From last time,

$$\hat{\tau}_i = \frac{k}{\lambda a} Q_i$$

Stat 565

2-1-18

1

Substitute this and solve for $\hat{\beta}_j$

$$SS_{TOT} = \sum_i \sum_j (y_{ij} - \bar{y}_{..})^2$$

$$= \sum_i \sum_j (y_{ij} - \hat{y}_{ij} + \hat{y}_{ij} - \bar{y}_{..})^2$$

$$= \sum_i \sum_j (y_{ij} - (\hat{\mu} + \hat{\tau}_i + \hat{\beta}_j) + (\hat{\mu} + \hat{\tau}_i + \hat{\beta}_j) - \bar{y}_{..})^2$$

$$= \ldots$$

2

$$SS_E = \sum_i \sum_j (y_{ij} - \hat{y}_{ij})^2$$

$$SS_{BLK} = \sum_{j=1}^{b} \frac{y_{.j}^2}{k} - \frac{y_{..}^2}{N} \quad \text{(unadjusted)}$$

If SS_{TRT} is found the usual way, then

$$SS_E + SS_{TRT} + SS_{BLK} \neq SS_{TOT}$$

Instead, define $SS_{TRT} = SS_{TOT} - SS_{BLK} - SS_E$

adjusted for blocks

ANOVA for BIBD

Source	SS	df	MS	F
TRT(adj)	$SS_{TRT}(adj)$	$a-1$	$\frac{SS}{df}$	$MS_{TRT}(adj)/MSE$
BLK	SS_{BLK}	$b-1$		—
ERR	SS_E	$N-a-b+1$		—
TOT	SS_{TOT}	$N-1$	—	—

Factorial Model

Simplest version: 2 treatment factors, 2 levels
no blocking

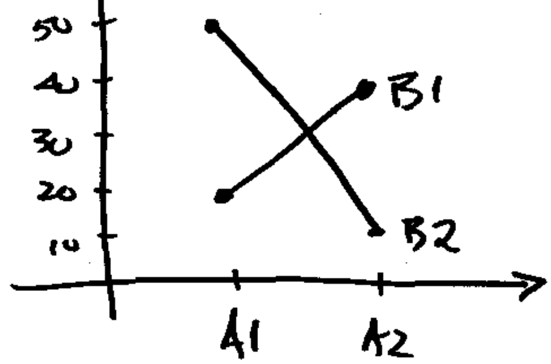
Ex:

Factor B

		B1	B2	
Factor A	A1	20	50	35
	A2	40	12	26
		30	31	

mean for this cell

Interaction plot



(5)

2-factor model:

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk} \qquad \begin{array}{l} i = 1, \dots, a \\ j = 1, \dots, b \\ k = 1, \dots, n \end{array}$$

$$E[\varepsilon_{ijk}] \equiv 0 \;, \; V[\varepsilon_{ijk}] = \sigma^2 \qquad N = abn$$

$$SSE = \sum_i \sum_j \sum_k \left(y_{ijk} - \mu - \tau_i - \beta_j - (\tau\beta)_{ij} \right)^2$$

Result. $\hat{\mu} = \bar{y}_{\cdot\cdot\cdot}$ $\qquad \hat{\beta}_j = \bar{y}_{\cdot j \cdot} - \bar{y}_{\cdot\cdot\cdot}$

$$\hat{\tau}_i = \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot}$$

$$\widehat{(\tau\beta)}_{ij} = \bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j \cdot} + \bar{y}_{\cdot\cdot\cdot}$$

(6)

ANOVA

Source	SS	df	MS	F
A	SS_A	$a-1$		MSA/MSE
B	SS_B	$b-1$	$\frac{SS}{df}$	MSB/MSE
AB	SS_{AB}	$(a-1)(b-1)$		$MSAB/MSE$
ERR	SS_E	$ab(n-1)$		——
TOT	SS_{TOT}	$N-1$	——	——

$$\begin{aligned} df_{AB} &= N - 1 - (a-1) - (b-1) - ab(n-1) \\ &= \cancel{N} - \cancel{1} - a + 1 - b + \cancel{1} - \cancel{N} + ab \\ &= (a-1)(b-1) \end{aligned}$$

3 hypothesis tests:

1. $H_0: \tau_1 = \tau_2 = \dots = \tau_a = 0$

2. $H_0: \beta_1 = \beta_2 = \dots = \beta_b = 0$

3. $H_0: (\tau\beta)_{ij} \equiv 0$ ← Do this one first

Decompose $SS_{TOT} = \sum_i \sum_j \sum_k (y_{ijk} - \bar{y}_{\cdots})^2$

$$= \sum_i \sum_j \sum_k (\underbrace{y_{ijk} - \hat{y}_{ijk}} + \underbrace{\hat{y}_{ijk} - \bar{y}_{\cdots}})^2$$

$$= \underbrace{\sum_i \sum_j \sum_k (y_{ijk} - \hat{y}_{ijk})^2}_{SS_E} + \underbrace{\sum_i \sum_j \sum_k (\hat{y}_{ijk} - \bar{y}_{\cdots})^2}_{SS_{TRT}} + 0$$

↑ cross product

$$\sum_i \sum_j \sum_k (\cancel{\hat{\mu}} + \hat{\tau}_i + \hat{\beta}_j + (\hat{\tau\beta})_{ij} - \cancel{\bar{y}_{\cdots}})^2$$

$$= \underbrace{bn \sum_i \hat{\tau}_i^2}_{SSA} + \underbrace{an \sum_j \hat{\beta}_j^2}_{SSB} + \underbrace{n \sum_i \sum_j (\hat{\tau\beta})_{ij}^2}_{SSAB} + 0$$

Find $E[SSA] = E\left[bn \sum_i \hat{\tau}_i^2\right]$ (9)

$$= E\left[bn \sum_i (\bar{y}_{i..} - \bar{y}_{...})^2\right]$$

$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}$

$\bar{y}_{i..} = \mu + \tau_i + \bar{\varepsilon}_{i..}$

$\bar{y}_{...} = \mu + \bar{\varepsilon}_{...}$

$$= E\left[bn \sum_i (\mu + \tau_i + \bar{\varepsilon}_{i..} - \mu - \bar{\varepsilon}_{...})^2\right]$$

(10)

$$= E\left[bn \left(\sum_i \tau_i^2 + \sum_i (\bar{\varepsilon}_{i..} - \bar{\varepsilon}_{...})^2 + 2\sum_i \tau_i (\bar{\varepsilon}_{i..} - \bar{\varepsilon}_{...})\right)\right]$$

$$= bn\left(\sum_i \tau_i^2 + E\Big(\underbrace{\sum_i (\bar{\varepsilon}_{i..} - \bar{\varepsilon}_{...})^2}_{(a-1)\times \text{ samp. var of } \bar{\varepsilon}_{1..}, \ldots \bar{\varepsilon}_{a..}}\Big) + 0\right)$$

$$= bn\left(\sum_i \tau_i^2 + (a-1)\frac{\sigma^2}{bn}\right)$$

$$\therefore E[SSA] = bn \sum_i \tau_i^2 + (a-1)\sigma^2$$

11

Define $MSA = \dfrac{SSA}{a-1}$

Then $E[MSA] = \dfrac{bn}{a-1} \sum_i \tau_i^2 + \sigma^2$

$[= \sigma^2$ under $H_0]$

Similarly, $E[MSB] = \dfrac{an}{b-1} \sum_j \beta_j^2 + \sigma^2$

and $E[MSAB] = \dfrac{n}{(a-1)(b-1)} \sum_i \sum_j (\tau\beta)_{ij}^2 + \sigma^2$

12

$$SSE = \sum_i \sum_j \sum_k (y_{ijk} - \hat{y}_{ijk})^2$$

$$= \sum_i \sum_j \sum_k \left(y_{ijk} - \left(\hat{\mu} + \hat{\tau}_i + \hat{\beta}_j + (\widehat{\tau\beta})_{ij} \right) \right)^2$$

$$= \sum_i \sum_j \sum_k \Big(y_{ijk} - \cancel{\bar{y}_{\cdots}} - (\cancel{\bar{y}_{i\cdot\cdot}} - \cancel{\bar{y}_{\cdots}}) - (\cancel{\bar{y}_{\cdot j \cdot}} - \cancel{\bar{y}_{\cdots}})$$
$$- \left(\bar{y}_{ij\cdot} - \cancel{\bar{y}_{i\cdot\cdot}} - \cancel{\bar{y}_{\cdot j\cdot}} + \cancel{\bar{y}_{\cdots}} \right) \Big)^2$$

$$= \sum_i \sum_j \sum_k (y_{ijk} - \bar{y}_{ij\cdot})^2$$

$$= (n-1) \sum_i \sum_j \left[\sum_k \frac{(y_{ijk} - \bar{y}_{ij\cdot})^2}{n-1} \right]$$

So $E[SSE] = (n-1)ab\,\sigma^2$

Define $MSE = \dfrac{SSE}{ab(n-1)}$

Then $E[MSE] = \sigma^2$

HW #4 due Thur 2/8

p. 178 # 4.23, 4.35

4.23. An industrial engineer is investigating the effect of four assembly methods (A, B, C, D) on the assembly time for a color television component. Four operators are selected for the study. Furthermore, the engineer knows that each assembly method produces such fatigue that the time required for the last assembly may be greater than the time required for the first, regardless of the method. That is, a trend develops in the required assembly time. To account for this source of variability, the engineer uses the Latin square design shown below. Analyze the data from this experiment ($\alpha = 0.05$) and draw appropriate conclusions.

Order of Assembly	Operator 1	2	3	4
1	$C = 10$	$D = 14$	$A = 7$	$B = 8$
2	$B = 7$	$C = 18$	$D = 11$	$A = 8$
3	$A = 5$	$B = 10$	$C = 11$	$D = 9$
4	$D = 10$	$A = 10$	$B = 12$	$C = 14$

4.35. The yield of a chemical process was measured using five batches of raw material, five acid concentrations, five standing times (A, B, C, D, E), and five catalyst concentrations ($\alpha, \beta, \gamma, \delta, \epsilon$). The Graeco-Latin square that follows was used. Analyze the data from this experiment (use $\alpha = 0.05$) and draw conclusions.

	Acid Concentration		
Batch	**1**	**2**	**3**
1	$A\alpha = 26$	$B\beta = 16$	$C\gamma = 19$
2	$B\gamma = 18$	$C\delta = 21$	$D\epsilon = 18$
3	$C\epsilon = 20$	$D\alpha = 12$	$E\beta = 16$
4	$D\beta = 15$	$E\gamma = 15$	$A\delta = 22$
5	$E\delta = 10$	$A\epsilon = 24$	$B\alpha = 17$

	Acid Concentration	
Batch	**4**	**5**
1	$D\delta = 16$	$E\epsilon = 13$
2	$E\alpha = 11$	$A\beta = 21$
3	$A\gamma = 25$	$B\delta = 13$
4	$B\epsilon = 14$	$C\alpha = 17$
5	$C\beta = 17$	$D\gamma = 14$