3 types of convergence

Defn:
Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of random variables. The sequence $\{X_n\}$ converges in probability to the random variable X if

$$\forall \varepsilon > 0, \quad \lim_{n\to\infty} P[\,|X_n - X| \geq \varepsilon\,] = 0.$$

(equivalently, $\forall \varepsilon > 0 \quad \lim_{n\to\infty} P[\,|X_n - X| < \varepsilon\,] = 1$)

Write $X_n \xrightarrow{P} X$

② Suppose that X is "degenerate", i.e. $P(X=c)=1$,
then write $X_n \xrightarrow{P} c$

Suppose that $\lim_{n\to\infty} E[X_n] = a$

and $\lim_{n\to\infty} V[X_n] = 0$

then $P[\,|X_n - a| \geq \varepsilon\,]$

$$= P[\,|X_n - E(X_n) + E(X_n) - a| \geq \varepsilon\,]$$

$$\leq P[\,|X_n - E(X_n)| + |E(X_n) - a| \geq \varepsilon\,]$$

③

$$= P\left[\,|X_n - E(X_n)| \geq \varepsilon - |E(X_n) - a|\,\right]$$

$$\leq \frac{Var(X_n)}{\left[\varepsilon - |E(X_n) - a|\right]^2}$$

$$P\left[|X_n - a| \geq \varepsilon\right] \leq \frac{Var(X_n)}{\left[\varepsilon - |E(X_n) - a|\right]^2}$$

Chebyshev's inequality:

$$P\left[|X-\mu| \geq \underbrace{t\sigma}_{c}\right] \leq \frac{1}{t^2}$$

$$P\left[|X-\mu| \geq c\right] \leq \frac{1}{(c/\sigma)^2}$$

$$P\left[|X-\mu| \geq c\right] \leq \frac{\sigma^2}{c^2}$$

$$\therefore \lim_{n\to\infty} P\left[|X_n - a| \geq \varepsilon\right] \leq \lim_{n\to\infty} \frac{Var(X_n)}{\left[\varepsilon - |E(X_n) - a|\right]^2} = \frac{0}{\varepsilon^2} = 0$$

$$\Rightarrow X_n \xrightarrow{P} a$$

④

Summary:

If $\lim_{n\to\infty} E(X_n) = a$ & $\lim_{n\to\infty} V(X_n) = 0$ } 2^{nd} order convergence

then $X_n \xrightarrow{P} a$

Let $X_1, \ldots, X_n, \ldots$ be iid with mean μ & variance σ^2

Consider $\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n, \ldots$

where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

(5)

Then $E[\bar{X}_n] = \mu$ and $V[\bar{X}_n] = \frac{\sigma^2}{n}$

So $\lim_{n\to\infty} E[\bar{X}_n] = \mu$ and $\lim_{n\to\infty} V[\bar{X}_n] = 0$

$\therefore \bar{X}_n \xrightarrow{P} \mu$

This is called the Weak Law of Large Numbers (WLLN)

(6)

Defn: The sequence $\{X_n\}$ converges almost surely to X if

$\forall\, \varepsilon > 0 \quad P\left(\lim_{n\to\infty} |X_n - X| < \varepsilon\right) = 1$

Write $X_n \xrightarrow{a.s.} X$

Theorem: $X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{P} X$

Pf: Done in the [illegible]-level sequence

The Strong Law of Large Numbers says $\bar{X}_n \xrightarrow{a.s.} \mu$

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Defn. Let $F_n(x)$ be the cdf for X_n.

If $F(x)$ is a valid cdf and if

$$\lim_{n\to\infty} F_n(x) = F(x) \quad \forall\ x \text{ that are points of continuity of } F(x)$$

then $F(x)$ is called the <u>limiting distribution</u> of $\{X_n\}$ and $\{X_n\}$ converges in distribution to the random variable X whose cdf is $F(x)$.

Write $X_n \xrightarrow{\mathcal{D}} X$

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<u>Examples of limiting distributions</u>

Let $X_1, X_2, \ldots, X_n, \ldots$ be iid $\mathrm{Unif}(0,\theta)$.

That is, $f_n(x) = \frac{1}{\theta} \quad 0 < x < \theta$

$$F_n(x) = \begin{cases} 0 & x \le 0 \\ x/\theta & 0 < x < \theta \\ 1 & x \ge \theta \end{cases}$$

Let $Y_n = \max\{X_1, \ldots, X_n\}$

and $Z_n = n(\theta - Y_n)$ Find the limiting distribution of Z_n.

(9)

$$G_n(y) = P[Y_n \le y] = P[\max\{X_1,\dots,X_n\} \le y]$$

$$= \left(P[X_i \le y]\right)^n = \begin{cases} 0 & y \le 0 \\ (y/\theta)^n & 0 < y < \theta \\ 1 & y \ge \theta \end{cases}$$

$$H_n(z) = P(Z_n \le z)$$

$$= P(n(\theta - Y_n) \le z)$$

$$= P(Y_n \ge \theta - z/n)$$

$$= 1 - P(Y_n < \theta - z/n)$$

$$= 1 - G_n(\theta - z/n)$$

(10)

$$H_n(z) = \begin{cases} 1 & z \geq n\theta \\ 1 - \left(1 - \frac{z}{n\theta}\right)^n & 0 < z < n\theta \\ 0 & z \le 0 \end{cases}$$

$$\lim_{n\to\infty} H_n(z) = \begin{cases} 0 & z \le 0 \\ 1 - e^{-z/\theta} & z > 0 \end{cases} \qquad \left(\left(1 + \tfrac{x}{n}\right)^n \to e^x\right)$$

This is a valid cdf (exponential distr.)

$$Z_n \xrightarrow{d} Z \quad \text{where } Z \sim \text{Exp}\left(\tfrac{1}{\theta}\right)$$

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Example: $X_1, X_2, \dots, X_n, \dots \sim$ iid $N(0,1)$

Find the limiting distribution of $\bar{X}_n$

We know $\bar{X}_n \sim N(0, \frac{1}{n})$

$$f_n(x) = \frac{1}{\sqrt{2\pi}\sqrt{\frac{1}{n}}} e^{-\frac{1}{2}\left(\frac{x-0}{\sqrt{1/n}}\right)^2}$$

$$= \sqrt{\frac{n}{2\pi}}\; e^{-\frac{1}{2}nx^2}$$

$$F_n(x) = \int_{-\infty}^{x} \sqrt{\frac{1}{2\pi}}\, e^{-\frac{1}{2}nt^2}\, dt$$

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let $u = \sqrt{n}\, t$

$du = \sqrt{n}\, dt$

$$F_n(x) = \int_{-\infty}^{\sqrt{n}\,x} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}u^2}\, du$$

$$\lim_{n\to\infty} F_n(x) = \begin{cases} 0 & x<0 \\ .5 & x=0 \\ 1 & x>0 \end{cases}$$

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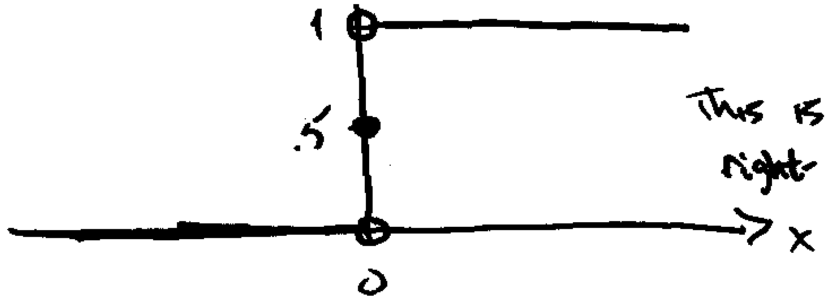
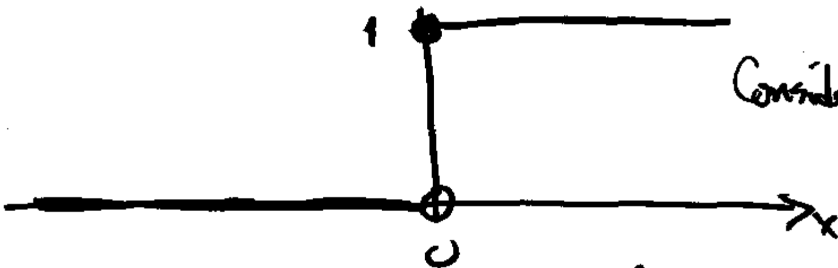


This is not right-continuous, so not a valid cdf.



Consider this instead.

this agrees with the graph on the top at every point of continuity.

This is the cdf of the r.v. X such that $P(X=0)=1$

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So $\bar{X}_n \xrightarrow{\mathcal{D}} 0$

Midterm Exam Tues 2/12

1 page of notes (front & back)

+ catalog of distributions

Conditional expectation & variance

Covariance & correlation

transformations of pairs of r.v.s

properties of $\overline{X}$ and S^2

normal, χ^2, t & F distributions

order statistics