Stat 562
1-24-19

①

Theorem: Assume $X_i \sim$ indep. $N(\mu_i, \sigma_i^2)$, $i=1,\dots,n$

Let $U = \sum_{i=1}^n a_i X_i$ and $V = \sum_{i=1}^n b_i X_i$.

Then U & V are independent iff $\mathrm{Cov}(U,V)=0$.

Proof: $\Rightarrow$: already done

$\Leftarrow$: Assume

$$
\begin{aligned}
0 &= \mathrm{Cov}(U,V) \\
&= \mathrm{Cov}\left(\sum_{i=1}^n a_i X_i, \sum_{i=1}^n b_i X_i\right) \\
&= \sum_{i=1}^n a_i \mathrm{Cov}\left(X_i, \sum_{j=1}^n b_j X_j\right)
\end{aligned}
$$

$$0 = \sum_{i=1}^n \sum_{j=1}^n a_i b_j \mathrm{Cov}(X_i, X_j)$$ ②

Consider the special case where $\mu_i \equiv 0$, $\sigma_i^2 \equiv 1$, $n=2$

$$U = a_1 X_1 + a_2 X_2 \quad , V = b_1 X_1 + b_2 X_2$$

Our assumption says $0 = a_1 b_1 + a_2 b_2$

$$X_1 = \frac{b_2 U - a_2 V}{D}, \quad X_2 = \frac{a_1 V - b_1 U}{D}$$ $D = a_1 b_2 - a_2 b_1$

$$J = \begin{vmatrix} b_2/D & -\frac{a_2}{D} \\ -b_1/D & a_1/D \end{vmatrix} = \frac{a_1 b_2 - a_2 b_1}{D^2} = \frac{1}{D}$$ ③

$$g(u,v) = f(x_1, x_2)\,|J|$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_1^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_2^2} \frac{1}{|D|}$$

$$= \frac{1}{2\pi} \frac{1}{|D|} e^{-\frac{1}{2}\left(\frac{1}{D^2}\left(b_2^2 u^2 + a_2^2 v^2 - 2a_2 b_2 uv\right) + \frac{1}{D^2}\left(a_1^2 v^2 + b_1^2 u^2 - 2a_1 b_1 uv\right)\right)}$$

Coefficient of uv is $\frac{-1}{D^2}(a_2 b_2 + a_1 b_1) = 0$

$\therefore$ $g(u,v)$ factors $\Rightarrow$ U & V are independent

④

The general case is straightforward, but the Jacobian involves an $n \times n$ matrix.

Let $Z \sim N(0,1)$ and $Y \sim \chi^2_r$, indep.

Let $T = \dfrac{Z}{\sqrt{Y/r}}$. This random variable has a <u>students t</u> distribution.

5

<u>Find $g(t)$</u>

Let $U = Y$ $\qquad\qquad$ $Y = U$

$\qquad\qquad\qquad\qquad Z = T\sqrt{\frac{U}{r}}$

$$J = \begin{vmatrix} \sqrt{\frac{u}{r}} & t\frac{1}{\sqrt{r}}\frac{1}{2}\frac{1}{\sqrt{u}} \\ 0 & 1 \end{vmatrix} = \sqrt{\frac{u}{r}}$$

$$g(t,u) = f(z,y)\,|J|$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \frac{1}{\Gamma(\frac{r}{2})2^{r/2}} y^{\frac{r}{2}-1} e^{-y/2} \sqrt{\frac{u}{r}}$$

6

$$= \frac{1}{\sqrt{2\pi}\,\Gamma(\frac{r}{2})2^{r/2}\sqrt{r}} u^{\frac{r}{2}-1} e^{-\frac{1}{2}\left(\frac{t^2u}{r}+u\right)} u^{1/2}$$

$$g(t,u) = \frac{1}{\sqrt{2\pi}\,\Gamma(\frac{r}{2})2^{\frac{r}{2}}\sqrt{r}} u^{\frac{r}{2}-\frac{1}{2}} e^{-\frac{u}{2}(1+t^2/r)}$$

$$g(t) = \int_0^\infty g(t,u)\,du$$

Let $w = \frac{1}{2}u(1+t^2/r)$

$dw = \frac{1}{2}(1+\frac{t^2}{r})\,du$

$$g(t) = \frac{1}{\sqrt{2\pi}\,\Gamma(\frac{r}{2})2^{\frac{r}{2}}\sqrt{r}} \int_0^\infty \left[\frac{2w}{(1+t^2/r)}\right]^{\frac{r-1}{2}} e^{-w} \frac{2\,dw}{(1+t^2/r)}$$

$$= \frac{1}{\sqrt{2\pi}\,\Gamma(\frac{r}{2})\,2^{\frac{r}{2}}\sqrt{r}} \; \frac{2^{\frac{r-1}{2}} \cdot 2}{\left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}}} \int_0^\infty w^{\frac{r-1}{2}} e^{-w}\, dw$$

$$\boxed{\text{Recall } \Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt}$$

$$g(t) = \frac{\cancel{2^{\frac{r+1}{2}}}\;\Gamma\left(\frac{r+1}{2}\right)}{\cancel{2^{\frac{r+1}{2}}}\;\Gamma\left(\frac{r}{2}\right)\sqrt{\pi r}\left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}}} \qquad -\infty < t < \infty$$

Why? Suppose $X_1, \dots, X_n \sim$ iid $N(\mu, \sigma^2)$

Then $\bar{X} \sim N(\mu, \sigma^2/n)$, $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

and $\bar{X}$ & S^2 are independent.

Then $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

Let $T = \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{\sigma^2}/(n-1)}}$ Then $T \sim t_{n-1}$

(9)

$$T = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \frac{\sigma}{S} = \frac{\bar{X}-\mu}{S/\sqrt{n}}$$

Find $E[T]$ & $V[T]$.

Recall #3.17 from Fall term:

If $X \sim \text{Gamma}(\alpha, \beta)$, then

$$E[X^k] = \frac{\beta^k \Gamma(\alpha+k)}{\Gamma(\alpha)}$$

$$E[T] = E\left[\frac{Z}{\sqrt{V/r}}\right] = \sqrt{r}\, E[Z\, V^{-1/2}]$$

(10)

$$= \sqrt{r}\, \underbrace{E[Z]}_{0} E[V^{-1/2}] \quad \text{by indep of } Z, V$$

$$E[T] = 0$$

$$V[T] = E[T^2] = E\left[\frac{Z^2}{V/r}\right] = r\, E[Z^2 V^{-1}]$$

$$= r\, \underbrace{E[Z^2]}_{1} E[V^{-1}] = r\, E[V^{-1}]$$

$$V \sim \chi^2_r \sim \text{Gamma}(\alpha = \tfrac{r}{2}, \beta = 2)$$

11

So $E[V^{-1}] = \dfrac{2^{-1}\Gamma(\alpha-1)}{\Gamma(\alpha)} = \dfrac{\Gamma(\frac{r}{2}-1)}{2\Gamma(\frac{r}{2})}$

$$V[T] = r\frac{\cancel{\Gamma(\frac{r}{2}-1)}}{2(\frac{r}{2}-1)\cancel{\Gamma(\frac{r}{2}-1)}}$$

$$V[T] = \frac{r}{r-2}$$

12

Let $U = \chi^2_{r_1}$ and $V = \chi^2_{r_2}$, indep.

Let $F = \dfrac{U/r_1}{V/r_2}$.

R.A. Fisher

Then F will have a <u>Snedecor's</u> <u>F</u> distribution.

Using the same technique,

$$g(f) = \frac{\Gamma(\frac{r_1+r_2}{2})\left(\frac{r_1}{r_2}\right)^{r_1/2} f^{\frac{r_1}{2}-1}}{\Gamma(r_1/2)\Gamma(r_2/2)\left(1+\frac{r_1}{r_2}f\right)^{\frac{r_1+r_2}{2}}}$$

(13)

Note: If $F \sim F(r_1, r_2)$,

then $\frac{1}{F} \sim F(r_2, r_1)$

Find the moments of the F distribution.

$$E[F^k] = E\left[\frac{U/r_1}{V/r_2}\right]^k = \frac{r_2^k}{r_1^k} E[U^k V^{-k}]$$

$$= \frac{r_2^k}{r_1^k} E[U^k]\, E[V^{-k}] \quad \text{by indp}$$

(14)

$$E[F^k] = \frac{r_2^k}{r_1^k} \, \frac{\cancel{2^k}\,\Gamma\left(\frac{r_1}{2}+k\right)}{\Gamma(r_1/2)} \cdot \frac{\cancel{2^{-k}}\,\Gamma\left(\frac{r_2}{2}-k\right)}{\Gamma(r_2/2)}$$

$$E[F] = \frac{r_2}{r_1} \, \frac{\Gamma\left(\frac{r_1}{2}+1\right)}{\Gamma\left(\frac{r_1}{2}\right)} \, \frac{\Gamma\left(\frac{r_2}{2}-1\right)}{\Gamma(r_2/2)}$$

$$= \frac{r_2}{r_1} \, \frac{\frac{\cancel{r_1}}{2}\cancel{\Gamma\left(\frac{r_1}{2}\right)}}{\cancel{\Gamma(r_1/2)}} \cdot \frac{\cancel{\Gamma\left(r_2/2-1\right)}}{\left(\frac{r_2}{2}-1\right)\cancel{\Gamma\left(\frac{r_2}{2}-1\right)}}$$

$$E[F] = \frac{r_2}{r_2-2}$$

Notice that if $T = \dfrac{Z}{\sqrt{V/r}}$,

then $T^2 = \dfrac{Z^2}{V/r}$ $\dfrac{\chi^2_1/1}{\chi^2_r/r}$

$$\sim F_{1,r}$$

HW #3 due 1/31

p. 259 #22, 24, 25

Stat 562 HW #3

5.22 Let X and Y be iid $n(0, 1)$ random variables, and define $Z = \min(X, Y)$. Prove that $Z^2 \sim \chi^2_1$.

5.24 Let $X_1, \dots, X_n$ be a random sample from a population with pdf

$$f_X(x) = \begin{cases} 1/\theta & \text{if } 0 < x < \theta \\ 0 & \text{otherwise.} \end{cases}$$

Let $X_{(1)} < \cdots < X_{(n)}$ be the order statistics. Show that $X_{(1)}/X_{(n)}$ and $X_{(n)}$ are independent random variables.

5.25 As a generalization of the previous exercise, let $X_1, \dots, X_n$ be iid with pdf

$$f_X(x) = \begin{cases} \frac{a}{\theta^a} x^{a-1} & \text{if } 0 < x < \theta \\ 0 & \text{otherwise.} \end{cases}$$

Let $X_{(1)} < \cdots < X_{(n)}$ be the order statistics. Show that $X_{(1)}/X_{(2)}, X_{(2)}/X_{(3)}, \dots, X_{(n-1)}/X_{(n)}$, and $X_{(n)}$ are mutually independent random variables. Find the distribution of each of them.