

## Some Asymptotic Results

Stat 563  
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### Slutsky's Theorem

Suppose  $X_n \xrightarrow{D} X$

And  $Y_n \xrightarrow{P} c$

Then (a)  $X_n + Y_n \xrightarrow{D} X + c$

(b)  $X_n Y_n \xrightarrow{D} cX$

Recall  $X_n \xrightarrow{P} X \iff$  ①

$\forall \epsilon > 0$

$$\lim_{n \rightarrow \infty} P[|X_n - X| \geq \epsilon] = 0$$

$X_n \xrightarrow{D} X \iff$

$$\lim_{n \rightarrow \infty} F_n(x) = F(x)$$

$\forall x :$

$x$  is a continuity point of  $F(x)$

Proof: (a)

Assume  $c = 0$

Let  $x$  be a point of continuity of  $F(x)$

$$\begin{aligned} P(X_n + Y_n \leq x) &= P(X_n + Y_n \leq x \cap |Y_n| \leq \epsilon) \\ &\quad + P(X_n + Y_n \leq x \cap |Y_n| > \epsilon) \end{aligned}$$

$$\leq P(X_n \leq x + \epsilon) + P(|Y_n| > \epsilon) \quad \text{①}$$

Since  $X_n \leq x - Y_n \cap -\epsilon \leq Y_n \leq \epsilon$

$$\Rightarrow X_n \leq x - Y_n \cap \epsilon \geq -Y_n \Rightarrow X_n \leq x + \epsilon$$

$$\text{Also, } P(X_n \leq \kappa - \varepsilon) = P(X_n \leq \kappa - \varepsilon \cap |Y_n| \leq \varepsilon) \quad (3) \\ + P(X_n \leq \kappa - \varepsilon \cap |Y_n| > \varepsilon)$$

$$\leq P(X_n + Y_n \leq \kappa) + P(|Y_n| > \varepsilon) \quad (2)$$

Since

$$X_n \leq \kappa - \varepsilon \cap -\varepsilon \leq Y_n \leq \varepsilon \Rightarrow X_n + Y_n \leq \kappa + \varepsilon \\ \Rightarrow X_n + Y_n \leq \kappa$$

Recap:

$$(1) \quad P(X_n + Y_n \leq \kappa) \leq P(X_n \leq \kappa + \varepsilon) + P(|Y_n| > \varepsilon)$$

$$(2) \quad P(X_n \leq \kappa - \varepsilon) \leq P(X_n + Y_n \leq \kappa) + P(|Y_n| > \varepsilon)$$

$$(2): \quad P(X_n \leq \kappa - \varepsilon) - P(|Y_n| > \varepsilon) \leq P(X_n + Y_n \leq \kappa)$$

Then with (1): As  $n \rightarrow \infty$

$$P(X_n \leq \kappa - \varepsilon) - P(|Y_n| > \varepsilon) \leq P(X_n + Y_n \leq \kappa) \leq P(X_n \leq \kappa + \varepsilon) + P(|Y_n| > \varepsilon)$$

$$\text{so } \lim_{n \rightarrow \infty} P(X_n + Y_n \leq \kappa) = F(\kappa)$$

$$\therefore X_n + Y_n \xrightarrow{P} X$$

(5)

To finish (a), what if  $c \neq 0$ ?

$$\begin{aligned} \text{Write } X_n + Y_n &= (X_n + c) + (Y_n - c) \\ &= W_n + V_n \end{aligned}$$

$$\text{Then } W_n \xrightarrow{P} X + c \text{ and } V_n \xrightarrow{P} 0$$

$$\Rightarrow W_n + V_n \xrightarrow{P} X + c \quad \text{This proves (a).}$$

(b) First consider  $c = 0$

(6)

$$\begin{aligned} P(|X_n Y_n| > \varepsilon) &= P(|X_n Y_n| > \varepsilon \cap |Y_n| \leq \frac{1}{N}) \\ &\quad + P(|X_n Y_n| > \varepsilon \cap |Y_n| > \frac{1}{N}) \\ &\leq P(|X_n| > \varepsilon N) + P(|Y_n| > \frac{1}{N}) \end{aligned}$$

$$\text{Since } |X_n Y_n| > \varepsilon \cap |Y_n| \leq \frac{1}{N}$$

$$\Rightarrow |X_n| > \frac{\varepsilon}{|Y_n|} \cap N \leq \frac{1}{|Y_n|}$$

$$\Rightarrow |X_n| > \varepsilon N$$

(7)

Take the limit as  $n \rightarrow \infty$

$$P(|X| > \varepsilon N) = 0$$

$$\text{Since } X_n \xrightarrow{d} X$$

$$\text{So } \lim_{n \rightarrow \infty} P(|X_n Y_n| > \varepsilon) \leq \underbrace{P(|X| > \varepsilon N)}_{\text{This can be made as close to 0 as desired by } N \text{ sufficiently large}} \quad \forall \varepsilon, N$$

$$\begin{aligned} \therefore X_n Y_n &\xrightarrow{P} 0 \\ \Rightarrow X_n Y_n &\xrightarrow{d} 0 \end{aligned}$$

To finish (b), what if  $c \neq 0$ ?

(8)

$$\begin{array}{ccccc} \text{Write } X_n Y_n & = & X_n (Y_n - c) & + & c X_n \\ & & \downarrow d & \downarrow P & \downarrow d \\ & & X & 0 & cX \\ & & \underbrace{\phantom{X}}_{\downarrow P} & & \\ & & 0 & & \end{array}$$

By Part (a), this converges in distribution to  $cX$ .

(9)

Theorem: See the regularity conditions in Sec 10.6

Let  $\hat{\theta}$  be the MLE of  $\theta$ .

$$\text{Then } \sqrt{n}(\hat{\theta} - \theta) \xrightarrow{D} N(0, \frac{1}{I(\theta)})$$

That is, MLEs are asymptotically unbiased, efficient, and normally distributed.

(10)

Proof: 1<sup>st</sup> order Taylor series with remainder term:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x^*) \frac{(x - x_0)^2}{2}$$

where  $x^*$  lies between  $x$  and  $x_0$ .

Do this for  $l'(\hat{\theta})$ , expanded around  $\theta$

$l(\theta)$  is the log-likelihood function

$$l'(\hat{\theta}) = l'(\theta) + l''(\theta)(\hat{\theta} - \theta) + l'''(\theta^*) \frac{(\hat{\theta} - \theta)^2}{2}$$

where  $\theta^*$  lies between  $\hat{\theta}$  and  $\theta$

$$-l'(\theta) = (\hat{\theta} - \theta) \left[ l''(\theta) + l'''(\theta^*) \frac{\hat{\theta} - \theta}{2} \right] \quad (11)$$

$$\hat{\theta} - \theta = \frac{-l'(\theta)}{l''(\theta) + l'''(\theta^*) \frac{\hat{\theta} - \theta}{2}}$$

$$\sqrt{n}(\hat{\theta} - \theta) = \frac{-l'(\theta)}{l''(\theta) + l'''(\theta^*) \frac{\hat{\theta} - \theta}{2}} \quad \sqrt{n} \frac{1}{\sqrt{n}} \frac{1}{\frac{1}{\sqrt{n}}}$$

$$\sqrt{n}(\hat{\theta} - \theta) = \frac{-\frac{1}{\sqrt{n}} l'(\theta)}{\frac{1}{n} l''(\theta) + \frac{1}{n} l'''(\theta^*) \frac{\hat{\theta} - \theta}{2}} \quad (12)$$