Midterm scores (out of 80)

Stat 862
2-21-19

①

2	5
3	05
4	35589
5	000058
6	035
7	0333568888
8	0000

mean 61.06 (76%)

median 63

②

Defn. $\{g(\vec{x}) : g \in G\}$ is a <u>group of transformations</u>

(1) closed under the operation of composition

(2) it contains an identity element

(3) $\forall g \in G$, $\exists\, g^{-1} \in G$ such that

$$g(g^{-1}(\vec{x})) = g^{-1}(g(\vec{x})) = \vec{x}$$

Defn: Let $T(\vec{x})$ be a statistic.

If $T(g(\vec{x})) = T(\vec{x})\ \forall g \in G$,

then T is invariant under G.

(3)

<u>Example</u>: $X_1, \dots, X_n \sim$ iid $N(\mu, \sigma^2)$

Let $g_a(\vec{x}) = (x_1 + a, x_2 + a, \dots, x_n + a)$

Let $G = \{g_a(\vec{x}) : a \in \mathbb{R}\}$ We can verify that this is a group

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$S^2(g(\vec{x})) = \frac{1}{n-1} \sum_{i=1}^{n} \left(x_i + a - (\bar{x} + a)\right)^2 = S^2(\vec{x})$$

$\therefore S^2$ is invariant under G.

(4)

For this particular G, any statistic that is invariant under it is called <u>location-invariant</u>.

Let $g_a(\vec{x}) = (a x_1, \dots, a x_n)$

and $G = \{g_a : a \neq 0\}$ We can verify that this is a group

$CV = \frac{S}{\bar{X}}$ is invariant under the group

Any statistic that is invariant under this group is called <u>scale-invariant</u>

⑤

Suppose that θ is a location parameter,

and $U(\vec{x})$ is a location invariant statistic.

$$f(x|\theta) = g(x-\theta)$$

$$X_1, \dots, X_n \sim \text{iid } g(x-\theta)$$

Let $W_i = X_i - \theta \quad i=1,\dots,n$

$$X_i = W_i + \theta$$

$$J = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & \cdots & 1 \end{vmatrix} = 1$$

⑥

$$\begin{aligned} h(w_1,\dots,w_n) &= f(x_1,\dots,x_n)\cdot 1 \\ &= \prod_{i=1}^{n} g(x_i-\theta) \\ &= \prod_{i=1}^{n} g(w_i), \text{ which is free of } \theta \end{aligned}$$

Also, $U(\vec{x})$ was location-invariant,

$$\text{so} \quad u(x_1,\dots,x_n) = u(x_1-\theta,\dots,x_n-\theta) = u(w_1,\dots,w_n),$$

so the distribution of $U(\vec{X})$ doesn't involve θ $\therefore U(\vec{X})$ is ancillary for θ.

(7)

Use Basu's Theorem to prove that

$\bar{X}$ and S^2 are independent if $X_i \sim iid\, N(\mu, \sigma^2)$

1.) μ is a location parameter

2.) S^2 is location-invariant

3.) S^2 is ancillary for μ

4.) $\bar{X}$ is a sufficient statistic for μ

5.) $N(\mu, \sigma^2)$ is complete

By Basu, $\bar{X}$ and S^2 are independent

(8)

Chapter 7 Estimation

Suppose that X has pdf or pmf

$f(x|\theta)$, $\theta \in \Omega$

Observe $X_1, \ldots, X_n \sim iid\ f(x|\theta)$

Method of Moments (MOM)

Equate the sample moments with the corresponding population moments

(1) Set $\bar{X} = \mu$, (2) $\frac{1}{n}\sum X_i^2 = \sigma^2 + \mu^2$

⑨

<u>Examples</u> $f(x|\theta) = \theta e^{-\theta x}$ $x>0$

Observe $X_1, \ldots, X_n$ iid

Use MOM to estimate θ.

$\mu = \frac{1}{\theta}$

Set $\bar{X} = \frac{1}{\theta}$ $\therefore$ $\hat{\theta}_{MOM} = \frac{1}{\bar{X}}$

Recall that $\bar{X} \xrightarrow{P} \mu$ (consistency)

So $\frac{1}{\bar{X}} \xrightarrow{P} \frac{1}{\mu}$ $\therefore$ $\frac{1}{\bar{X}} = \hat{\theta}$ is a consistent est. of μ

$E(\hat{\theta}) = E\left[\frac{1}{\bar{X}}\right] \neq \frac{1}{E[\bar{X}]} = \frac{1}{\mu}$ so $\hat{\theta}$ is <u>biased</u>

⑩

<u>Example</u>: $f(x|\theta) = \frac{1}{\theta}$ $0 < x < \theta$

Unif$(0,\theta)$

MOM: Set $\bar{X} = \frac{\theta}{2}$

$\mu = \frac{0+\theta}{2} = \frac{\theta}{2}$

$\Rightarrow \hat{\theta}_{MOM} = 2\bar{X}$

$E[\hat{\theta}_{MOM}] = 2E[\bar{X}] = 2\mu = 2\left(\frac{\theta}{2}\right) = \theta$

<u>Example</u>: $f(x|\mu) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ σ known

MOM: Set $\bar{X} = \mu$ So $\hat{\mu}_{MOM} = \bar{X}$

What if σ is unknown? (11)

(1) Set $\bar{X} = \mu$ $\qquad \hat{\mu} = \bar{X}$

(2) $\frac{1}{n}\sum x_i^2 = \sigma^2 + \mu^2$

$$\frac{1}{n}\sum X_i^2 = \sigma^2 + \bar{X}^2$$

$$\hat{\sigma}^2_{MM} = \frac{1}{n}\sum X_i^2 - \bar{X}^2$$

$$= \frac{1}{n}\left(\sum X_i^2 - n\bar{X}^2\right)$$

$$= \frac{n-1}{n} S^2 = S_n^2$$

(12)

Maximum Likelihood Method

$$f(\vec{x}\mid\theta) = \prod_{i=1}^{n} f(x_i\mid\theta)$$

$$= L(\theta)$$, the likelihood function

Find the value of θ that maximizes $L(\theta)$.

Example: $f(x\mid\theta) = \theta e^{-\theta x}$

$$L(\theta) = \prod_{i=1}^{n} f(x_i\mid\theta) = \theta^n e^{-\theta\sum x_i}$$

Note: $\ln L(\theta)$ will have its maximum (13)

at the same θ.

$$\ell(\theta) = \ln L(\theta) = \ln\left(\theta^n e^{-\theta \Sigma x_i}\right)$$

$$= n \ln\theta - \theta \Sigma x_i$$

$$\ell'(\theta) = \frac{n}{\theta} - \Sigma x_i \overset{\text{set}}{=} 0$$

$$\hat{\theta} = \frac{n}{\Sigma x_i} = \frac{1}{\bar{X}}$$

$$\ell''(\theta) = -\frac{n}{\theta^2} < 0 \quad \text{So } \hat{\theta}_{MLE} = \frac{1}{\bar{X}}$$

(14)

<u>Example</u>: $f(x|\theta) = \frac{1}{\theta} \quad 0 < x \le \theta$

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} = \frac{1}{\theta^n}$$

$$\ell(\theta) = \ln L(\theta) = -n \ln\theta$$

$$\ell'(\theta) = -\frac{n}{\theta} \overset{\text{set}}{=} 0 \quad \underline{\text{No solution}}$$

Note: $L(\theta)$ was a decreasing function of θ.

So, to maximize $L(\theta)$, we should use the smallest possible value of θ. $\hat{\theta}_{MLE} = X_{(n)}$

HW #6 due 2/28

6.30

7.6

10

11

Stat 562 HW #6

6.30 Let $X_1, \ldots, X_n$ be a random sample from the pdf $f(x|\mu) = e^{-(x-\mu)}$, where $-\infty < \mu < x < \infty$.

(a) Show that $X_{(1)} = \min_i X_i$ is a complete sufficient statistic.

(b) Use Basu's Theorem to show that $X_{(1)}$ and S^2 are independent.

7.6 Let $X_1, \ldots, X_n$ be a random sample from the pdf

$$f(x|\theta) = \theta x^{-2}, \quad 0 < \theta \leq x < \infty.$$

(a) What is a sufficient statistic for θ?

(b) Find the MLE of θ.

(c) Find the method of moments estimator of θ.

7.10 The independent random variables $X_1, \ldots, X_n$ have the common distribution

$$P(X_i \leq x|\alpha, \beta) = \begin{cases} 0 & \text{if } x < 0 \\ (x/\beta)^\alpha & \text{if } 0 \leq x \leq \beta \\ 1 & \text{if } x > \beta, \end{cases}$$

where the parameters α and β are positive.

(a) Find a two-dimensional sufficient statistic for (α, β).

(b) Find the MLEs of α and β.

(c) The length (in millimeters) of cuckoos' eggs found in hedge sparrow nests can be modeled with this distribution. For the data

22.0, 23.9, 20.9, 23.8, 25.0, 24.0, 21.7, 23.8, 22.8, 23.1, 23.1, 23.5, 23.0, 23.0,

find the MLEs of α and β.

7.11 Let $X_1, \ldots, X_n$ be iid with pdf

$$f(x|\theta) = \theta x^{\theta-1}, \quad 0 \leq x \leq 1, \quad 0 < \theta < \infty.$$

(a) Find the MLE of θ, and show that its variance $\to 0$ as $n \to \infty$.

(b) Find the method of moments estimator of θ.