Example: $X_1, \dots, X_n \sim \text{iid } N(\theta, 1)$

Stat 563

4-16-19

①

$H_0: \theta = \theta_0$

$H_1: \theta \neq \theta_0$

let θ_1 be any value of $\theta \neq \theta_0$

$$\Lambda = \frac{L(\theta_0)}{L(\theta_1)} = \frac{\cancel{\left(\frac{1}{\sqrt{2\pi}}\right)^n} e^{-\frac{1}{2}\sum(x_i-\theta_0)^2}}{\cancel{\left(\frac{1}{\sqrt{2\pi}}\right)^n} e^{-\frac{1}{2}\sum(x_i-\theta_1)^2}} \overset{\text{set}}{\leq} c$$

$$e^{-\frac{1}{2}\left[\cancel{\sum x_i^2} - 2\theta_0 \sum x_i + n\theta_0^2 - \left(\cancel{\sum x_i^2} - 2\theta_1 \sum x_i + n\theta_1^2\right)\right]} \leq c$$

②

$$-2\theta_0 \sum x_i + n\theta_0^2 + 2\theta_1 \sum x_i - n\theta_1^2 \geq c'$$

$$2\bar{x}(\theta_1 - \theta_0) + \theta_0^2 - \theta_1^2 \geq c''$$

$$\bar{x}(\theta_1 - \theta_0) \geq c'''$$

Our reject region is $\bar{x} \geq k$ if $\theta_1 > \theta_0$

or $\bar{x} \leq k$ if $\theta_1 < \theta_0$

$\therefore$ There is no single rejection region that gives a most powerful test $\forall\, \theta_1$.

That is, there is no UMP test.

③

Defn: $L(\theta)$ has monotone likelihood ratio (MLR) in $T = g(X_1, \ldots, X_n)$ if

$\forall\, \theta_1 < \theta_2$, $\dfrac{L(\theta_2)}{L(\theta_1)}$ is a monotone function of T.

Some families of distributions with this property:

Normal (σ known), Poisson, Binomial,

most exponential families

Assume that T is a sufficient statistic for θ.

and that $L(\theta)$ has MLR in T

④

Consider $H_0: \theta = \theta_0$

$H_1: \theta = \theta_1$ where $\theta_1 > \theta_0$

Neyman-Pearson theorem says that the MP test is the one which rejects H_0 when

$$\Lambda = \frac{L(\theta_0)}{L(\theta_1)} \leq c$$

But our additional assumption says

$$\frac{L(\theta_1)}{L(\theta_0)} = g(t), \text{ which is } \uparrow \text{ or } \downarrow$$

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Case 1: $g(t) \uparrow$

Then Λ is $\downarrow$ in t

So $\Lambda \le c \iff \frac{1}{g(t)} \le c$

$\iff g(t) \ge c''$

$\iff t \ge c'''$

Case 2: $g(t) \downarrow$

Then Λ is $\uparrow$ in t

$\Lambda \le c \iff \frac{1}{g(t)} \le c$

6

$g(t) \ge c''$

$t \le c'''$

This is the Karlin-Rubin theorem, which is a corollary to the Neyman-Pearson theorem.

Summary: If $L(\theta)$ has MLR in T (sufficient), then the MP test of $H_0: \theta = \theta_0$, $H_1: \theta = \theta_1$

will be of the form $T \le c$ ($g(t) \downarrow$) or $T \ge c$ ($g(t) \uparrow$)

(7)

Bayesian Hypothesis Tests

We now assume that θ is a random variable whose distribution is known, and θ is independent of $X_1, \ldots, X_n$.

Example (normal-normal)

$$X_1, \ldots, X_n \sim \text{iid } N(\theta, \underbrace{\sigma^2}_{\text{known}})$$

$$\theta \sim N(\underbrace{\mu, \tau^2}_{\text{both known}})$$

(8)

Prior distribution

$$h(\theta) = \frac{1}{\tau\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\theta-\mu}{\tau}\right)^2}$$

$$L(\theta) = f(x_1, \ldots, x_n \mid \theta)$$
$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\theta}{\sigma}\right)^2}$$

the joint distr. of $(X_1, \ldots, X_n, \theta)$ is $L(\theta)h(\theta)$

Recall that the posterior distribution of $\theta \mid \vec{x}$ is proportional to $L(\theta)h(\theta)$

⑨

$$L(\theta) h(\theta) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \frac{1}{\tau\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\Sigma(x_i-\theta)^2} e^{-\frac{1}{2\tau^2}(\theta-\mu)^2}$$

This is proportional to a normal density

with mean $\frac{n\tau^2\bar{x}+\sigma^2\mu}{n\tau^2+\sigma^2}$ and

variance $\frac{\sigma^2\tau^2}{n\tau^2+\sigma^2}$

Decision rule: Reject H_0 if

$$P(\theta \in \Omega_0^c \mid \vec{x}) \geq P(\theta \in \Omega_0 \mid \vec{x})$$

⑩

Suppose that we were testing

$H_0: \theta \leq \theta_0$ vs $H_1: \theta > \theta_0$

Rule: reject H_0 if

$$P(\theta > \theta_0 \mid \vec{x}) \geq .5$$

$$\parallel$$

$$P\left(\frac{\theta - \frac{n\tau^2\bar{x}+\sigma^2\mu}{n\tau^2+\sigma^2}}{\sqrt{\frac{\sigma^2\tau^2}{n\tau^2+\sigma^2}}} > \frac{\theta_0 - \dots}{\dots} \,\Big|\, \vec{x}\right)$$

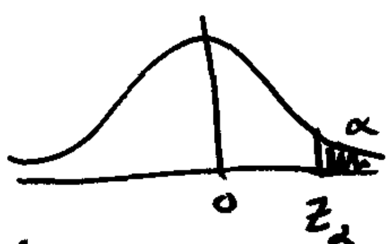
$$P\left(Z > \frac{\theta_0 - \dots}{\dots}\right) \geq .5$$

$$\Rightarrow \quad \frac{\theta_0 - \frac{n\tau^2\bar{x}+\sigma^2\mu}{n\tau^2+\sigma^2}}{\sqrt{\frac{\sigma^2\tau^2}{n\tau^2+\sigma^2}}} \le 0$$

$$\Rightarrow \quad \bar{x} \ge \theta_0 + \frac{\sigma^2(\theta_0-\mu)}{n\tau^2}$$

Compare this to the LRT, where we would have rejected H_0 if $\bar{x} \ge c$

Under H_0: $\dfrac{\bar{x}-\theta_0}{\sigma/\sqrt{n}} \sim N(0,1)$



Reject H_0 if

$$\frac{\bar{x}-\theta_0}{\sigma/\sqrt{n}} \ge z_\alpha \qquad \bar{x} \ge \theta_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$$