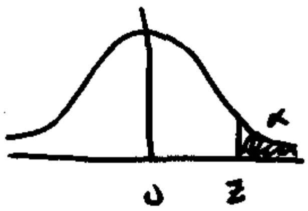
Stat 563
4-11-19
①

Additional comments about the example from last time:

$H_0: \mu \le \mu_0$
$H_1: \mu > \mu_0$



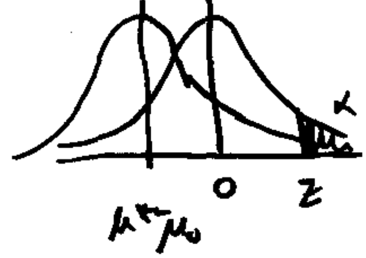
Our solution was to reject H_0 when

$$\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \ge z$$

What if the true mean was actually $\mu^* < \mu_0$?

Then $E[\bar{x}] = \mu^*$

So $E[\bar{x}-\mu_0] = \mu^* - \mu_0 < 0$



So the new tail area to the right of z is $< \alpha$. ②

Now $\sup_{\Omega_0} P(\Lambda \le c) = \alpha$

Defn: The power function of a test is

$$\pi(\theta) = P(\text{Reject } H_0 \mid \theta) \quad \text{for } \theta \in \Omega_0^c$$

Power $= 1 - \text{Prob.}(\text{Type II error})$

Note: this is a function of θ

③

Defn. Let C be a class of hypothesis tests of size α for

$H_0: \theta \in \Omega_0$

$H_1: \theta \notin \Omega_0$

$\alpha = \text{Prob}(\text{Type I})$
$= $ "size"
$= $ "level of significance"

Then a test in C is (UMP) uniformly most powerful if its power function $\pi(\theta)$ is greater than or equal to $\pi'(\theta)$, the power function for any other test in C, $\forall\, \theta \in \Omega_0^c$.

④

Neyman-Pearson Theorem

Suppose we are testing $H_0: \theta = \theta_0$
$H_1: \theta = \theta_1$

(i.e. $\Omega = \{\theta_0, \theta_1\}$, $\Omega_0 = \{\theta_0\}$)

And our rejection region is

$$R = \{\vec{x} : \Lambda \leq c\}$$

and c is chosen so that $P_{H_0}(\vec{x} \in R) = \alpha$.

Then this test is MP.

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Proof: Let A be the rejection region for any other test of size α

We need to show

$$P(\vec{x} \in R \mid \theta_1) \geq P(\vec{x} \in A \mid \theta_1),$$

i.e. $$\int_R f(\vec{x} \mid \theta_1)\, d\vec{x} \geq \int_A f(\vec{x} \mid \theta_1)\, d\vec{x}$$

$$= \int_R L(\theta_1)\, d\vec{x} \geq \int_A L(\theta_1)\, d\vec{x}$$

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Consider $$\int_R L(\theta_1)\, d\vec{x} - \int_A L(\theta_1)\, d\vec{x} = \bigstar$$

We need to show that $\bigstar \geq 0$

$$\bigstar = \cancel{\int_{R \cap A} L(\theta_1)\, d\vec{x}} + \int_{R \cap A^c} L(\theta_1)\, d\vec{x} - \left[\cancel{\int_{A \cap R} L(\theta_1)\, d\vec{x}} + \int_{A \cap R^c} L(\theta_1)\, d\vec{x} \right]$$

$$= \int_{R \cap A^c} L(\theta_1)\, d\vec{x} - \int_{A \cap R^c} L(\theta_1)\, d\vec{x}$$

⑦

Suppose $\vec{x} \in R$. Then $\Lambda \le c$

$$\frac{L(\theta_0)}{L(\theta_1)} \le c$$

$$L(\theta_1) \ge \frac{1}{c} L(\theta_0)$$

So $$\int_{R \cap A^c} L(\theta_1) d\vec{x} \ge \frac{1}{c} \int_{R \cap A^c} L(\theta_0) d\vec{x}$$

Suppose $\vec{x} \notin R$. Then $\Lambda > c$

So $L(\theta_1) < \frac{1}{c} L(\theta_0)$

⑧

Now $$\int_{A \cap R^c} L(\theta_1) d\vec{x} < \frac{1}{c} \int_{A \cap R^c} L(\theta_0) d\vec{x}$$

So $$\star \ge \frac{1}{c} \int_{R \cap A^c} L(\theta_0) d\vec{x} - \frac{1}{c} \int_{A \cap R^c} L(\theta_0) d\vec{x}$$

$$= \frac{1}{c} \left[\int_{R \cap A^c} L(\theta_0) d\vec{x} + \int_{R \cap A} L(\theta_0) d\vec{x} - \int_{R \cap A} L(\theta_0) d\vec{x} - \int_{A \cap R^c} L(\theta_0) d\vec{x} \right]$$

$$= \frac{1}{c}\left[\int_R L(\theta_0)\,d\vec{x} - \int_A L(\theta_0)\,d\vec{x} \right]$$

$$= \frac{1}{c}\left[\int_R f(\vec{x}\,|\,\theta_0)\,d\vec{x} - \int_A f(\vec{x}\,|\,\theta_0)\,d\vec{x} \right]$$

$$= \frac{1}{c}\left[P(\vec{x} \in R \,|\, \theta_0) - P(\vec{x} \in A \,|\, \theta_0) \right]$$

$$= \frac{1}{c}\left[\alpha - \alpha\right] = 0$$

$\therefore \bigstar \geq 0$

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Recap: For testing $H_0: \theta = \theta_0$, $H_1: \theta = \theta_1$, the L.R.T. is most powerful.

Consider the simple null vs. composite alternative hypotheses.

$H_0: \theta = \theta_0$
$H_1: \theta > \theta_0$

Neyman-Pearson Thm says that LRT is most powerful for $H_0: \theta = \theta_0$, $H_1: \theta = \theta_1$ for any $\theta_1 > \theta_0$

(11)

If it happens that the structure of the test does not involve the actual value of θ_1, then the LRT would be UMP.

<u>Example</u>: $X_1, \ldots, X_{10} \sim$ iid Poisson(θ)

$$f(x|\theta) = \frac{e^{-\theta}\theta^x}{x!}$$

$x = 0, 1, 2, \ldots$

$H_0: \theta = .1$

$H_1: \theta > .1$

Consider $H_0: \theta = .1$ where $\theta_1 > .1$
$H_1: \theta = \theta_1$

$$L(\theta) = \frac{e^{-10\theta}\theta^{\Sigma x_i}}{\Pi(x_i!)}$$

(12)

$$\Lambda = \frac{L(\theta_0)}{L(\theta_1)} = \frac{\dfrac{e^{-10(.1)}(.1)^{\Sigma x_i}}{\cancel{\Pi(x_i!)}}}{\dfrac{e^{-10(\theta_1)}\theta_1^{\Sigma x_i}}{\cancel{\Pi(x_i!)}}}$$

$$\Lambda = \frac{e^{-1}(.1)^{\Sigma x_i}}{e^{-10\theta_1}\theta_1^{\Sigma x_i}} \overset{set}{\leq} c$$

$$\frac{1}{(10\theta_1)^{\Sigma x_i}} \leq c'$$

$$\sum x_i \ln\left(\frac{1}{10\theta_1}\right) \le c''$$ (13)

$$\sum x_i \ge c'''$$

Under H_0, $\sum_{i=1}^{10} x_i \sim \text{Poisson}(1)$

Choose c''' so that $P(\sum x_i \ge c''') = \alpha$

Note that this no longer involves θ_1, so we have a test that is most powerful $\forall\, \theta_1 \notin \Omega_0$, so it is UMP.

(14)

Table of Poisson(1) distribution:

k	$P(X \ge k)$
0	1
1	.632
2	.264
3	.08
4	.019

Question: how do you construct a test with $\alpha = .05$?

Let $p = \dfrac{.05 - .019}{.08 - .019} = \dfrac{31}{61}$

Decision rule: If $\sum x_i \ge 4$, Reject H_0

If $\sum x_i \le 2$, Fail to reject H_0.

If $\sum x_i = 3$, flip a coin whose "heads" prob is p.

Reject H_0 if you get "heads" (15)

$$P_{H_0}(\text{Reject } H_0) = P_{H_0}(\Sigma x_i \geq 4) + p\,P(\Sigma x_i = 3)$$

$$= .019 + \frac{31}{61}(.08 - .019) = .05$$

HW #2 due 4/18

Chp 8 # 5, 17, 19

Stat 563 HW #2

8.5 A random sample, $X_1, \ldots, X_n$, is drawn from a Pareto population with pdf

$$f(x|\theta,\nu) = \frac{\theta\nu^\theta}{x^{\theta+1}} I_{[\nu,\infty)}(x), \quad \theta > 0, \quad \nu > 0.$$

(a) Find the MLEs of θ and ν.

(b) Show that the LRT of

$$H_0: \theta = 1,\ \nu \text{ unknown}, \qquad \text{versus} \qquad H_1: \theta \neq 1,\ \nu \text{ unknown},$$

has critical region of the form $\{\mathbf{x}: T(\mathbf{x}) \leq c_1 \text{ or } T(\mathbf{x}) \geq c_2\}$, where $0 < c_1 < c_2$ and

$$T = \log\left[\frac{\prod_{i=1}^{n} X_i}{(\min_i X_i)^n}\right].$$

(c) Show that, under H_0, $2T$ has a chi squared distribution, and find the number of degrees of freedom. (*Hint:* Obtain the joint distribution of the $n-1$ nontrivial terms $X_i/(\min_i X_i)$ conditional on $\min_i X_i$. Put these $n-1$ terms together, and notice that the distribution of T given $\min_i X_i$ does not depend on $\min_i X_i$, so it is the unconditional distribution of T.)

8.17 Suppose that $X_1, \ldots, X_n$ are iid with a beta$(\mu, 1)$ pdf and $Y_1, \ldots, Y_m$ are iid with a beta$(\theta, 1)$ pdf. Also assume that the Xs are independent of the Ys.

(a) Find an LRT of $H_0: \theta = \mu$ versus $H_1: \theta \neq \mu$.

(b) Show that the test in part (a) can be based on the statistic

$$T = \frac{\Sigma \log X_i}{\Sigma \log X_i + \Sigma \log Y_i}.$$

(c) Find the distribution of T when H_0 is true, and then show how to get a test of size $\alpha = .10$.

8.19 The random variable X has pdf $f(x) = e^{-x}, x > 0$. One observation is obtained on the random variable $Y = X^\theta$, and a test of $H_0: \theta = 1$ versus $H_1: \theta = 2$ needs to be constructed. Find the UMP level $\alpha = .10$ test and compute the Type II Error probability.