Stat 563
6-4-19

From last time:

①

$$P(\sqrt{n}(m_n - M) \le a) = P\left(m_n \le M + \frac{a}{\sqrt{n}}\right)$$
$$= P\left(\sum_{i=1}^{n} Y_i \ge \frac{n+1}{2}\right)$$
$$= P\Bigg(\underbrace{\frac{\sum Y_i - np_n}{\sqrt{np_nq_n}}}_{\downarrow \mathcal{D} \atop N(0,1)} \ge \frac{\frac{n+1}{2} - np_n}{\sqrt{np_nq_n}}\Bigg)$$

Since $\sum Y_i \sim \text{Binom}(n, p_n)$

where $p_n = P\left(X_i \le M + \frac{a}{\sqrt{n}}\right)$

②

$$\lim_{n\to\infty} \frac{\frac{n+1}{2} - np_n}{\sqrt{np_nq_n}} = \lim_{n\to\infty} \frac{n\left(\frac{1}{2} - p_n\right) + \frac{1}{2}}{\sqrt{np_nq_n}}$$

$$= \lim_{n\to\infty} \frac{n\left(\frac{1}{2} - p_n\right)}{\sqrt{np_nq_n}} + 0$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}\left(\frac{1}{2} - p_n\right)}{\sqrt{p_nq_n}} = \lim_{n\to\infty} \frac{1}{\sqrt{p_nq_n}} \cdot \lim_{n\to\infty} \sqrt{n}\left(\tfrac{1}{2} - p_n\right)$$

$$= 2 \lim_{n\to\infty} \frac{\frac{1}{2} - p_n}{1/\sqrt{n}} = 2 \lim_{n\to\infty} \frac{-\frac{dp_n}{dn}}{-\frac{1}{2} n^{-3/2}}$$

since $p_n \to \frac{1}{2}$

③

what is $\frac{dp_n}{dn}$? $\quad p_n = P(X_i \le M + \frac{a}{\sqrt{n}})$

$$= F(M + \frac{a}{\sqrt{n}})$$

$$\frac{dp_n}{dn} = f(M + \frac{a}{\sqrt{n}}) \cdot a\left(-\frac{1}{2}\right) n^{-3/2}$$

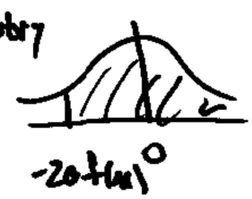
$$2 \lim_{n\to\infty} \cancel{2n^{3/2}} f(M + \frac{a}{\sqrt{n}})\, a\left(-\frac{1}{\cancel{2}}\right)\cancel{n^{-3/2}} = -2f(M)a$$

④

So $\lim_{n\to\infty} P[\sqrt{n}(M_n - M) \le a]$

$$= P[Z \ge -2af(M)] \quad \text{where } Z \sim N(0,1)$$

$$= P[Z \le 2af(M)] \quad \text{by symmetry}$$



Let $z = 2af(M)$

$$a = \frac{z}{2f(M)}$$

$$\lim_{n\to\infty} P[\sqrt{n}(M_n - M) \le \frac{z}{2f(M)}] = P[Z \le z]$$

$$= \Phi(z)$$

⑤

$$\lim_{n\to\infty} P\left[\frac{m_n - M}{\frac{1}{2\sqrt{n}f(M)}} \le z \right] = \Phi(z)$$

That is, $\dfrac{m_n - M}{\left(\frac{1}{2\sqrt{n}f(M)}\right)} \xrightarrow{\mathcal{D}} N(0,1)$

For large n, $m_n \approx N\left(M, \dfrac{1}{4n f^2(M)}\right)$

The sample median is $\begin{cases} \text{asymptotically normal} \\ \text{" unbiased} \\ \text{consistent} \end{cases}$

⑥

Consider the specific case of the normal distribution. We have $\mu = M$

$$f(M) = f(\mu) = \frac{1}{\sigma\sqrt{2\pi}} \qquad f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

For large n, $m_n \approx N\left(\mu, \dfrac{1}{4n\frac{1}{\sigma^2 2\pi}}\right)$

$$\approx N\left(\mu, \frac{\pi\sigma^2}{2n}\right)$$

Compare this to $\bar{x} \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$

(approximate)
relative efficiency of m_n to $\bar{x}$

$$= \frac{V(\bar{x})}{V(m_n)} = \frac{\sigma^2/n}{\frac{\pi}{2}\frac{\sigma^2}{n}} = \frac{2}{\pi}$$

The asymptotic rel. eff. of m_n to $\bar{x}$ is $\frac{2}{\pi} \approx .64$

C.I. for μ based on $\bar{x}$:

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

C.I. for μ based on m_n:

$$m_n \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\sqrt{\frac{\pi}{2}}$$