}) = Cov\left(\beta + \frac{1}{E_{xx}} \sum_i \sum_j (x_{ij} - \bar{x}_{i\cdot})\varepsilon_{ij}, \bar{\varepsilon}_{i\cdot}\right)$$

$$= \frac{1}{E_{xx}} \sum_i \sum_j (x_{ij} - \bar{x}_{i\cdot}) Cov(\varepsilon_{ij}, \bar{\varepsilon}_{i\cdot})$$

$$= \frac{1}{E_{xx}} \sum_i \sum_j (x_{ij} - \bar{x}_{i\cdot}) Cov\Big(\underbrace{\varepsilon_{ij}, \frac{1}{n}\sum_{j=1}^{n} \varepsilon_{ij}}_{\frac{1}{n}\sigma^2}\Big)$$

$$= \frac{1}{n}\sigma^2 \frac{1}{E_{xx}} \sum_i \sum_j (x_{ij} - \bar{x}_{i\cdot}) = 0$$

⑥

$$\text{So } V(\hat{\mu}_i) = \sigma^2\left[\frac{1}{n} + \frac{(\bar{x}_{i\cdot} - \bar{x}_{\cdot\cdot})^2}{E_{xx}}\right]$$

Finally, consider $\hat{\mu}_i - \hat{\mu}_j$

$$E(\hat{\mu}_i - \hat{\mu}_j) = \mu_i - \mu_j = (\mu + \tau_i) - (\mu + \tau_j)$$
$$= \tau_i - \tau_j$$

$$V(\hat{\mu}_i - \hat{\mu}_j) = V(\hat{\mu}_i) + V(\hat{\mu}_j) - 2\underbrace{Cov(\hat{\mu}_i, \hat{\mu}_j)}_{**}$$

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$$** = Cov\Big(\mu + \tau_i + (\beta - \hat\beta)(\bar x_{i\cdot} - \bar x_{\cdot\cdot}) + \bar\varepsilon_{i\cdot},$$

$$\mu + \tau_j + (\beta - \hat\beta)(\bar x_{j\cdot} - \bar x_{\cdot\cdot}) + \bar\varepsilon_{j\cdot}\Big)$$

$$= (\bar x_{i\cdot} - \bar x_{\cdot\cdot})(\bar x_{j\cdot} - \bar x_{\cdot\cdot}) Cov(\beta - \hat\beta, \beta - \hat\beta)$$

$$+ (\bar x_{i\cdot} - \bar x_{\cdot\cdot}) \cancel{Cov(\beta - \hat\beta, \bar\varepsilon_{j\cdot})}^{0} \text{, as before}$$

$$+ (\bar x_{j\cdot} - \bar x_{\cdot\cdot}) \cancel{Cov(\bar\varepsilon_{i\cdot}, \beta - \hat\beta)}^{0}$$

$$+ \cancel{Cov(\bar\varepsilon_{i\cdot}, \bar\varepsilon_{j\cdot})}\ 0$$

$$= (\bar x_{i\cdot} - \bar x_{\cdot\cdot})(\bar x_{j\cdot} - \bar x_{\cdot\cdot}) \frac{\sigma^2}{E_{xx}}$$

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So $V(\hat\mu_i - \hat\mu_j)$

$$= \sigma^2\left[\frac{1}{n} + \frac{(\bar x_{i\cdot} - \bar x_{\cdot\cdot})^2}{E_{xx}}\right] + \sigma^2\left[\frac{1}{n} + \frac{(\bar x_{j\cdot} - \bar x_{\cdot\cdot})^2}{E_{xx}}\right]$$

$$- 2(\bar x_{i\cdot} - \bar x_{\cdot\cdot})(\bar x_{j\cdot} - \bar x_{\cdot\cdot}) \frac{\sigma^2}{E_{xx}}$$

$$= \sigma^2\left[\frac{2}{n} + \frac{(\bar x_{i\cdot} - \bar x_{\cdot\cdot})^2 + (\bar x_{j\cdot} - \bar x_{\cdot\cdot})^2 - 2(\bar x_{i\cdot} - \bar x_{\cdot\cdot})(\bar x_{j\cdot} - \bar x_{\cdot\cdot})}{E_{xx}}\right]$$

$$= \sigma^2\left[\frac{2}{n} + \frac{(\bar x_{i\cdot} - \bar x_{j\cdot})^2}{E_{xx}}\right]$$

$$\therefore \quad \frac{\hat{\mu}_i - \hat{\mu}_j - (\tau_i - \tau_j)}{\sqrt{\sigma^2\left[\frac{2}{n} + \frac{(\bar{x}_{i\cdot} - \bar{x}_{j\cdot})^2}{E_{xx}}\right]}} \approx N(0,1)$$

$$\frac{\hat{\mu}_i - \hat{\mu}_j - (\tau_i - \tau_j)}{\sqrt{MSE\left[\frac{2}{n} + \frac{(\bar{x}_{i\cdot} - \bar{x}_{j\cdot})^2}{E_{xx}}\right]}} \approx t_{df_E}$$