Graeco-Latin Square Design

Stat 565
1-30-18

①

2 treatment factors + 2 blocking factors
All 4 factors have p levels

$p=4$ Example:

row \ column	1	2	3	4
1	$A\alpha$	$B\beta$	$C\gamma$	$D\delta$
2	$B\delta$	$A\gamma$	$D\beta$	$C\alpha$
3	$C\beta$	$D\alpha$	$A\delta$	$B\gamma$
4	$D\gamma$	$C\delta$	$B\alpha$	$A\beta$

②

Necessary conditions:

- The Roman letters must form a Latin Square
- The Greek " " " " " "
- Each Roman/Greek letter combination appears exactly once

1st treatment, row block

Model : $$y_{ijkl} = \mu + \tau_i + \gamma_j + \alpha_k + \beta_l + \varepsilon_{ijkl}$$

2nd treatment, col block

③

Parameter estimates & SS decomposition

Follow the same pattern as in Latin Squares

Source	df
TRT1	$p-1$
TRT2	$p-1$
BLK1	$p-1$
BLK2	$p-1$
ERR	$(p-1)(p-3)$
TOT	$N-1 = p^2-1$

$$p^2-1-4(p-1)$$
$$= p^2-4p+3$$
$$(p-1)(p-3)$$

Balanced Incomplete Block Design (BIBD) ④

Start with a RCBD, but now allow some missing cells

1 treatment factor with a levels

1 blocking factor with b levels

But we will not conduct ab runs

- In each block, k levels of the treatment will be tested ($k < a$)
- Each level of the treatment appears in r blocks ($r < b$)

- Each pair of treatment levels will appear in λ blocks (5)

<u>Example</u>: 5 different fertilizers $a=5$

10 fields $b=10$

A RCBD would require 50 observations.

Suppose that each field can only accommodate 3 fertilizers. $k=3$

Find r & λ: $N = bk = ar$ $\therefore r=6$

$30 = 10 \cdot 3 = 5 \cdot r$

(6)

- A particular treatment level appears in r different blocks
- In each block that our particular treatment level appears, k levels are being tested, so our particular level appears with $(k-1)$ other levels
- So our particular level appears in the same block with $r(k-1)$ other treatment levels
- There are $(a-1)$ other treatment levels

⑦

$$\therefore \quad \lambda = \frac{r(k-1)}{a-1}$$

$$\lambda = \frac{6(3-1)}{5-1} = 3 \quad \text{(must be an integer)}$$

Each pair of fertilizers will appear together in 3 fields

(Note: If $a = b$, then the BIBD is symmetric)

⑧

Model: $y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij}$

$i = 1, \dots, a$

$j = 1, \dots, b$

$N = ar = bk \neq ab$

$$\sum_{i=1}^{a} \tau_i = 0 = \sum_{j=1}^{b} \beta_j$$

$E(\varepsilon_{ij}) \equiv 0$

$V(\varepsilon_{ij}) \equiv \sigma^2$

$$SSE = \sum_i \sum_j (y_{ij} - \mu - \tau_i - \beta_j)^2$$

$$\frac{\partial SSE}{\partial \mu} = \sum_i \sum_j \cancel{2}(y_{ij} - \mu - \tau_i - \beta_j)(-\cancel{1}) \stackrel{set}{=} 0$$ 9

$$y_{..} - N\mu - r\cancel{\sum \tau_i}^{0} - k\cancel{\sum_j \beta_j}^{0} = 0$$

$$\hat{\mu} = \frac{y_{..}}{N} = \bar{y}_{..}$$

$$\frac{\partial SSE}{\partial \hat{\tau}_i} = \sum_j \cancel{2}(y_{ij} - \mu - \tau_i - \beta_j)(-\cancel{1}) \stackrel{set}{=} 0$$

$$y_{i.} - r\mu - r\tau_i - \sum_{j=1}^{b} n_{ij}\beta_j = 0$$

$$n_{ij} = \begin{cases} 1 & \text{trmt } i \text{ is in blk } j \\ 0 & \text{o.w.} \end{cases}$$

10

$$\hat{\tau}_i = \bar{y}_{i.} - \bar{y}_{..} - \frac{1}{r}\sum_{j=1}^{b} n_{ij}\hat{\beta}_j$$

Similarly,

$$\hat{\beta}_j = \bar{y}_{.j} - \bar{y}_{..} - \frac{1}{k}\sum_{i=1}^{a} n_{ij}\hat{\tau}_i$$

Substitute:

$$\hat{\tau}_i = \bar{y}_{i.} - \bar{y}_{..} - \frac{1}{r}\sum_{j=1}^{b} n_{ij}\left(\bar{y}_{.j} - \bar{y}_{..} - \frac{1}{k}\sum_{i=1}^{a} n_{ij}\hat{\tau}_i\right)$$

$$= \bar{y}_{i\cdot} - \cancel{\bar{y}_{\cdot\cdot}} - \frac{1}{r}\sum_{j=1}^{b} n_{ij}\bar{y}_{\cdot j} + \cancel{\frac{1}{r}\bar{y}_{\cdot\cdot}\sum_{j=1}^{b} n_{ij}} \qquad (11)$$

$$+ \frac{1}{rk}\sum_{j=1}^{b}\sum_{m=1}^{a} n_{ij}n_{mj}\hat{\tau}_m$$

$$\hat{\tau}_i = \bar{y}_{i\cdot} - \frac{1}{r}\sum_{j=1}^{b} n_{ij}\bar{y}_{\cdot j} +$$

$$\left[\frac{1}{rk}\underbrace{\sum_{j=1}^{b}\sum_{m=1}^{a}}_{m\neq i} n_{ij}n_{mj}\hat{\tau}_m + \frac{1}{rk}\sum_{j=1}^{b} n_{ij}^2\hat{\tau}_i\right]$$

$$\hat{\tau}_i\left(1-\frac{1}{k}\right) = \bar{y}_{i\cdot} - \frac{1}{r}\sum_{j=1}^{b} n_{ij}\bar{y}_{\cdot j} + \frac{1}{rk}\sum_{\substack{m=1\\ m\neq i}}^{a}\left(\hat{\tau}_m \underbrace{\sum_{j=1}^{b} n_{ij}n_{mj}}_{\lambda}\right)$$

$$(12)$$

$$\hat{\tau}_i\left(1-\frac{1}{k}\right) = \bar{y}_{i\cdot} - \frac{1}{r}\sum_{j=1}^{b} n_{ij}\bar{y}_{\cdot j} + \frac{\lambda}{rk}\underbrace{\sum_{\substack{m=1\\ m\neq i}}^{a}\hat{\tau}_m}_{-\hat{\tau}_i}$$

$$\hat{\tau}_i\underbrace{\left(1-\frac{1}{k}+\frac{\lambda}{rk}\right)} = \bar{y}_{i\cdot} - \frac{1}{r}\sum_{j=1}^{b} n_{ij}\bar{y}_{\cdot j}$$

$$\frac{rk-r+\lambda}{rk} = \frac{r(k-1)+\lambda}{rk} = \frac{\lambda(a-1)+\lambda}{rk} = \frac{\lambda a}{rk}$$

$$\therefore\ \hat{\tau}_i = \frac{rk}{\lambda a}\left(\bar{y}_{i\cdot} - \frac{1}{r}\sum_{j=1}^{b} n_{ij}\bar{y}_{\cdot j}\right)$$

or $\hat{\tau}_i = \frac{k}{\lambda a}\underbrace{\left(y_{i\cdot} - \frac{1}{k}\sum_{j=1}^{b} n_{ij} y_{\cdot j} \right)}_{\text{Call this } Q_i}$

$$\hat{\tau}_i = \frac{k}{\lambda a} Q_i$$