Stat 562
2-7-19
1

Theorem: $X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} X$

Proof: Assume $X_n \xrightarrow{P} X$

$$F_n(x) = P(X_n \le x)$$
$$= P(X_n \le x \cap |X_n - X| < \varepsilon)^{(1)} + P(X_n \le x \cap |X_n - X| \ge \varepsilon)^{(2)}$$

2

(1): $\underline{X_n \le x} \cap -\varepsilon < X_n - X < \varepsilon$

$$-X_n - \varepsilon < -X < -X_n + \varepsilon$$
$$\underline{X_n + \varepsilon > X} > X_n - \varepsilon$$

$$\Rightarrow X < x + \varepsilon$$

(2) $X_n \le x \cap |X_n - X| \ge \varepsilon$

$$\Rightarrow |X_n - X| \ge \varepsilon$$

So $F_n(x) \le P(X < x + \varepsilon) + P(|X_n - X| \ge \varepsilon)$

③

$$\lim_{n\to\infty} F_n(x) \le P(X < x+\varepsilon) + \underbrace{\lim_{n\to\infty} P(|X_n - X| \ge \varepsilon)}_{0}$$

So $\lim_{n\to\infty} F_n(x) \le P(X < x+\varepsilon)$

Similarly,

$$P(X_n > x) = P(X_n > x \cap |X_n - X| < \varepsilon)$$
$$+ P(X_n > x \cap |X_n - X| \ge \varepsilon)$$

④

$$1 - F_n(x) \le P(X > x-\varepsilon) + P(|X_n - X| \ge \varepsilon)$$

$$\lim_{n\to\infty} (1 - F_n(x)) \le P(X > x-\varepsilon) + 0$$

$$\lim_{n\to\infty} F_n(x) \ge 1 - P(X > x-\varepsilon) = P(X \le x-\varepsilon)$$
$$= F(x-\varepsilon)$$

$$F(x-\varepsilon) \le \lim_{n\to\infty} F_n(x) \le P(X < x+\varepsilon) \quad \forall \varepsilon > 0$$

If x is a point of continuity of $F(x)$, then $\lim_{n\to\infty} F_n(x) = F(x)$

$$\therefore \quad X_n \xrightarrow{\mathcal{D}} X$$

In summary,

$$X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{\mathcal{D}} X$$

<u>Theorem</u>: $X_n \xrightarrow{P} c$ iff $X_n \xrightarrow{\mathcal{D}} c$

<u>Proof</u>: ($\Rightarrow$) follows from theorem

($\Leftarrow$) was Ch 5 #41

<u>Slutsky's theorem</u>

Suppose $X_n \xrightarrow{\mathcal{D}} X$ and $Y_n \xrightarrow{P} c$.

Then $X_n Y_n \xrightarrow{\mathcal{D}} cX$

and $X_n + Y_n \xrightarrow{\mathcal{D}} X + c$

<u>Proof</u>: done in the 600-level sequence

(7)

The Central Limit theorem

Let $X_1, X_2, \ldots, X_n, \ldots$ be iid random variables

with $E[X_i] = \mu$ & $V[X_i] = \sigma^2 \quad \forall i$

Let $Z_i = \frac{X_i - \mu}{\sigma}$ so $E[Z_i] = 0$

and $V[Z_i] = 1$

Let $M(t)$ be the m.g.f. for Z_i

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$$M_{\sum_{i=1}^{n} Z_i}(t) = (M(t))^n \quad \text{by indep.}$$

Let $Y_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i$

$$M_{Y_n}(t) = \left(M\left(\tfrac{1}{\sqrt{n}} t\right)\right)^n$$

$$= \left(E\left(e^{\frac{t}{\sqrt{n}} Z}\right)\right)^n$$

$$= \left(E\left(1 + \frac{tZ}{\sqrt{n}} + \frac{t^2 Z^2}{2n} + O\left(\frac{1}{n^{3/2}}\right)\right)\right)^n$$

$$M_{Y_n}(t) = \left(1 + 0 + \frac{t^2}{2n} + O\left(\frac{1}{n^{3/2}}\right)\right)^n$$ (9)

$$\ln M_{Y_n}(t) = n \ln\left(1 + \frac{t^2}{2n} + O\left(\frac{1}{n^{3/2}}\right)\right)$$
$$= \frac{\ln\left(1 + \frac{t^2}{2n} + O\left(\frac{1}{n^{3/2}}\right)\right)}{1/n}$$
$$\lim_{n\to\infty} \ln M_{Y_n}(t) = \lim_{n\to\infty} \frac{\frac{1}{\left(1 + t^2/2n + O(n^{-3/2})\right)} \cdot \left(-\frac{t^2}{2n^2} + O\left(\frac{1}{n^{5/2}}\right)\right)}{-1/n^2}$$

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$$= \lim_{n\to\infty} \frac{\frac{t^2}{2} + O\left(\frac{1}{n^{1/2}}\right)}{1 + \frac{t^2}{2n} + O\left(\frac{1}{n^{3/2}}\right)} = \frac{\frac{t^2}{2}}{1}$$

$$\ln\left(\lim_{n\to\infty} M_{Y_n}(t)\right) = \frac{t^2}{2}$$

$$\lim_{n\to\infty} M_{Y_n}(t) = e^{t^2/2}$$, which is the m.g.f. of $N(0,1)$

We would like to conclude that $Y_n \xrightarrow{\mathcal{D}} N(0,1)$

Unfortunately, convergence of m.g.f.s does not imply convergence in distribution without more conditions.

(11)

However, with characteristic functions $E[e^{itX}]$, convergence does imply convergence in distribution.

What was Y_n?

$$Y_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)$$

$$= \frac{1}{\sigma\sqrt{n}} \left(\Sigma X_i - n\mu\right) = \frac{n}{\sigma\sqrt{n}} (\bar{X} - \mu)$$

$$= \frac{\bar{X} - \mu}{(\sigma/\sqrt{n})}$$

(12)

So the Central Limit Theorem says

$$\frac{\bar{X} - \mu}{(\sigma/\sqrt{n})} \xrightarrow{\mathcal{D}} N(0,1) \quad !!!!$$

Midterm problem types

— Suppose $X_1, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$

Given some function of these, find the distribution

- Given 2 random variables & their joint distribution, and given a function of them, find its distribution.

- Given a joint pdf, find the covariance, correlation, conditional expectation, conditional variance

- Find the distribution of $X_{(k)}$ or find the joint distribution of $X_{(j)}, X_{(k)}$