Stat 523
4-18-19
1

Additional comment on the Bayesian hypothesis test from last time.

Instead of rejecting H_0 when $P(H_1) \geq \frac{1}{2}$,
consider this:

Suppose the cost of a Type I error is C_1
" " " " " " II " " C_2

These are the losses

Recall that the risk is the expected loss.

$$\text{Risk} = C_1 P(\text{Type I}) + C_2 P(\text{Type II})$$

2

$$\text{Risk} = C_1 P(H_0 \text{ is true} \cap \text{Reject } H_0) + C_2 P(H_1 \text{ is true} \cap \text{Fail to reject } H_0)$$

$$= C_1 P(H_0 \text{ is true}) P(\text{Reject } H_0 \mid H_0 \text{ is true}) + C_2 P(H_1 \text{ is true}) P(\text{Fail to reject } H_0 \mid H_1 \text{ is true})$$

Then the Bayesian rule will be adjusted to minimize this risk.

Power & p-values

3

Recall: $\alpha = \text{Prob}(\text{Rej } H_0 \mid H_0 \text{ true})$

$\beta = \text{Prob}(\text{Fail to rej. } H_0 \mid H_1 \text{ is true})$

The power of the test is $1-\beta =$

$\text{Prob}(\text{rej } H_0 \mid H_1 \text{ is true})$

Example: $H_0 : \mu = \mu_0$ $\quad X_1, \ldots, X_n \sim \text{iid } N(\mu, \sigma^2)$

$H_1 : \mu > \mu_0$ $\quad \sigma^2$ known

L.R.T. said to reject H_0 when $\bar{x} > c$

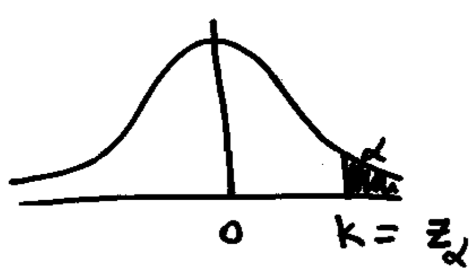
4

Given that H_0 is true,

$$\bar{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)$$

$$\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$$

$\bar{X} > c$ is equivalent to $\dfrac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > k$



⑤

Our rejection rule is $\dfrac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > Z_\alpha$

Select a specific $\mu_1 > \mu_0$

$$\text{Power} = P(\text{Reject } H_0 \mid \mu = \mu_1)$$

$$= P\left(\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > Z_\alpha \,\middle|\, \mu = \mu_1\right)$$

$$= P\left(\bar{X} > \mu_0 + Z_\alpha \frac{\sigma}{\sqrt{n}} \,\middle|\, \mu = \mu_1\right)$$

$$= P\left(\frac{\bar{X}-\mu_1}{\sigma/\sqrt{n}} > \frac{\mu_0-\mu_1+Z_\alpha \sigma/\sqrt{n}}{\sigma/\sqrt{n}} \,\middle|\, \mu = \mu_1\right)$$

⑥

$$= P\left(\frac{\bar{X}-\mu_1}{\sigma/\sqrt{n}} > Z_\alpha - \frac{(\mu_1-\mu_0)\sqrt{n}}{\sigma} \,\middle|\, \mu = \mu_1\right)$$

$$\text{Power} = P\left(Z > Z_\alpha - \frac{\delta\sqrt{n}}{\sigma}\right) \quad \text{where } \delta = \mu_1 - \mu_0$$

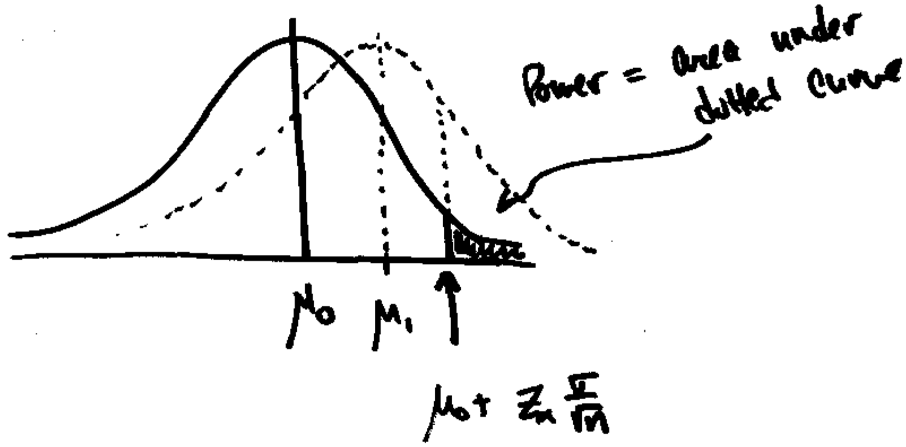
Note: as $n \uparrow$ Power $\uparrow$

as $\mu_1 \uparrow$ Power $\uparrow$

as $\alpha \downarrow$ Power $\downarrow$

If σ is large, power will be small

7

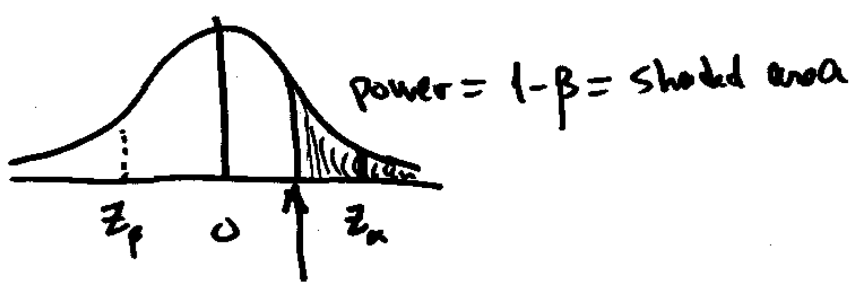


__Defn__: A test whose power never drops below α is called __unbiased__

8

__Problem__: For a specific δ, find the sample size necessary to achieve a given power.

$$\text{Power} = P\left(Z > z_\alpha - \frac{\delta\sqrt{n}}{\sigma}\right)$$



$$z_\alpha - \frac{\delta\sqrt{n}}{\sigma} = z_{1-\beta} = -z_\beta$$

(9)

So set $Z_\alpha - \frac{\delta\sqrt{n}}{\sigma} = -Z_\beta$ + solve for n

$$Z_\alpha + Z_\beta = \frac{\delta\sqrt{n}}{\sigma}$$

$$n = \left[\frac{(Z_\alpha + Z_\beta)\sigma}{\delta}\right]^2$$

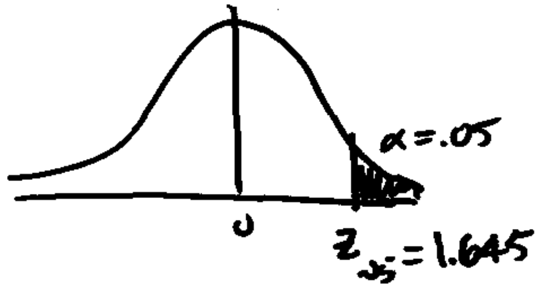
(10)

<u>Defn</u>: The <u>p-value</u> is the probability that the test statistic $W(\vec{X})$ is more extreme than the observed value $w(\vec{x})$, given H_0.

<u>Example, continued</u>

$H_0: \mu = 6$ $\quad \alpha = .05$
$H_1: \mu > 6$

Suppose we observe $\bar{x} = 6.7$, $n = 18$, $\sigma = 2$



$$\text{Test stat} = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{6.7 - 6}{2/\sqrt{18}} = 1.48$$

we fail to reject H_0, since $T.S. < z_\alpha$

12

Alternative way of expressing the rejection rule for <u>any</u> hypothesis test:

Reject H_0 if p-value $< \alpha$.

HW # 3 due 4/25

Chp 8 # 28, 33

8.28 Let $f(x|\theta)$ be the logistic location pdf

$$f(x|\theta) = \frac{e^{(x-\theta)}}{(1+e^{(x-\theta)})^2}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

(a) Show that this family has an MLR.
(b) Based on one observation, X, find the most powerful size α test of $H_0: \theta = 0$ versus $H_1: \theta = 1$. For $\alpha = .2$, find the size of the Type II Error.
(c) Show that the test in part (b) is UMP size α for testing $H_0: \theta \le 0$ versus $H_1: \theta > 0$. What can be said about UMP tests in general for the logistic location family?

8.33 Let $X_1, \ldots, X_n$ be a random sample from the uniform$(\theta, \theta + 1)$ distribution. To test $H_0: \theta = 0$ versus $H_1: \theta > 0$, use the test

$$\text{reject } H_0 \text{ if } Y_n \ge 1 \text{ or } Y_1 \ge k,$$

where k is a constant, $Y_1 = \min\{X_1, \ldots, X_n\}, Y_n = \max\{X_1, \ldots, X_n\}$.

(a) Determine k so that the test will have size α.
(b) Find an expression for the power function of the test in part (a).
(c) Prove that the test is UMP size α.
(d) Find values of n and k so that the UMP .10 level test will have power at least .8 if $\theta > 1$.