Satterthwait's Approximate F-test

Stat 566
4-9-15

(1)

Construct 2 new MS terms

$$MS' = MS_t + \dots + MS_u$$
$$MS'' = MS_v + \dots + MS_w$$

no terms in common

Such $E(MS') - E(MS'') =$ some constant times the desired variance component

From Tuesday's example, let $MS' = MSA + MS_{ABC}$
$MS'' = MS_{AB} + MS_{AC}$

(2)

Then $E(MS') - E(MS'') = bcn \dfrac{\sum \tau_i^2}{a-1}$

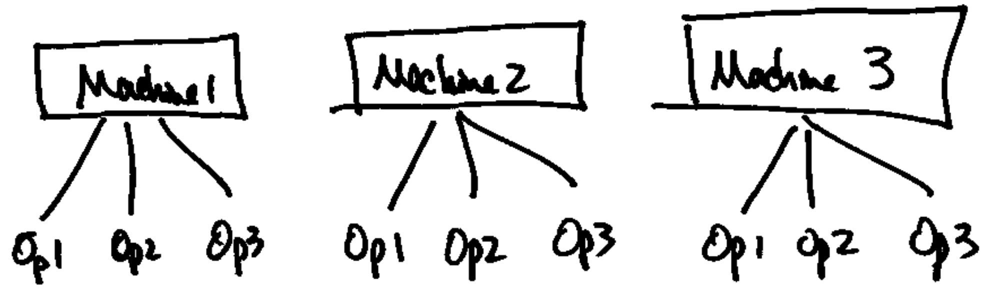
Satterthwaite proved that $\dfrac{MS'}{MS''}$ has an approximate F distribution with df p, q,

where $$p = \frac{(MS')^2}{\frac{MS_t^2}{df_t} + \dots + \frac{MS_u^2}{df_u}}, \quad q = \frac{(MS'')^2}{\frac{MS_v^2}{df_v} + \dots + \frac{MS_w^2}{df_w}}$$

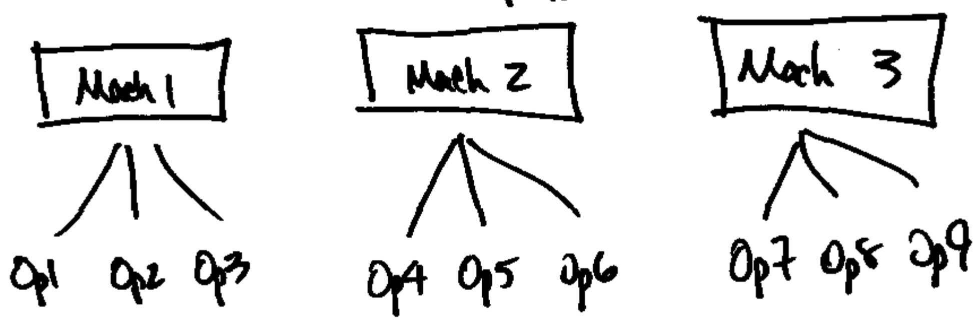
3

Nested designs

Machine + operator are crossed



operator nested within machine



4

$$y_{ijk} = \mu + \underset{\substack{\uparrow \\ \text{machine}}}{\tau_i} + \underset{\substack{\uparrow \\ \text{operator nested within machine}}}{\beta_{j(i)}} + \varepsilon_{(ij)k}$$

Parameter estimates:

Assume both factors are fixed

$$\sum_{i=1}^{a} \tau_i = 0, \quad \sum_{j=1}^{b} \beta_{j(i)} = 0 \quad \forall i$$

$$SSE = \sum_i \sum_j \sum_k \left(y_{ijk} - (\mu + \tau_i + \beta_{j(i)}) \right)^2$$

$$\frac{\partial SSE}{\partial \mu} = \sum\sum\sum \cancel{2}\left(y_{ijk} - \mu - \tau_i - \beta_{j(i)}\right)(\cancel{-1}) \overset{\text{set}}{=} 0$$ 5

$$y_{\cdots} - N\mu - 0 - 0 = 0$$

$$\hat{\mu} = \frac{y_{\cdots}}{N} = \bar{y}_{\cdots}$$

$$\frac{\partial SSE}{\partial \tau_i} = \sum_j \sum_k \cancel{2}\left(y_{ijk} - \mu - \tau_i - \beta_{j(i)}\right)(\cancel{-1}) \overset{\text{set}}{=} 0$$

$$y_{i\cdot\cdot} - bn\mu - bn\tau_i - 0 = 0$$

$$\hat{\tau}_i = \frac{y_{i\cdot\cdot} - bn\bar{y}_{\cdots}}{bn}$$

$$= \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots}$$

$$\frac{\partial SSE}{\partial \beta_{j(i)}} = \sum_k \cancel{2}\left(y_{ijk} - \mu - \tau_i - \beta_{j(i)}\right)(\cancel{-1}) \overset{\text{set}}{=} 0$$ 6

$$= y_{ij\cdot} - n\mu - n\tau_i - n\beta_{j(i)} = 0$$

$$\hat{\beta}_{j(i)} = \frac{y_{ij\cdot} - n\bar{y}_{\cdots} - n(\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots})}{n}$$

$$= \bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot}$$

Compare to crossed model: $\hat{\mu} = \bar{y}_{\cdots}$, $\hat{\tau}_i = \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots}$

$\hat{\beta}_j = \bar{y}_{\cdot j\cdot} - \bar{y}_{\cdots}$, $\widehat{\tau\beta}_{ij} = \bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdots}$

SS & df, using formulas from Tuesday

$df_A = a-1$

$df_{B(A)} = (b-1)a = ab - a$

$$SS_A = \sum_i \frac{y_{i\cdot\cdot}^2}{bn} - \frac{y_{\cdot\cdot\cdot}^2}{abn}$$

$$SS_{B(A)} = \sum_i \sum_j \frac{y_{ij\cdot}^2}{n} - \sum_i \frac{y_{i\cdot\cdot}^2}{bn}$$

Ex: A: machine fixed

B: operator random, nested within machine

	F a i	R b j	R n k	EMS
τ_i	0	b	n	$bn \frac{\sum \tau_i^2}{a-1} + n\sigma_\beta^2 + \sigma^2$
$\beta_{j(i)}$	1	1	n	$n\sigma_\beta^2 + \sigma^2$
$\varepsilon_{(ij)k}$	1	1	1	σ^2

F test for A: $MSA / MSB(A)$

F test for B(A): $MSB(A) / MSE$

(9)

Example: y measures hardness of the metal

Test 2 different alloy formulations

Test 3 different heats for each alloy

Select 2 ingots of each alloy

Run 2 tests on each ingot.

A: alloy $a=2$ fixed
B: heat $b=3$ fixed, nested within A
C: ingot $c=2$ random, nested within alloy + heat
$n=2$

(10)

$$y_{ijkl} = \mu + \tau_i + \beta_{j(i)} + \gamma_{k(ij)} + \varepsilon_{(ijk)l}$$

	F 2 i	F 3 j	R 2 k	R 2 l	EMS
τ_i	0	3	2	2	$12\frac{\sum \tau_i^2}{1} + 2\sigma_\gamma^2 + \sigma^2$
$\beta_{j(i)}$	1	0	2	2	$4\frac{\sum\sum \beta_{j(i)}^2}{4} + 2\sigma_\gamma^2 + \sigma^2$
$\gamma_{k(ij)}$	1	1	1	2	$2\sigma_\gamma^2 + \sigma^2$
$\varepsilon_{(ijk)l}$	1	1	1	1	σ^2

A: $MSA/MSC(AB)$

B(A): $MS_{B(A)}/MSC(AB)$

C(AB): $MS_{C(AB)}/MSE$

HW #2 13.17
due 4/16

10 parts, three operators, and three replicates. The data are shown in Table P13.2.

■ **TABLE P13.2**
Power Module Thermal Test Equipment Data for Problem 13.2

	Inspector 1			Inspector 2			Inspector 3		
	Test			Test			Test		
Part No.	1	2	3	1	2	3	1	2	3
1	37	38	37	41	41	40	41	42	41
2	42	41	43	42	42	42	43	42	43
3	30	31	31	31	31	31	29	30	28
4	42	43	42	43	43	43	42	42	42
5	28	30	29	29	30	29	31	29	29
6	42	42	43	45	45	45	44	46	45
7	25	26	27	28	28	30	29	27	27
8	40	40	40	43	42	42	43	43	41
9	25	25	25	27	29	28	26	26	26
10	35	34	34	35	35	34	35	34	35

(**a**) Analyze the data from this experiment, assuming that both parts and operators are random effects.

(**b**) Estimate the variance components using the analysis of variance method.

(**c**) Estimate the variance components using the REML method. Use the confidence intervals on the variance components to assist in drawing conclusions.

13.3. Reconsider the data in Problem 5.8. Suppose that both factors, machines and operators, are chosen at random.

(**a**) Analyze the data from this experiment.

(**b**) Find point estimates of the variance components using the analysis of variance method.

13.4. Reconsider the data in Problem 5.15. Suppose that both factors are random.

(**a**) Analyze the data from this experiment.

(**b**) Estimate the variance components using the ANOVA method.

13.5. Suppose that in Problem 5.13 the furnace positions were randomly selected, resulting in a mixed model experiment. Reanalyze the data from this experiment under this new assumption. Estimate the appropriate model components using the ANOVA method.

13.6. Reanalyze the measurement systems experiment in Problem 13.1, assuming that operators are a fixed factor. Estimate the appropriate model components using the ANOVA method.

13.7. Reanalyze the measurement system experiment in Problem 13.2, assuming that operators are a fixed factor. Estimate the appropriate model components using the ANOVA method.

13.8. In Problem 5.8, suppose that there are only four machines of interest, but the operators were selected at random.

(**a**) What type of model is appropriate?

(**b**) Perform the analysis and estimate the model components using the ANOVA method.

13.9 Rework Problem 13.5 using the REML method.

13.10 Rework Problem 13.6 using the REML method.

13.11 Rework Problem 13.7 using the REML method.

13.12 Rework Problem 13.8 using the REML method.

13.13. By application of the expectation operator, develop the expected mean squares for the two-factor factorial, mixed model. Use the restricted model assumptions. Check your results with the expected mean squares given in Equation 13.9 to see that they agree.

13.14. Consider the three-factor factorial design in Example 13.5. Propose appropriate test statistics for all main effects and interactions. Repeat for the case where A and B are fixed and C is random.

13.15. Consider the experiment in Example 13.6. Analyze the data for the case where A, B, and C are random.

13.16. Derive the expected mean squares shown in Table 13.11.

13.17. Consider a four-factor factorial experiment where factor A is at a levels, factor B is at b levels, factor C is at c levels, factor D is at d levels, and there are n replicates. Write down the sums of squares, the degrees of freedom, and the expected mean squares for the following cases. Assume the restricted model for all mixed models. You may use a computer package such as Minitab.

(**a**) A, B, C, and D are fixed factors.

(**b**) A, B, C, and D are random factors.

(**c**) A is fixed and B, C, and D are random.

(**d**) A and B are fixed and C and D are random.

(**e**) A, B, and C are fixed and D is random.

Do exact tests exist for all effects? If not, propose test statistics for those effects that cannot be directly tested.

13.18. Reconsider cases (c), (d), and (e) of Problem 13.17. Obtain the expected mean squares assuming the unrestricted model. You may use a computer package such as Minitab. Compare your results with those for the restricted model.

13.19. In Problem 5.19, assume that the three operators were selected at random. Analyze the data under these conditions and draw conclusions. Estimate the variance components.

13.20. Consider the three-factor factorial model

$$y_{ijk} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\beta\gamma)_{jk} + \epsilon_{ijk} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, c \end{cases}$$