Stat 564
10-10-17

From last time:

①

$$SSE = Y'Y - 2Y'X\hat{\beta} + \hat{\beta}'X'X\hat{\beta}$$

The solution was $\hat{\beta} = (X'X)^{-1}X'Y$

What is SSE, evaluated at the least-squares solution for $\hat{\beta}$?

$$SSE = Y'Y - 2Y'X(X'X)^{-1}X'Y + Y'X(X'X)^{-1}\cancel{X'X(X'X)^{-1}}X'Y$$

$$= Y'Y - Y'X(X'X)^{-1}X'Y$$

②

let $H = X(X'X)^{-1}X'$

So $SSE = Y'Y - Y'HY$

$$= Y'(I_n - H)Y$$

Also, $\hat{Y} = X\hat{\beta} = X(X'X)^{-1}X'Y$

$$= HY$$

"hat" matrix

If Y is a random vector $\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ (3)

then $E[Y] = \begin{bmatrix} E[y_1] \\ \vdots \\ E[y_n] \end{bmatrix}$

and $Cov(Y) = \begin{bmatrix} V[y_1] & & Cov(y_i, y_j) \\ & \ddots & \\ & & V[y_n] \end{bmatrix}$

Properties: (4)

$$E[a'Y] = a'E[Y]$$

$$E[AY] = A\,E[Y]$$

$$Cov(a'Y) = a'\,Cov(Y)\,a$$

$$Cov(AY) = A\,Cov(Y)\,A'$$

Use these to find $E[\hat{\beta}]$ & $Cov(\hat{\beta})$

$$E(\hat{\beta}) = E\left((X'X)^{-1}X'Y\right) = (X'X)^{-1}X'\,E(Y)$$

⑤

$$= (X'X)^{-1}X'E(X\beta + \varepsilon)$$

$$= (X'X)^{-1}X'X\beta = \beta$$ So the L.S. estimators are unbiased

$$\text{Cov}(\hat\beta) = \text{Cov}((X'X)^{-1}X'Y)$$
$$= (X'X)^{-1}X'\,\text{Cov}(Y)\,X(X'X)^{-1}$$
$$= (X'X)^{-1}X'\,\text{Cov}(X\beta+\varepsilon)\,X(X'X)^{-1}$$
$$= (X'X)^{-1}X'\,\text{Cov}(\varepsilon)\,X(X'X)^{-1}$$

⑥

$$= (X'X)^{-1}X'\underbrace{\begin{bmatrix}\sigma^2 & & 0\\ & \sigma^2 & \\ 0 & \ddots & \sigma^2\end{bmatrix}}_{\sigma^2 I}X(X'X)^{-1}$$

$$= \sigma^2(X'X)^{-1}\cancel{X'X(X'X)^{-1}}$$

$$\text{Cov}(\hat\beta) = \sigma^2(X'X)^{-1}$$

<u>Fact</u>: If the random vector Y has a multivariate normal distribution, and A is a symmetric matrix,

(7)

then $\frac{Y'AY}{\sigma^2}$ has a χ^2 distribution with df = rank A

Assume $\vec{\varepsilon}$ has a multivariate normal distribution

$SSE = Y'(I-H)Y$

So $\frac{Y'(I-H)Y}{\sigma^2} \sim \chi^2$ with df = rank(I-H)

(8)

$H = X(X'X)^{-1}X'$ X was $n \times \underbrace{(k+1)}_{p}$

$H' = X(X'X)^{-1}X'$
$= H$ so H is symmetric

$H^2 = X(X'X)^{-1}\cancel{X'X(X'X)^{-1}}X'$
$= H$ so H is idempotent

The <u>spectral decomposition</u> of a matrix

If A is a square matrix, $A = P \Lambda P^{-1}$

where P is a matrix whose columns (9)
are the eigenvectors of A and

Λ is a diagonal matrix containing
the eigenvalues

Also, if A is symmetric, then

$$A = P \Lambda P' \quad (\text{i.e. } \underset{\text{orthogonal}}{P' = P^{-1}})$$

What if A is symmetric & idempotent? (10)

$$A^2 = P \Lambda \underbrace{P' P}_{I} \Lambda P'$$

$$= P \Lambda^2 P' \qquad \text{But } A^2 = A$$

$$= P \Lambda P'$$

By the uniqueness of eigenvalues,

Λ^2 must equal Λ

So the eigenvalues of A are all 0s & 1s

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Property of the trace of a matrix:
↑sum of diagonal elements

$$tr(AB) = tr(BA) \text{ if } AB \text{ \& } BA \text{ are both defined}$$

$$tr(H) = tr(\overset{A}{\underset{n\times p}{X}}\overbrace{(X'X)^{-1}\underset{p\times n}{X'}}^{B})$$

$$= tr((X'X)^{-1}\underbrace{X'X}_{p\times p}) = tr(I_p) = p$$

Also, $tr(H) = tr(P\Lambda P') = tr(\Lambda \cancel{P'P}) = tr(\Lambda)$

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$tr(H) =$ sum of the eigenvalues of H
$=$ # of eigenvalues that are 1

$\therefore$ rank of H is p

What about $I-H$?

$(I-H)' = I' - H' = I - H$ symmetric

$(I-H)(I-H) = I - H - \cancel{H} + H^2$ idempotent

So the $tr(I-H) = rank(I-H)$

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But $tr(I_n - H) = tr\, I_n - tr\, H$
$$= n - p$$

$\therefore\ rank(I - H) = n - p$

So $$\frac{Y'(I-H)Y}{\sigma^2} \sim \chi^2_{n-p}$$

$$E\left[\frac{Y'(I-H)Y}{\sigma^2}\right] = n - p$$

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Therefore $$E\left[\frac{\overbrace{Y'(I-H)Y}^{4.6}}{n-p}\right] = \sigma^2$$

Define $$MSE = \frac{SSE}{n-p}$$

and MSE is an unbiased estimator of σ^2

Geometry: Y and $\hat{Y}$ are both $n \times 1$ vectors

Also $\hat{Y} = X\hat{\beta}$

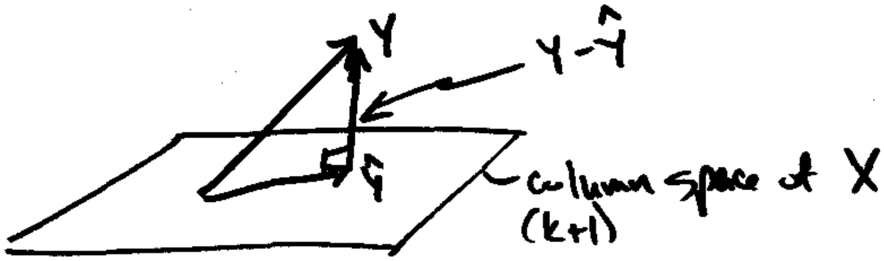
$$= \begin{bmatrix} 1 & x_{11} & \cdots & x_{1k} \\ 1 & \vdots & & \\ \vdots & \vdots & & \\ 1 & x_{n1} & \cdots & x_{nk} \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \vdots \\ \hat{\beta}_k \end{bmatrix}$$

$$\vec{1} \;|\; \vec{X}_1 \;|\; \cdots \;|\; \vec{X}_k$$

$$\hat{Y} = \left[\vec{1} \;\middle|\; \vec{X}_1 \;\middle|\; \cdots \;\middle|\; \vec{X}_k \right] \begin{bmatrix} \hat{\beta}_0 \\ \vdots \\ \hat{\beta}_k \end{bmatrix}$$

$$\hat{Y}_{n \times 1} = \vec{1}\hat{\beta}_0 + \vec{X}_1 \hat{\beta}_1 + \cdots + \vec{X}_k \hat{\beta}_k \tag{16}$$

So $\hat{Y}$ is always in the $(k+1)$-dimensional subspace spanned by the columns of the design matrix X.



Why is that a right angle?

$$\hat{Y}'(Y-\hat{Y}) = (HY)'(Y-HY)$$
$$= Y'H(Y-HY)$$
$$= Y'HY - Y'H^2Y = 0$$

So $\hat{Y}$ is actually the projection of the Y vector onto the column space of X.