

Stat 563
5-9-19

From 4/30:

$$V[\hat{\beta}_1] = \sigma^2 \sum_{i=1}^n c_i^2, \quad \sum c_i x_i = 1, \quad \sum c_i = 0 \quad (1)$$

if we wanted to minimize $\sum c_i^2$

The Lemma said

$$\frac{(\sum a_i v_i)^2}{\sum a_i^2 / w_i} \text{ is maximized when}$$

$$a_i = k w_i (v_i - \bar{v}_w)$$

$$\text{where } \bar{v}_w = \frac{\sum w_i v_i}{\sum w_i}$$

In the lemma, let $v_i = x_i$, $a_i = c_i$, $w_i \equiv 1$ (2)

$$\text{Then } \frac{(\sum c_i x_i)^2}{\sum c_i^2} \text{ is maximized when}$$
$$c_i = k(x_i - \bar{x})$$

$$\text{But } \sum c_i x_i = 1$$

$$\text{So } \sum c_i^2 \text{ is minimized when } c_i = k(x_i - \bar{x})$$

$$\sum c_i x_i = 1 \Rightarrow \sum k(x_i - \bar{x})x_i = 1$$

$$k \underbrace{\sum x_i (x_i - \bar{x})}_{S_{xx}} = 1$$

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$$\text{So } k = \frac{1}{S_{xx}}$$

$$\Rightarrow c_i = \frac{1}{S_{xx}} (x_i - \bar{x})$$

$$\text{And } \hat{\beta}_1 = \sum c_i y_i = \sum \frac{1}{S_{xx}} (x_i - \bar{x}) y_i$$

$$= \frac{1}{S_{xx}} \underbrace{\sum (x_i - \bar{x}) y_i}_{S_{xy}} = \frac{S_{xy}}{S_{xx}}$$

$\therefore \hat{\beta}_1$ is the BLUE of β_1

We know $E[\hat{\beta}_1] = \beta_1$

(4)

$$V[\hat{\beta}_1] = \sigma^2 \sum c_i^2 \quad \begin{matrix} c_i = k(x_i - \bar{x}) \\ k = \frac{1}{S_{xx}} \end{matrix}$$

$$= \sigma^2 \frac{1}{S_{xx}^2} \underbrace{\sum (x_i - \bar{x})^2}_{S_{xx}}$$

$$= \frac{\sigma^2}{S_{xx}}$$

$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$. This is also a linear estimator.

$$= \frac{1}{n} \sum y_i - \frac{S_{xy}}{S_{xx}} \bar{x}$$

$$= \frac{1}{n} \sum y_i - \frac{1}{S_{xx}} \bar{x} \sum (x_i - \bar{x}) y_i \quad (5)$$

$$= \sum_{i=1}^n \left(\frac{1}{n} - \frac{\bar{x}(x_i - \bar{x})}{S_{xx}} \right) y_i \quad \therefore \underline{\text{linear}}$$

$$E(\hat{\beta}_0) = E[\bar{y} - \hat{\beta}_1 \bar{x}] = E[\bar{y}] - \bar{x} E[\hat{\beta}_1]$$

$$= E\left[\frac{1}{n} \sum_{i=1}^n (\beta_0 + \beta_1 x_i + \varepsilon_i) \right] - \bar{x} \beta_1$$

$$= \beta_0 + \beta_1 \bar{x} + 0 - \beta_1 \bar{x} = \beta_0$$

$$V[\hat{\beta}_0] = V\left[\sum_{i=1}^n \left(\frac{1}{n} - \frac{\bar{x}(x_i - \bar{x})}{S_{xx}} \right) y_i \right] \quad (6)$$

$$= \sum_{i=1}^n \left(\frac{1}{n} - \frac{\bar{x}(x_i - \bar{x})}{S_{xx}} \right)^2 \underbrace{V[y_i]}_{\sigma^2}$$

$$= \sigma^2 \sum_{i=1}^n \left(\frac{1}{n^2} + \frac{\bar{x}^2 (x_i - \bar{x})^2}{S_{xx}^2} - \frac{2\bar{x}(x_i - \bar{x})}{n S_{xx}} \right)$$

$$= \sigma^2 \left[\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}^2} \underbrace{\sum (x_i - \bar{x})^2}_{S_{xx}} - \frac{2\bar{x}}{n S_{xx}} \underbrace{\sum (x_i - \bar{x})}_0 \right]$$

$$= \sigma^2 \left[\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} \right]$$

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Since $\hat{\beta}_1$ and $\hat{\beta}_0$ are both linear combinations of the y_i 's and the y_i 's were indep normal random variables, they are both normally distributed.

$$\therefore \hat{\beta}_1 \sim N\left(\beta_1, \frac{\sigma^2}{S_{xx}}\right)$$

$$\text{and } \hat{\beta}_0 \sim N\left(\beta_0, \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)\right)$$

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Our MLE of σ^2 was

$$\hat{\sigma}^2 = \frac{S_{yy} - \frac{S_{xy}^2}{S_{xx}}}{n}$$

Find $E[\hat{\sigma}^2]$

$$E[S_{yy}] = E\left[\sum y_i^2 - n\bar{y}^2\right]$$

$$\begin{aligned} E[y_i^2] &= V[y_i] + (E[y_i])^2 \\ &= \sigma^2 + (\beta_0 + \beta_1 x_i)^2 \end{aligned}$$

$$E[\bar{y}^2] = V[\bar{y}] + (E[\bar{y}])^2 = \frac{\sigma^2}{n} + (\beta_0 + \beta_1 \bar{x})^2$$

$$\left[\begin{aligned} S_{yy} &= \sum (y_i - \bar{y})^2 \\ &= \sum \bar{y} (y_i - \bar{y}) \\ &= \sum y_i^2 - \frac{(\sum y_i)^2}{n} \end{aligned} \right.$$

$$\text{Now } E[S_{yy}] = \sum (\sigma^2 + (\beta_0 + \beta_1 x_i)^2)$$

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$$= n \left[\frac{\sigma^2}{n} + (\beta_0 + \beta_1 \bar{x})^2 \right]$$

$$= n\sigma^2 + n\beta_0^2 + \beta_1^2 \sum x_i^2 + 2\beta_0 \beta_1 \sum x_i$$

$$= n\sigma^2 + n\beta_0^2 + n\beta_1^2 \bar{x}^2 + 2n\beta_0 \beta_1 \bar{x}$$

$$= (n-1)\sigma^2 + \beta_1^2 \underbrace{\left[\sum x_i^2 - n\bar{x}^2 \right]}_{S_{xx}}$$

$$E[S_{yy}] = (n-1)\sigma^2 + \beta_1^2 S_{xx}$$

$$E\left[\frac{S_{xy}^2}{S_{xx}} \right] = E\left[\frac{S_{xy}^2}{S_{xx}^2} S_{xx} \right]$$

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$$= S_{xx} E[\hat{\beta}_1^2]$$

$$= S_{xx} \left[V[\hat{\beta}_1] + (E[\hat{\beta}_1])^2 \right]$$

$$= S_{xx} \left[\frac{\sigma^2}{S_{xx}} + \beta_1^2 \right]$$

$$E[\hat{\sigma}^2] = E\left[\frac{S_{yy} - \frac{S_{xy}^2}{S_{xx}}}{n} \right]$$

$$= \frac{1}{n} \left[(n-1)\sigma^2 + \beta_1^2 S_{xx} - \sigma^2 - \beta_1^2 S_{xx} \right] = \frac{n-2}{n} \sigma^2$$

$$\text{Define } S^2 = \frac{1}{n-2} \hat{\sigma}^2 = \frac{S_{yy} - \frac{S_{xy}^2}{S_{xx}}}{n-2}$$

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$$\text{Then } E[S^2] = \sigma^2$$

Take-home midterm make-up exam
is due in class Tuesday 5-14