Stat 502
1-8-19

Review:

(1)

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$ ← this is a real number

$$E[X|y] = \int_{-\infty}^{\infty} x f(x|y)\,dx$$ (Defined on 11/27/18)

this is a function of y

<u>Defn</u>: $E[X|Y]$ is a new random variable, and is a function of the r.v. Y

(2)

<u>Defn</u>: $V[X|Y] = E\left[(X - E[X|Y])^2 \mid Y\right]$

Find $E(E[X|Y])$

$$= \int_{-\infty}^{\infty} \underbrace{E[X|y]}\, g(y)\,dy$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x f(x|y)\,dx\right] g(y)\,dy$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x \frac{f(x,y)}{g(y)}\,dx\right] g(y)\,dy$$

(3)

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x,y)\, dy \right] x\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x)\, x\, dx = E[X]$$

$$\therefore \quad E\left(E[X|Y]\right) = E[X]$$

"Law of iterated expectations"

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$$V[X] = E\left[(X - \mu_X)^2\right]$$

$$= E\left[\left(\underbrace{X - E[X|Y]} + \underbrace{E[X|Y] - \mu_X}\right)^2\right]$$

$$= E\left[(X - E[X|Y])^2\right] \quad (1)$$

$$+ E\left[(E[X|Y] - \mu_X)^2\right] \quad (2)$$

$$+ 2E\left[(X - E[X|Y])(E[X|Y] - \mu_X)\right] \quad (3)$$

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(3): $$2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x - E[X|y])(E[X|y]-\mu_x)\, f(x,y)\,dx\,dy$$

$$= 2\int_{-\infty}^{\infty}(E[X|y]-\mu_x)\underbrace{\left[\int_{-\infty}^{\infty}(x-E[X|y])\, f(x,y)\,dx\right]}\, dy = 0$$

$$\underbrace{\int_{-\infty}^{\infty} x\, f(x,y)\,dx}_{\int_{-\infty}^{\infty} x\, f(x|y)\, g(y)\,dx} - E[X|y]\underbrace{\int_{-\infty}^{\infty} f(x,y)\,dx}_{g(y)}$$

$$g(y)\, E[X|y]$$

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(2): $$E\left[(E[X|Y]-\mu_x)^2\right]$$

Let $W = E[X|Y]$

Then $E[W] = E(E[X|Y])$
$= E[X]$
$= \mu_x$

$$= E\left[(W - E[W])^2\right]$$

$$= V[W]$$

$$= V[E[X|Y]]$$

(1): $$E\left[(X-E[X|Y])^2\right] = E\left(E\left[(X-E[X|Y])^2 \,|\, Y\right]\right)$$

(using law of iterated expectations)

(1): $E[V[X|Y]]$

$$\therefore\ V[X] = E[V[X|Y]] + V[E[X|Y]]$$

Suppose that X and Y are independent r.v.s

Let $g(X)$ and $h(Y)$ be 2 new r.v.s

Then $E[g(X)h(Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y) \underbrace{f(x,y)}_{f_X(x) f_Y(y) \text{ by indp}} dx\,dy$

$$= \int_{-\infty}^{\infty} h(y) f_Y(y) \underbrace{\left[\int_{-\infty}^{\infty} g(x) f_X(x)\, dx\right]}_{E[g(X)]} dy$$

$$= E[g(X)]\, E[h(Y)]$$

In particular, if X & Y are independent,

then $E[XY] = E[X]E[Y]$.

Defn: The covariance of X and Y is

$$E[(X-\mu_X)(Y-\mu_Y)] = Cov(X,Y) = \sigma_{XY}$$

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Alternate formula:

$$\sigma_{XY} = E[(X-\mu_x)(Y-\mu_Y)]$$
$$= E[XY - X\mu_Y - \mu_x Y + \mu_x\mu_Y]$$
$$= E[XY] - \mu_Y \underbrace{E[X]}_{\mu_x} - \mu_x \underbrace{E[Y]}_{\mu_Y} + \mu_x\mu_Y$$
$$= E[XY] - \mu_x\mu_Y$$

Note: $\mathrm{Cov}(X,X) = E[X^2] - \mu_x^2 = V[X]$

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Note: If X and Y are independent, then

$$\sigma_{XY} = E[XY] - \mu_x\mu_Y = E[X]E[Y] - \mu_x\mu_Y = 0$$

Defn: The correlation between X and Y is

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{V[X]V[Y]}}$$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y}$$

(11)

Let X and Y be independent r.v.s

Let $W = X + Y$

Find the moment generating function of W

$$M_W(t) = E[e^{tW}]$$
$$= E[e^{t(X+Y)}] = E[e^{tX} e^{tY}]$$
$$= E[e^{tX}] E[e^{tY}] = M_X(t) M_Y(t)$$

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Example: Let X have a Poisson distribution with parameter μ_1

Let Y have a Poisson distribution with parameter μ_2, independent of X

Let $W = X + Y$

$$M_W(t) = M_X(t) M_Y(t) = e^{\mu_1(e^t - 1)} e^{\mu_2(e^t - 1)}$$
$$= e^{(\mu_1 + \mu_2)(e^t - 1)}$$

So $W \sim \text{Poisson}(\mu_1 + \mu_2)$