The Delta Method

Stat 562
3-5-19

①

In an example on 2/21/19, for $f(x) = \theta e^{-\theta x}$ $(x>0)$,

we found $\hat{\theta} = \frac{1}{\bar{X}}$

Use a 2^{nd} order Taylor series

$$g(x) = g(x_0) + g'(x_0)(x - x_0) + g''(x_0)\frac{(x-x_0)^2}{2} + R$$

Consider $g(x) = \frac{1}{x}$ $\quad g'(x) = -\frac{1}{x^2}$ $\quad g'' = \frac{2}{x^3}$

②

$$g(x) = \frac{1}{x} \approx \frac{1}{x_0} - \frac{1}{x_0^2}(x - x_0) + \frac{2}{x_0^3}\frac{(x-x_0)^2}{2}$$

Choose $x_0 = E[X] = \mu$

$$\text{So } \hat{\theta} = \frac{1}{\bar{X}} \approx \frac{1}{\mu} - \frac{1}{\mu^2}(\bar{x} - \mu) + \frac{\cancel{2}}{\mu^3}\frac{(\bar{x}-\mu)^2}{\cancel{2}}$$

$$E[\hat{\theta}] \approx \frac{1}{\mu} - 0 + \frac{V[\bar{x}]}{\mu^3}$$

$$= \frac{1}{\mu} + \frac{\sigma^2/n}{\mu^3}$$

Also, we know

$\mu = \frac{1}{\theta}$, $\sigma^2 = \frac{1}{\theta^2}$

$$E[\hat{\theta}] \approx \theta + \frac{\theta^3}{\theta^2 n} = \theta\left(1+\frac{1}{n}\right)$$

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To approximate the variance, use just the 1^{st} order Taylor series:

$$V[\hat{\theta}] \approx V\left[\frac{1}{\mu} - \frac{1}{\mu^2}(\bar{X}-\mu)\right]$$

$$= \frac{1}{\mu^4} V[\bar{X}] = \frac{1}{\mu^4}\frac{\sigma^2}{n} = \theta^4 \frac{1}{\theta^2 \cdot n}$$

$$V[\hat{\theta}] \approx \frac{\theta^2}{n}$$

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<u>Example</u>: Say $\hat{\theta} = \dfrac{2\bar{x}^2}{\bar{x}+1}$

Estimate the expected value & variance of $\hat{\theta}$.

$$g(x) = \frac{2x^2}{x+1} \qquad g'(x) = \frac{(x+1)4x - 2x^2}{(x+1)^2} = \frac{2x^2+4x}{(x+1)^2}$$

$$g''(x) = \frac{(x+1)^2(4x+4) - (2x^2+4x)2(x+1)}{(x+1)^4} = \frac{4}{(x+1)^2}$$

$$\hat{\theta} = \frac{2\bar{x}^2}{\bar{x}+1} \approx \frac{2\mu^2}{\mu+1} + \frac{2\mu^2+4\mu}{(\mu+1)^2}(\bar{x}-\mu) + \frac{4}{(\mu+1)^3}\frac{(\bar{x}-\mu)^2}{2}$$

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$$E[\hat{\theta}] \approx \frac{2\mu^2}{\mu+1} + 0 + \frac{2}{(\mu+1)^3}\frac{\sigma^2}{n}$$

$$V[\hat{\theta}] \approx \left[\frac{2\mu^2+4\mu}{(\mu+1)^2}\right]^2 \frac{\sigma^2}{n}$$

Example: Assume $X_1, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$

$$\theta = \frac{\sigma}{\mu} = \text{Coefficient of variation}$$

So the MLE of $\theta = \hat{\theta}_{MLE} = \dfrac{\hat{\sigma}_{MLE}}{\hat{\mu}_{MLE}}$

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Let $\hat{\theta} = \dfrac{S}{\bar{X}}$. Estimate the expected value & variance.

Let $g(x,y) = \dfrac{\sqrt{y}}{x}$

$$g(x,y) \approx g(x_0,y_0) + \frac{\partial g}{\partial x}(x_0,y_0)(x-x_0)$$
$$+ \frac{\partial^2 g}{\partial x^2}(x_0,y_0)\frac{(x-x_0)^2}{2} + \frac{\partial g}{\partial y}(x_0,y_0)(y-y_0)$$
$$+ \frac{\partial^2 g}{\partial y^2}(x_0,y_0)\frac{(y-y_0)^2}{2} + \frac{\partial^2 g}{\partial y\,\partial x}(x_0,y_0)(x-x_0)(y-y_0)$$

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$$g(x,y) = \frac{\sqrt{y}}{x} \qquad \frac{\partial g}{\partial x} = -\frac{\sqrt{y}}{x^2} \qquad \frac{\partial^2 g}{\partial x^2} = \frac{2\sqrt{y}}{x^3}$$

$$\frac{\partial g}{\partial y} = \frac{1}{x}\frac{1}{2}y^{-1/2} \qquad \frac{\partial^2 g}{\partial y^2} = \frac{1}{2x}\left(-\frac{1}{2}\right)y^{-3/2}$$

$$\frac{\partial^2 g}{\partial y\,\partial x} = -\frac{1}{x^2}\frac{1}{2}y^{-\frac{1}{2}}$$

$$\frac{\sqrt{y}}{x} \approx \frac{\sqrt{\mu_y}}{\mu_x} - \frac{\sqrt{\mu_y}}{\mu_x^2}(x-\mu_x) + \frac{2\sqrt{\mu_y}}{\mu_x^3}\frac{(x-\mu_x)^2}{2}$$

$$+ \frac{1}{2\mu_x\sqrt{\mu_y}}(y-\mu_y) - \frac{1}{4\mu_x\,\mu_y^{3/2}}\frac{(y-\mu_y)^2}{2} - \frac{(x-\mu_x)(y-\mu_y)}{2\mu_x^2\sqrt{\mu_y}}$$

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Remember: $Y = S^2$ so $\mu_y = \sigma^2$

$X = \bar{X}$ so $\mu_x = \mu$

$$E\left[\frac{S}{\bar{X}}\right] \approx \frac{\sigma}{\mu} + \frac{\sigma}{\mu^3}V[\bar{X}] - \frac{1}{8\mu\sigma^3}V(S^2)$$

$$- \frac{1}{2\mu^2\sigma}\mathrm{Cov}(\bar{X}, S^2)$$

But $V[\bar{X}] = \frac{\sigma^2}{n}$, $\mathrm{Cov}(\bar{X}, S^2) = 0$

We know $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$. $V\left[\frac{(n-1)S^2}{\sigma^2}\right] = 2(n-1)$

$$\frac{(n-1)^2}{\sigma^4} V[s^2] = 2(n-1)$$

$$\text{so} \quad V[s^2] = \frac{2\sigma^4}{n-1}$$

Now $$E\left[\frac{s}{\bar{x}}\right] \approx \frac{\sigma}{\mu} + \frac{\sigma}{\mu^3}\frac{\sigma^2}{n} - \frac{1}{8\mu\sigma^3}\cdot\frac{2\sigma^4}{n-1}$$

$$= \frac{\sigma}{\mu} + \frac{\sigma^3}{n\mu^3} - \frac{\sigma}{(n-1)4\mu}$$

$$V\left[\frac{s}{\bar{x}}\right] \approx \frac{\sigma^2}{\mu^4} V[\bar{x}] + \frac{1}{4\mu^2\sigma^2} V[s^2] + \underbrace{0}_{\text{Covariance term}}$$

$$V\left[\frac{s}{\bar{x}}\right] \approx \frac{\sigma^2}{\mu^4}\frac{\sigma^2}{n} + \frac{1}{4\mu^2\sigma^2}\frac{2\sigma^4}{n-1}$$

$$= \frac{\sigma^4}{n\mu^4} + \frac{\sigma^2}{2(n-1)\mu^2}$$

If $\hat{\theta}$ is asymptotically unbiased and $V[\hat{\theta}] \to 0$ as $n \to \infty$, then $\hat{\theta}$ has 2nd order consistency, which implies $\hat{\theta} \xrightarrow{P} \theta$

Evaluating Estimators

Defn: For an estimator T of θ,

$$MSE(T) = E[(T-\theta)^2]$$

mean squared error

Note: $MSE(T) = E[(\underbrace{T - E(T)} + \underbrace{E(T) - \theta})^2]$

$= E[(T-E(T))^2 + (E(T)-\theta)^2 + 2(T-E(T))(E(T)-\theta)]$

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$= V[T] + [\underbrace{E(T)-\theta}_{Bias(T)}]^2 + 0$

So $MSE(T) = V(T) + Bias^2(T)$

Which is a better estimator of σ^2: S^2 or S_n^2?

Assume $X_1, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$

Know: $E[S^2] = \sigma^2$ $\quad Bias(S^2) = 0$

$V[S^2] = \dfrac{2\sigma^4}{n-1}$ $\quad MSE(S^2) = \dfrac{2\sigma^4}{n-1}$

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$$S_n^2 = \frac{n-1}{n} S^2$$

$$E[S_n^2] = \frac{n-1}{n}\sigma^2$$

$$\text{Bias}(S_n^2) = \frac{n-1}{n}\sigma^2 - \sigma^2 = \frac{n-1-n}{n}\sigma^2 = -\frac{\sigma^2}{n}$$

$$V[S_n^2] = \left(\frac{n-1}{n}\right)^2 V(S^2) = \left(\frac{n-1}{n}\right)^2 \frac{2\sigma^4}{n-1} = \frac{2\sigma^4(n-1)}{n^2}$$

$$MSE[S_n^2] = \frac{2\sigma^4(n-1)}{n^2} + \frac{\sigma^4}{n^2} = \frac{2n-1}{n^2}\sigma^4$$

Compare $\frac{2\cancel{\sigma^4}}{n-1}$ with $\frac{2n-1}{n^2}\cancel{\sigma^4}$

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$2n^2$ vs $(2n-1)(n-1)$

0 vs $-3n+1$

$\therefore MSE(S^2) > MSE(S_n^2)$

$\uparrow$

wins!!