Stat 524
10-5-17

from last time:

$$R^2 = \frac{SSR}{SST}$$ = "coefficient of determination" ①

$$= \frac{S_{xy}^2/S_{xx}}{S_{yy}} = \frac{S_{xy}^2}{S_{xx}S_{yy}}$$

$$\sqrt{R^2} = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}} = r$$ = "correlation coefficient"

Analysis of Variance (ANOVA) table ②

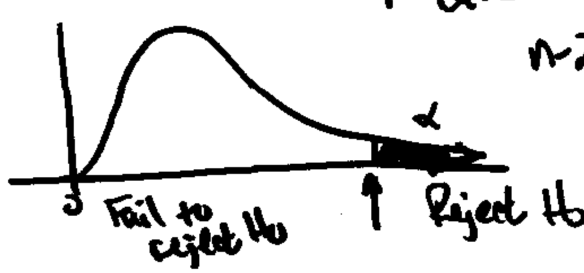
for simple linear regression

| Source | SS | df | MS $= \frac{SS}{df}$ | F |
|---|---|---|---|---|
| REG | SSR | 1 | MSR | MSR/MSE |
| ERR | SSE | n-2 | MSE | |
| TOT | SST | n-1 | — | |

→ test statistic

F table 1 num df
n-2 den df

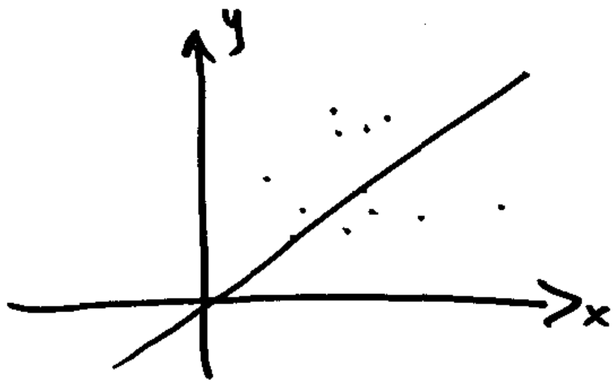
$H_0: \beta_1 = 0$
$H_1: \beta_1 \neq 0$



③

New problem:

Find the best line through a set of points, with the condition that it passes through $(0,0)$.



Model:

$$y_i = \beta_1 x_i + \varepsilon_i$$

$$SSE = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

④

$$SSE = \sum_{i=1}^{n} (y_i - \beta_1 x_i)^2$$

$$\frac{\partial SSE}{\partial \beta_1} = \sum_{i=1}^{n} \cancel{2}(y_i - \beta_1 x_i)(\cancel{-}x_i) \stackrel{\text{set}}{=} 0$$

$$= \sum x_i y_i - \beta_1 \sum x_i^2 = 0$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

(5)

The general linear model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k + \varepsilon$$

There are $n$ observations

$$y_1 = \beta_0 + \beta_1 x_{11} + \beta_2 x_{12} + \ldots + \beta_k x_{1k} + \varepsilon_1$$

$$\vdots$$

$$y_n = \beta_0 + \beta_1 x_{n1} + \beta_2 x_{n2} + \ldots + \beta_k x_{nk} + \varepsilon_n$$

(6)

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1k} \\ 1 & x_{21} & \cdots & x_{2k} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 1 & x_{n1} & \cdots & x_{nk} \end{bmatrix}_{n\times(k+1)} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}_{(k+1)\times 1} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}_{n\times 1}$$

$$Y = X \ \beta + \varepsilon$$

$$Y = X\beta + \varepsilon$$

design matrix

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Still want to minimize SSE

$$SSE = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

$$Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \quad \text{let } \hat{Y} = \begin{bmatrix} \hat{y}_1 \\ \vdots \\ \hat{y}_n \end{bmatrix}$$

$$\text{so } Y - \hat{Y} = \begin{bmatrix} y_1 - \hat{y}_1 \\ \vdots \\ y_n - \hat{y}_n \end{bmatrix}$$

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$$(Y - \hat{Y})'(Y - \hat{Y}) = [y_1 - \hat{y}_1, \ldots, y_n - \hat{y}_n] \begin{bmatrix} y_1 - \hat{y}_1 \\ \vdots \\ y_n - \hat{y}_n \end{bmatrix}$$

$$= \sum (y_i - \hat{y}_i)^2 = SSE$$

Also, $\hat{Y} = X\hat{\beta}$

so $SSE = (Y - X\hat{\beta})'(Y - X\hat{\beta})$

(9)

**Review:** If $f(x_1, \dots, x_k)$ is a real-valued function, then

$$\frac{\partial f}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_k} \end{bmatrix}$$

**Special case #1:** $$\frac{\partial (\vec{a}'\vec{x})}{\partial \vec{x}} = \frac{\partial \left( \sum_{i=1}^{k} a_i x_i \right)}{\partial \vec{x}}$$

(10)

$$= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix} = \vec{a}$$

**Special case #2** $$\frac{\partial (\vec{x}' \overset{\text{symmetric}}{A} \vec{x})}{\partial \vec{x}}$$

$$= \frac{\partial}{\partial \vec{x}} \left[ (x_1, \dots, x_k) \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \\ a_{k1} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \right]$$

(11)

$$= \frac{\partial}{\partial \vec{x}} \left[ (x_1, \ldots, x_k) \begin{bmatrix} a_{11}x_1 + \ldots + a_{1k}x_k \\ a_{k1}x_1 + \ldots + a_{kk}x_k \end{bmatrix} \right]$$

$$= \frac{\partial}{\partial \vec{x}} \left[ x_1(a_{11}x_1 + \ldots + a_{1k}x_k) + \ldots + x_k(a_{k1}x_1 + \ldots + a_{kk}x_k) \right]$$

$$= \frac{\partial}{\partial \vec{x}} \left[ \sum_{i=1}^{k} a_{ii}x_i^2 + 2\sum_{i<j}\sum a_{ij}x_i x_j \right]$$

$$= \begin{bmatrix} 2a_{11}x_1 + 2a_{12}x_2 + \ldots + 2a_{1k}x_k \\ \vdots \\ 2a_{kk}x_k + 2a_{1k}x_1 + \ldots + 2a_{k(k-1)}x_{k-1} \end{bmatrix}$$

(12)

$$= 2 \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ & & & \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}$$

$$= 2A\vec{x}$$

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$$SSE = (Y - X\hat{\beta})'(Y - X\hat{\beta})$$

$$= Y'Y - Y'X\hat{\beta} - (X\hat{\beta})'Y + (X\hat{\beta})'(X\hat{\beta})$$

$$= Y'Y - Y'X\hat{\beta} - \hat{\beta}'X'Y + \hat{\beta}'X'X\hat{\beta}$$

$$SSE = Y'Y - 2Y'X\hat{\beta} + \hat{\beta}'X'X\hat{\beta}$$

$$\frac{\partial SSE}{\partial \hat{\beta}} = -2X'Y + 2X'X\hat{\beta} \overset{set}{=} 0$$

$$X'X\hat{\beta} = X'Y$$

$$\hat{\beta} = (X'X)^{-1}X'Y$$

, assuming $X'X$ has an inverse

***2.18*** *On March 1, 1984, the Wall Street Journal published a survey of television advertisements conducted by Video Board Tests, Inc., a New York ad-testing company that interviewed 4000 adults. These people were regular product users who were asked to cite a commercial they had seen for that product category in the past week. In this case, the response is the number of millions of retained impressions per week. The regressor is the amount of money spent by the firm on advertising. The data follow.*

***a.*** *Fit the simple linear regression model to these data.*
***b.*** *Is there a significant relationship between the amount a company spends on advertising and retained impressions? Justify your answer statistically.*
***c.*** *Construct the 95% confidence and prediction bands for these data.*
***d.*** *Give the 95% confidence and prediction intervals for the number of retained impressions for MCI.*

| Firm | Amount Spent (millions) | Returned Impressions per week (millions) |
|---|---|---|
| Miller Lite | 50.1 | 32.1 |
| Pepsi | 74.1 | 99.6 |
| Stroh's | 19.3 | 11.7 |
| Federal Express | 22.9 | 21.9 |
| Burger King Coca- | 82.4 | 60.8 |
| Cola | 40.1 | 78.6 |
| McDonald's | 185.9 | 92.4 |
| MCI | 26.9 | 50.7 |
| Diet Cola | 20.4 | 21.4 |
| Ford | 166.2 | 40.1 |
| Levi's | 27 | 40.8 |
| Bud Lite | 45.6 | 10.4 |
| ATT Bell | 154.9 | 88.9 |
| Calvin Klein | 5 | 12 |
| Wendy's Polaroid | 49.7 26.9 | 29.2 38 |
| Shasta | 5.7 | 10 |
| Meow Mix | 7.6 | 12.3 |
| Oscar Meyer Crest | 9.2 32.4 | 23.4 71.1 |
| Kibbles N Bits | 6.1 | 4.4 |

**2.25** *Consider the simple linear regression model* $y = \beta_0 + \beta_1 x + \varepsilon$, *with* $E(\varepsilon) = 0$, $Var(\varepsilon) = \sigma^2$, *and* $\varepsilon$ *uncorrelated.*

***a.*** *Show that* $\mathrm{Cov}(\hat{\beta}_0, \hat{\beta}_1) = -\bar{x}\sigma^2 / S_{xx}$.

***b.*** *Show that* $\mathrm{Cov}(\bar{y}, \hat{\beta}_1) = 0$.