

## The Rao-Blackwell Theorem

Stat 563  
4-4-19

Let  $X_1, \dots, X_n \sim \text{iid } f(x|\theta)$

①

Let  $T$  be a sufficient statistic for  $\theta$ .

Let  $W$  be an unbiased estimator of  $\tau(\theta)$ .

Let  $\phi(T) = E[W|T]$

Then  $\phi(T)$  is an unbiased estimator of  $\tau(\theta)$

and  $V[\phi(T)] \leq V[W] \quad \forall \theta$ .

②

Proof: 
$$\begin{aligned} E[\phi(T)] &= E[E[W|T]] \\ &= E[W] = \tau(\theta) \end{aligned}$$

$$V[W] = \underbrace{E[V[W|T]]}_{\geq 0} + \underbrace{V[E[W|T]]}_{V[\phi(T)]}$$

$$V[W] \geq V[\phi(T)]$$

Example:  $X_1, X_2 \sim \text{iid } \frac{1}{\theta} e^{-x/\theta} \quad x > 0$  (3)

$$\text{Let } W = X_1 \quad E[W] = \theta \quad V[W] = \theta^2$$

Find a sufficient statistic for  $\theta$ .

$$\begin{aligned} L(\theta) &= \frac{1}{\theta} e^{-\frac{x_1}{\theta}} \frac{1}{\theta} e^{-\frac{x_2}{\theta}} \\ &= \frac{1}{\theta^2} e^{-\frac{1}{\theta}(x_1 + x_2)} \cdot 1 \end{aligned}$$

So  $T = X_1 + X_2$  is sufficient for  $\theta$ .

Find  $E[W|T]$

(4)

Start with the joint distribution of  $W, T$ .

$$\begin{array}{ll|l} W = X_1 & X_1 = W & X_1 > 0 \\ T = X_1 + X_2 & X_2 = T - W & X_2 > 0 \\ & & \hline & & W > 0 \\ & & t - w > 0 \\ & & t > w \end{array}$$
$$|J| = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

$$\begin{aligned} g(w, t) &= f(x_1, x_2) \cdot 1 = \frac{1}{\theta^2} e^{-\frac{1}{\theta}(x_1 + x_2)} \\ &= \frac{1}{\theta^2} e^{-t/\theta} \quad 0 < w < t \end{aligned}$$

$$g(w|t) = \frac{g(w,t)}{h(t)} = \frac{\frac{1}{\theta^2} e^{-t/\theta}}{\frac{1}{\theta^2} t e^{-t/\theta}} \quad (5)$$

Because  $T \sim \text{Gamma}(\alpha=2, \beta=\theta)$

$$= \frac{1}{t}, \quad 0 < w < t$$

That is,  $W|T \sim \text{Unif}(0, T)$

$$\text{So } E[W|T] = \frac{0+T}{2} = \frac{T}{2} = \frac{X_1+X_2}{2} = \bar{X}$$

Note  $E[\bar{X}] = \theta$  and  $V[\bar{X}] = \frac{\theta^2}{2}$

A simpler approach might be to

- ① Find a sufficient statistic for  $\theta$ , called  $T$
- ② Find a function of  $T$  that is an unbiased estimator of  $\theta$ .

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Defn:  $W$  is a minimum variance unbiased (MVUB) estimator of  $\theta$  if  $E[W] = \theta$  and its variance is less than or equal to the variance of any other unbiased estimator of  $\theta$ ,  $\forall \theta$ .

⑦

Theorem: If  $W$  is a MVUE of  $\theta$ ,  
then  $W$  is unique.

Proof: Let  $W'$  also be MVUE of  $\theta$ .

$$\text{Let } W^* = \frac{1}{2}(W + W')$$

$$E[W^*] = \frac{1}{2}[E(W) + E(W')] = \theta$$

$$\begin{aligned} V[W^*] &= \frac{1}{4} [ \underline{V[W] + V[W']} + 2 \text{Cov}(W, W') ] \\ &= \frac{1}{4} [ 2V[W] + 2 \text{Cov}(W, W') ] \end{aligned}$$

$$V[W^*] = \frac{1}{2}V[W] + \frac{1}{2}\text{Cov}(W, W')$$

⑧

$$\text{Recall } \rho^2 = \frac{\text{Cov}^2(W, W')}{V[W]V[W']} \leq 1$$

$$\text{So } \text{Cov}^2(W, W') \leq (V[W])^2$$

$$|\text{Cov}(W, W')| \leq V[W] \quad \star$$

$$\text{Now } V[W^*] \leq \frac{1}{2}V[W] + \frac{1}{2}V[W]$$

$$V[W^*] \leq V[W]$$

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If, for any  $\theta$ ,  $V[W^*] < V[W]$ ,  
our assumption that  $W$  was MVUE  
would be violated.

$$\therefore V[W^*] = V[W] \quad \forall \theta$$

Equivalently,  $\rho = \text{Cor}(W, W') = 1$

Cauchy-Schwarz says  $W' = aW + b$

By  $\star$ ,  $\text{Cov}(W, W') = V[W]$

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 $\text{Cov}(W, W') = \text{Cov}(W, aW + b)$

$$= a \text{Cov}(W, W) + \text{Cov}(W, b)$$

$$= a V[W] + 0$$

$$\therefore a = 1$$

$$\text{So } W' = W + b$$

$$E[W'] = E[W] + b$$

$$\theta = \theta + b \quad \therefore b = 0$$

$$\therefore W = W'$$

## The Lehmann-Scheffé Theorem

(11)

Let  $T$  be a complete sufficient statistic for  $\theta$ . If there exists  $W$ , a function of  $T$ , that is an unbiased estimator of  $\tau(\theta)$ , then  $W$  is the unique MVUE of  $\tau(\theta)$ .

Proof: Assume  $W$  exists.

Let  $W'$  be another unbiased estimator of  $\tau(\theta)$ .

By the Rao-Blackwell Theorem,

(12)

$E[W'|T]$  will be an unbiased estimator of  $\tau(\theta)$  and its variance will be  $\leq V[W']$ .

Also note that  $E[W'|T]$  is a function of  $T$ .

Let  $\phi(T)$  and  $\psi(T)$  be functions of  $T$  that are both unbiased estimators of  $\tau(\theta)$ .

$$\begin{aligned} E[\varphi(T) - \psi(T)] &= E[\varphi(T)] - E[\psi(T)] \\ &= \tau(\theta) - \tau(\theta) = 0 \quad \forall \theta \end{aligned} \quad (13)$$

By completeness, this implies  
 $\varphi(T) - \psi(T) = 0$

So  $W$  and  $E[W'|T]$  must be equal.

$$\text{Then } V[W] = V[E[W'|T]] \leq V[W']$$

So  $W$  is MVUE and is unique.

HW #1 due Thurs 4/11

Chp 7 # 12, 48, 59

(14)

**7.12** Let  $X_1, \dots, X_n$  be a random sample from a population with pmf

$$P_\theta(X = x) = \theta^x(1 - \theta)^{1-x}, \quad x = 0 \text{ or } 1, \quad 0 \leq \theta \leq \frac{1}{2}.$$

- (a) Find the method of moments estimator and MLE of  $\theta$ .
- (b) Find the mean squared errors of each of the estimators.
- (c) Which estimator is preferred? Justify your choice.

**7.48** Suppose that  $X_i, i = 1, \dots, n$ , are iid Bernoulli( $p$ ).

- (a) Show that the variance of the MLE of  $p$  attains the Cramér–Rao Lower Bound.
- (b) For  $n \geq 4$ , show that the product  $X_1 X_2 X_3 X_4$  is an unbiased estimator of  $p^4$ , and use this fact to find the best unbiased estimator of  $p^4$ .

**7.59** Let  $X_1, \dots, X_n$  be iid  $n(\mu, \sigma^2)$ . Find the best unbiased estimator of  $\sigma^p$ , where  $p$  is a known positive constant, not necessarily an integer.