Stat 563
4-2-19

(1)

Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be unbiased estimators of θ.

Defn: The relative efficiency of $\hat{\theta}_1$ to $\hat{\theta}_2$ is

$$\frac{V(\hat{\theta}_2)}{V(\hat{\theta}_1)}$$

Defn: The efficiency of $\hat{\theta}_1$ is $\dfrac{CRLB}{V(\hat{\theta}_1)}$

Defn: $\hat{\theta}_1$ is efficient if the efficiency is 1.

(2)

Defn: $\hat{\theta}$ is asymptotically unbiased

if $\lim_{n\to\infty} \text{Bias}(\hat{\theta}) = 0$

Defn: If $\hat{\theta}_1$ and $\hat{\theta}_2$ are asymptotically unbiased estimators of θ, then the asymptotic relative efficiency of $\hat{\theta}_1$ to $\hat{\theta}_2$

is $\lim_{n\to\infty} \dfrac{V(\hat{\theta}_2)}{V(\hat{\theta}_1)}$.

③

Defn: If $\hat{\theta}_1$ is asymptotically unbiased, then its asymptotic efficiency is

$$\lim_{n\to\infty} \frac{CRLB}{V(\hat{\theta}_1)}.$$

Example: $X_1, \ldots, X_n \sim$ iid $Exp(\lambda)$

Compare $\hat{\lambda}_{MLE}$ to an unbiased version of it.

④

$$L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda \sum x_i}$$

$$\ell(\lambda) = \ln L(\lambda) = n \ln \lambda - \lambda \sum x_i$$

$$\ell'(\lambda) = \frac{n}{\lambda} - \sum x_i \overset{\text{set}}{=} 0$$

$$\hat{\lambda}_{MLE} = \frac{n}{\sum x_i} = \frac{1}{\bar{x}}$$

$$E[\hat{\lambda}_{MLE}] = E\left[\frac{n}{\sum x_i}\right] = E\left[\frac{n}{Y}\right] = n\,E[Y^{-1}]$$

$$Y = \sum X_i \sim Gamma(\alpha = n, \beta = \tfrac{1}{\lambda})$$

(5)

$$E[Y^{-1}] = \beta^{-1}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha)} = \lambda\frac{(n-2)!}{(n-1)!} = \frac{\lambda}{n-1}$$

$$E[\hat{\lambda}_{MLE}] = n\,E[Y^{-1}] = \frac{n}{n-1}\lambda$$

$$\text{Bias}(\hat{\lambda}_{MLE}) = \frac{n}{n-1}\lambda - \lambda = \frac{\lambda}{n-1}$$

$\therefore$ $\hat{\lambda}_{MLE}$ is biased, but asymptotically unbiased.

Define $\hat{\lambda}_u = \frac{n-1}{n}\hat{\lambda}_{MLE} = \frac{n-1}{n}\frac{n}{\Sigma x_i} = \frac{n-1}{\Sigma x_i}$

(6)

$$V(\hat{\lambda}_{MLE}) = V\left(\frac{n}{\Sigma x_i}\right) = V\left(\frac{n}{Y}\right)$$

$$= n^2 V[Y^{-1}]$$

$$= n^2\Big[\underbrace{E[Y^{-2}]}_{\downarrow} - \big(\underbrace{E[Y^{-1}]}_{\frac{\lambda}{n-1}}\big)^2\Big]$$

$$\beta^{-2}\frac{\Gamma(\alpha-2)}{\Gamma(\alpha)}$$

$$\lambda^2\frac{(n-3)!}{(n-1)!}$$

$$\frac{\lambda^2}{(n-1)(n-2)}$$

(7)

$$V(\hat{\lambda}_{MLE}) = n^2\left[\frac{\lambda^2}{(n-1)(n-2)} - \frac{\lambda^2}{(n-1)^2}\right]$$

$$= \frac{n^2\lambda^2}{(n-1)^2(n-2)}$$

$$V(\hat{\lambda}_u) = V\left(\frac{n-1}{n}\hat{\lambda}_{MLE}\right) = \left(\frac{n-1}{n}\right)^2 \frac{n^2\lambda^2}{(n-1)^2(n-2)}$$

$$= \frac{\lambda^2}{n-2}$$

the asymptotic relative efficiency of $\hat{\lambda}_u$ to $\hat{\lambda}_{MLE}$

(8)

is $$\lim_{n\to\infty} \frac{\frac{n^2\lambda^2}{(n-1)^2\cancel{(n-2)}}}{\frac{\lambda^2}{\cancel{n-2}}} = \lim_{n\to\infty} \frac{n^2}{(n-1)^2} = 1$$

Find the Cramér-Rao Lower Bound (CRLB):

$$f(x) = \lambda e^{-\lambda x}$$

$$\ln f(x) = \ln\lambda - \lambda x$$

$$\frac{\partial \ln f}{\partial \lambda} = \frac{1}{\lambda} - x$$

$$\frac{\partial^2 \ln f}{\partial \lambda^2} = -\frac{1}{\lambda^2}$$

(9)

$$I(\lambda) = E\left[\frac{\partial \ln f}{\partial \lambda}\right]^2 = -E\left[\frac{\partial^2 \ln f}{\partial \lambda^2}\right]$$

Fisher Information

$$E\left[\frac{1}{\lambda} - X\right]^2 = V(X) = \frac{1}{\lambda^2}$$

$$-E\left[-\frac{1}{\lambda^2}\right] = \frac{1}{\lambda^2}$$

$$CRLB = \frac{1}{n I(\lambda)} = \frac{1}{n \cdot \frac{1}{\lambda^2}} = \frac{\lambda^2}{n}$$

(10)

The efficiency of $\hat{\lambda}_u$ is $\dfrac{CRLB}{V(\hat{\lambda}_u)} = \dfrac{\frac{\lambda^2}{n}}{\frac{\lambda^2}{n-2}} = \dfrac{n-2}{n}$

the asymptotic efficiency of $\hat{\lambda}_u = 1$

the asymptotic efficiency of $\hat{\lambda}_{MLE}$ is

$$\lim_{n\to\infty} \frac{CRLB}{V(\hat{\lambda}_{MLE})} = \frac{\frac{\lambda^2}{n}}{\frac{n^2\lambda^2}{(n-1)^2(n-2)}} = \lim_{n\to\infty} \frac{(n-1)^2(n-2)}{n^3}$$

$$= 1$$

$$MSE(\hat{\theta}) = B^2(\hat{\theta}) + V(\hat{\theta})$$

$$MSE(\hat{\lambda}_u) = V(\hat{\lambda}_u) = \frac{\lambda^2}{n-2}$$

$$MSE(\hat{\lambda}_{MLE}) = \left(\frac{\lambda}{n-1}\right)^2 + \frac{n^2\lambda^2}{(n-1)^2(n-2)}$$

$$= \lambda^2 \frac{(n-2) + n^2}{(n-1)^2(n-2)}$$

$$= \lambda^2 \frac{n^2+n-2}{(n-1)^2(n-2)} = \frac{\lambda^2 (n+2)}{(n-1)(n-2)}$$

$\therefore MSE(\hat{\lambda}_{MLE}) > MSE(\hat{\lambda}_u)$