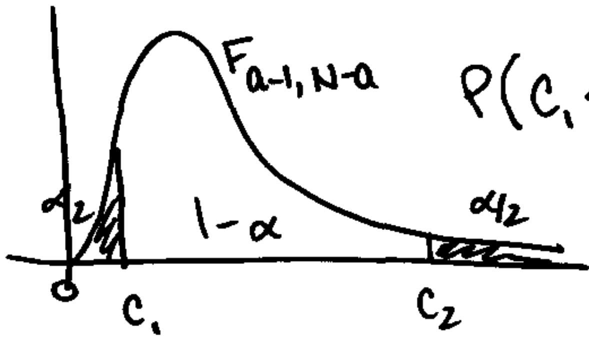
Stat 566
4-2-15

Continued from last time:

①

We wanted a confidence interval for $\dfrac{\sigma_\tau^2}{\sigma_\tau^2+\sigma^2}$.

We had this: $\dfrac{MS_{TRT}}{MSE}\,\dfrac{\sigma^2}{\sigma^2+n\sigma_\tau^2} \sim F_{a-1,N-a}$



$$P\left(c_1 < \frac{MS_{TRT}}{MSE}\,\frac{\sigma^2}{\sigma^2+n\sigma_\tau^2} < c_2\right) = 1-\alpha$$

②

$$\underbrace{\frac{c_1\,MSE}{MS_{TRT}}}_{b_1} < \frac{\sigma^2}{\sigma^2+n\sigma_\tau^2} < \underbrace{\frac{c_2\,MSE}{MS_{TRT}}}_{b_2}$$ with conf. $1-\alpha$

$$\frac{1}{b_1} > \frac{\sigma^2+n\sigma_\tau^2}{\sigma^2} > \frac{1}{b_2}$$

$$\underbrace{\frac{\frac{1}{b_1}-1}{n}}_{d_1} > \frac{\sigma_\tau^2}{\sigma^2} > \underbrace{\frac{\frac{1}{b_2}-1}{n}}_{d_2}$$

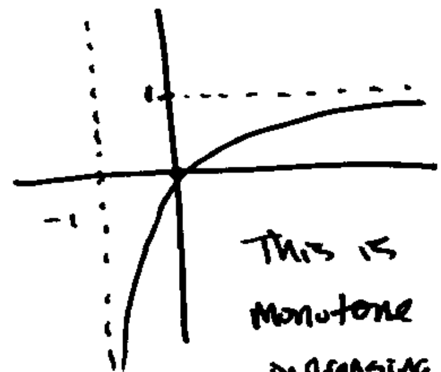
$$d_1 > \frac{\sigma_\tau^2}{\sigma^2} > d_2 \quad \text{with conf. } 1-\alpha$$ ③

$$f(d_1) > f\left(\frac{\sigma_\tau^2}{\sigma^2}\right) > f(d_2)$$

$$f\left(\frac{\sigma_\tau^2}{\sigma^2}\right) = \frac{\sigma_\tau^2/\sigma^2}{1+\frac{\sigma_\tau^2}{\sigma^2}}$$

$$\boxed{f(d_1) > \frac{\sigma_\tau^2}{\sigma^2+\sigma_\tau^2} > f(d_2)}$$

$$f(x) = \frac{x}{1+x}$$



This is monotone increasing

2-way ANOVA, both effects are random ④

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk} \qquad \begin{array}{l} i=1,\dots,a \\ j=1,\dots,b \\ k=1,\dots,n \end{array}$$

$$E(\tau_i) \equiv 0 \equiv E(\beta_j) \equiv E[(\tau\beta)_{ij}] \equiv E(\varepsilon_{ijk}) \qquad N = abn$$

$$V(\tau_i) \equiv \sigma_\tau^2,\ V(\beta_j) \equiv \sigma_\beta^2,\ V[(\tau\beta)_{ij}] \equiv \sigma_{\tau\beta}^2,\ V(\varepsilon_{ijk}) \equiv \sigma^2,$$

all r.v.s are independent

Similar to last time, $MSA = \dfrac{nb\sum_{i=1}^{a}(\bar{y}_{i..}-\bar{y}_{...})^2}{a-1}$

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Find its expected value.

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}$$

$$\bar{y}_{i..} = \mu + \tau_i + \bar{\beta}_. + (\overline{\tau\beta})_{i.} + \bar{\varepsilon}_{i..}$$

$$\bar{y}_{...} = \mu + \bar{\tau}_. + \bar{\beta}_. + (\overline{\tau\beta})_{..} + \bar{\varepsilon}_{...}$$

$$\bar{y}_{i..} - \bar{y}_{...} = (\tau_i - \bar{\tau}_.) + \left[(\overline{\tau\beta})_{i.} - (\overline{\tau\beta})_{..}\right] + (\bar{\varepsilon}_{i..} - \bar{\varepsilon}_{...})$$

$$E(\bar{y}_{i..} - \bar{y}_{..})^2 = E(\tau_i - \bar{\tau}_.)^2 + E\left[(\overline{\tau\beta})_{i.} - (\overline{\tau\beta})_{..}\right]^2 + E(\bar{\varepsilon}_{i..} - \bar{\varepsilon}_{..})^2 + 0 + 0 + 0$$

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$$E(MS_A) = nb\left[E\left(\frac{\Sigma(\tau_i - \bar{\tau}_.)^2}{a-1}\right) + E\left(\frac{\Sigma\left[(\overline{\tau\beta})_{i.} - (\overline{\tau\beta})_{..}\right]^2}{a-1}\right) + E\left(\frac{\Sigma(\bar{\varepsilon}_{i..} - \bar{\varepsilon}_{...})^2}{a-1}\right)\right]$$

$$= nb\left[\sigma_\tau^2 + \frac{\sigma_{\tau\beta}^2}{b} + \frac{\sigma^2}{bn}\right]$$

$$= nb\sigma_\tau^2 + n\sigma_{\tau\beta}^2 + \sigma^2$$

Similarly, $E(MS_B) = na\sigma_\beta^2 + n\sigma_{\tau\beta}^2 + \sigma^2$

Also $E(MS_{AB}) = n\sigma_{\tau\beta}^2 + \sigma^2$

To test $H_0: \sigma_\tau^2 = 0$, use $F = \dfrac{MSA}{MSAB}$ (7)

$H_0: \sigma_\beta^2 = 0$, use $F = \dfrac{MSB}{MSAB}$

$H_0: \sigma_{\tau\beta}^2 = 0$, use $F = \dfrac{MSAB}{MSE}$

The mixed model: A fixed, B random

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk} \qquad \begin{array}{l} i = 1, \dots, a \\ j = 1, \dots, b \\ k = 1, \dots, n \\ N = abn \end{array}$$

$$\sum_{i=1}^{a} \tau_i = 0, \; E(\beta_j) \equiv 0, \; V(\beta_j) \equiv \sigma_\beta^2$$

$$E[(\tau\beta)_{ij}] \equiv 0, \; V[(\tau\beta)_{ij}] \equiv \sigma_{\tau\beta}^2 \quad E(\varepsilon_{ijk}) \equiv 0, \; V(\varepsilon_{ijk}) \equiv \sigma^2$$

(8)

$$\sum_{i=1}^{a} (\tau\beta)_{ij} = 0 \quad \forall j$$ is an additional assumption

With this assumption, we have the <u>restricted</u> version of the mixed model.

$$E(MSA) = nb\left[\sum_i \frac{\tau_i^2}{a-1} + \frac{\sigma_{\tau\beta}^2}{b} + \frac{\sigma^2}{nb} \right]$$

$$= nb \frac{\sum \tau_i^2}{a-1} + n\sigma_{\tau\beta}^2 + \sigma^2$$

$$E(MSB) = na\left[E\left(\frac{\sum(\beta_j - \bar{\beta}_.)^2}{b-1} \right) + E\left(\frac{\sum \left[\overline{(\tau\beta)}_{.j} - \overline{(\tau\beta)}_{..} \right]^2}{b-1} \right) + E\left(\frac{\sum(\bar{\varepsilon}_{.j.} - \bar{\varepsilon}_{...})^2}{b-1} \right) \right]$$

(9)

$$= na\left[\sigma^2_\beta + 0 + \frac{\sigma^2}{na} \right]$$

$$= na\sigma^2_\beta + \sigma^2$$

$E(MSAB) = n\sigma^2_{\tau\beta} + \sigma^2$, as before.

$H_0: \tau_i \equiv 0$ use $F = \frac{MSA}{MSAB}$

$H_0: \sigma^2_\beta = 0$ use $F = \frac{MSB}{MSE}$

$H_0: \sigma^2_{\tau\beta} = 0$ use $F = \frac{MSAB}{MSE}$

(10)

HW #1 due 4/9

13.1, 13.6

EXAMPLE 13.8

To illustrate the modified large-sample method, reconsider the three-factor mixed model in Example 13.6. We will find an approximate 95 percent lower confidence interval on $\sigma^2_{\tau\beta}$. Recall that the point estimate of $\sigma^2_{\tau\beta}$ is

$$\hat{\sigma}^2_{\tau\beta} = \frac{MS_{AB} - MS_{ABC}}{cn} = \frac{202.00 - 13.84}{(3)(2)} = 31.359$$

Therefore, in the notation of Equation 13.29, $c_1 = c_2 = \frac{1}{6}$, and

$$\begin{aligned}
G_1 &= 1 - \frac{1}{F_{0.05,6,\infty}} = 1 - \frac{1}{2.1} = 0.524 \\
H_2 &= \frac{1}{F_{0.95,12,\infty}} - 1 = \frac{1}{0.435} - 1 = 1.30 \\
G_{12} &= \frac{(F_{0.05,6,12} - 1)^2 - (G_1)^2F^2_{0.05,6,12} - (H_2)^2}{F_{0.05,6,12}} \\
&= \frac{(3.00 - 1)^2 - (0.524)^2(3.00)^2 - (1.3)^2}{3.00} = -0.054 \\
G^*_{1t} &= 0
\end{aligned}$$

From Equation 13.30

$$\begin{aligned}
V_L &= G_1^2c_1^2MS^2_{AB} + H_2^2c_2^2MS^2_{ABC} + G_{12}c_1c_2MS_{AB}MS_{ABC} \\
&= (0.524)^2(1/6)^2(202.00)^2 + (1.3)^2(1/6)^2(13.84)^2 + (-0.054)(1/6)(1/6)(202.00)(13.84) \\
&= 316.02
\end{aligned}$$

So an approximate 95 percent lower confidence limit on $\sigma^2_{\tau\beta}$ is

$$L = \hat{\sigma}^2_{\tau\beta} - \sqrt{V_L} = 31.36 - \sqrt{316.02} = 13.58$$

This result is consistent with the results of the exact F test for this effect.

13.8 Problems

13.1. An experiment was performed to investigate the capability of a measurement system. Ten parts were randomly selected, and two randomly selected operators measured each part three times. The tests were made in random order, and the data are shown in Table P13.1.

■ **TABLE P13.1**
Measurement Systems Data for Problem 13.1

	Operator 1 Measurements			Operator 2 Measurements		
Part No.	**1**	**2**	**3**	**1**	**2**	**3**
1	50	49	50	50	48	51
2	52	52	51	51	51	51
3	53	50	50	54	52	51
4	49	51	50	48	50	51
5	48	49	48	48	49	48
6	52	50	50	52	50	50
7	51	51	51	51	50	50
8	52	50	49	53	48	50
9	50	51	50	51	48	49
10	47	46	49	46	47	48

(**a**) Analyze the data from this experiment.

(**b**) Estimate the variance components using the ANOVA method.

13.2. An article by Hoof and Berman ("Statistical Analysis of Power Module Thermal Test Equipment Performance," *IEEE Transactions on Components, Hybrids, and Manufacturing Technology* Vol. 11, pp. 516–520, 1988) describes an experiment conducted to investigate the capability of measurements in thermal impedance (C°/w × 100) on a power module for an induction motor starter. There are

10 parts, three operators, and three replicates. The data are shown in Table P13.2.

■ **TABLE P13.2**
Power Module Thermal Test Equipment Data for Problem 13.2

	Inspector 1			Inspector 2			Inspector 3		
	Test			Test			Test		
Part No.	1	2	3	1	2	3	1	2	3
1	37	38	37	41	41	40	41	42	41
2	42	41	43	42	42	42	43	42	43
3	30	31	31	31	31	31	29	30	28
4	42	43	42	43	43	43	42	42	42
5	28	30	29	29	30	29	31	29	29
6	42	42	43	45	45	45	44	46	45
7	25	26	27	28	28	30	29	27	27
8	40	40	40	43	42	42	43	43	41
9	25	25	25	27	29	28	26	26	26
10	35	34	34	35	35	34	35	34	35

(**a**) Analyze the data from this experiment, assuming that both parts and operators are random effects.

(**b**) Estimate the variance components using the analysis of variance method.

(**c**) Estimate the variance components using the REML method. Use the confidence intervals on the variance components to assist in drawing conclusions.

13.3. Reconsider the data in Problem 5.8. Suppose that both factors, machines and operators, are chosen at random.

(**a**) Analyze the data from this experiment.

(**b**) Find point estimates of the variance components using the analysis of variance method.

13.4. Reconsider the data in Problem 5.15. Suppose that both factors are random.

(**a**) Analyze the data from this experiment.

(**b**) Estimate the variance components using the ANOVA method.

13.5. Suppose that in Problem 5.13 the furnace positions were randomly selected, resulting in a mixed model experiment. Reanalyze the data from this experiment under this new assumption. Estimate the appropriate model components using the ANOVA method.

13.6. Reanalyze the measurement systems experiment in Problem 13.1, assuming that operators are a fixed factor. Estimate the appropriate model components using the ANOVA method.

13.7. Reanalyze the measurement system experiment in Problem 13.2, assuming that operators are a fixed factor. Estimate the appropriate model components using the ANOVA method.

13.8. In Problem 5.8, suppose that there are only four machines of interest, but the operators were selected at random.

(**a**) What type of model is appropriate?

(**b**) Perform the analysis and estimate the model components using the ANOVA method.

13.9 Rework Problem 13.5 using the REML method.

13.10 Rework Problem 13.6 using the REML method.

13.11 Rework Problem 13.7 using the REML method.

13.12 Rework Problem 13.8 using the REML method.

13.13. By application of the expectation operator, develop the expected mean squares for the two-factor factorial, mixed model. Use the restricted model assumptions. Check your results with the expected mean squares given in Equation 13.9 to see that they agree.

13.14. Consider the three-factor factorial design in Example 13.5. Propose appropriate test statistics for all main effects and interactions. Repeat for the case where *A* and *B* are fixed and *C* is random.

13.15. Consider the experiment in Example 13.6. Analyze the data for the case where *A*, *B*, and *C* are random.

13.16. Derive the expected mean squares shown in Table 13.11.

13.17. Consider a four-factor factorial experiment where factor *A* is at *a* levels, factor *B* is at *b* levels, factor *C* is at *c* levels, factor *D* is at *d* levels, and there are *n* replicates. Write down the sums of squares, the degrees of freedom, and the expected mean squares for the following cases. Assume the restricted model for all mixed models. You may use a computer package such as Minitab.

(**a**) *A*, *B*, *C*, and *D* are fixed factors.

(**b**) *A*, *B*, *C*, and *D* are random factors.

(**c**) *A* is fixed and *B*, *C*, and *D* are random.

(**d**) *A* and *B* are fixed and *C* and *D* are random.

(**e**) *A*, *B*, and *C* are fixed and *D* is random.

Do exact tests exist for all effects? If not, propose test statistics for those effects that cannot be directly tested.

13.18. Reconsider cases (c), (d), and (e) of Problem 13.17. Obtain the expected mean squares assuming the unrestricted model. You may use a computer package such as Minitab. Compare your results with those for the restricted model.

13.19. In Problem 5.19, assume that the three operators were selected at random. Analyze the data under these conditions and draw conclusions. Estimate the variance components.

13.20. Consider the three-factor factorial model

$$y_{ijk} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\beta\gamma)_{jk} + \epsilon_{ijk} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, c \end{cases}$$