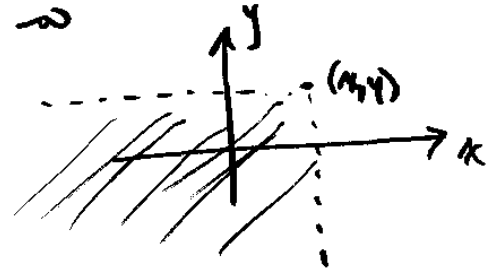
Stat 562
1-15-19

Defn. For a random vector $\begin{bmatrix} X \\ Y \end{bmatrix}$, the cdf (cumulative distribution function) ①

is $F(x,y) = P(X \le x \cap Y \le y)$.

In the continuous case,

$$F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t)\,ds\,dt$$



②

Properties of Cov(X,Y)

1) $$\begin{aligned} Cov(aX, bY) &= E[abXY] - E[aX]E[bY] \\ &= ab\,E[XY] - ab\,E[X]E[Y] \\ &= ab\,Cov(X,Y) \end{aligned}$$

2) $$\begin{aligned} Cov(X, Y+Z) &= E[X(Y+Z)] - E[X]E[Y+Z] \\ &= \underline{E[XY]} + \underline{\underline{E[XZ]}} - \left[\underline{E[X]E[Y]} + \underline{\underline{E[X]E[Z]}}\right] \\ &= Cov(X,Y) + Cov(X,Z) \end{aligned}$$

3.) $\mathrm{Cov}(X, c) = E[cX] - E[c]E[X]$ ③

$= cE[X] - cE[X] = 0$

Properties (1) and (2) make $\mathrm{Cov}(X,Y)$ a <u>bilinear</u> <u>operator</u>.

Consider $\{X - \mu_X \mid X \text{ is a random variable}\}$

that is, the set of all random variables with mean 0.

④

This is a well-defined vector space, where the scalars are the real numbers.

The Covariance operator will satisfy the definition of an inner product on this vector space.

The Cauchy-Schwarz inequality says

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$$

⑤

Apply this to the covariance operator:

$$\left|\operatorname{Cov}(X-\mu_x, Y-\mu_Y)\right| \le \sqrt{\operatorname{Cov}(X-\mu_x, X-\mu_x)} \cdot \sqrt{\operatorname{Cov}(Y-\mu_Y, Y-\mu_Y)}$$

$$\left|\operatorname{Cov}(X,Y)\right| \le \sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}$$

$$|\sigma_{XY}| \le \sigma_X \sigma_Y \qquad \left|\frac{\sigma_{XY}}{\sigma_X \sigma_Y}\right| \le 1$$

$$\therefore \; |\rho_{XY}| \le 1$$

⑥

Cauchy-Schwarz also says that equality occurs iff $\vec{u} = c\vec{v}$ for some c

So $|\rho_{XY}| = 1$ iff $Y - \mu_Y = c(X - \mu_X)$

$$Y = cX + \mu_Y - c\mu_X$$

$\therefore$ The correlation equals ± 1 iff Y is a perfect linear function of X.

⑦

Note: $V[X+Y] = E[(X+Y)^2] - (E[X+Y])^2$

$$= E[X^2+Y^2+2XY] - (E[X]+E[Y])^2$$

$$= \underbrace{E[X^2] - (E[X])^2} + \underbrace{E[Y^2] - (E[Y])^2}$$

$$+ \underbrace{2E[XY] - 2E[X]E[Y]}$$

$$= V[X] + V[Y] + 2\,\mathrm{Cov}(X,Y)$$

If X & Y are uncorrelated, then

$$V[X+Y] = V[X] + V[Y]$$

⑧

Assume that the random variables $X_1, X_2, \ldots, X_n$ are <u>independent</u> and <u>identically</u> <u>distributed</u> (iid)

<u>Defn</u>: A <u>statistic</u> T is a function of the random variables $X_1, \ldots, X_n$.

<u>Defn</u>: The <u>sampling distribution</u> of T is the probability distribution of T.

(9)

Defn: $\bar{X}$ = sample mean = $\frac{1}{n}\sum_{i=1}^{n} X_i$

S^2 = sample variance = $\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$

S = sample standard deviation = $\sqrt{S^2}$

Alternate forms for $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i^2+\bar{X}^2-2X_i\bar{X}\right)$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n}X_i^2 + n\bar{X}^2 - 2\bar{X}\underbrace{\sum_{i=1}^{n}X_i}_{n\bar{X}}\right]$$

(10)

$$S^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n}X_i^2 - n\bar{X}^2\right]$$

Theorem: $\sum_{i=1}^{n}(x_i-a)^2$ is minimized when $a=\bar{x}$

Pf: Let $g(a) = \sum_{i=1}^{n}(x_i-a)^2$

$$g'(a) = \sum_{i=1}^{n}\cancel{2}(x_i-a)(-\cancel{1}) \overset{\text{set}}{=} 0$$

$$\sum x_i - na = 0$$

$$a = \frac{1}{n}\sum x_i = \bar{x}$$

(11)

<u>Theorem</u>: If X and Y are independent, then $g(X)$ and $h(Y)$ are independent.

<u>Proof</u>: let $U = g(X)$ and $V = h(Y)$

$$F(u,v) = P(U \le u \cap V \le v)$$
$$= P(g(X) \le u \cap h(Y) \le v)$$
$$= P(X \in A \cap Y \in B)$$

where $A = \{x : g(x) \le u\}$ and $B = \{y : h(y) \le v\}$

(12)

Let $I_A(x) = \begin{cases} 1 & x \in A \\ 0 & \text{otherwise} \end{cases}$

$$I_B(y) = \begin{cases} 1 & y \in B \\ 0 & \text{otherwise} \end{cases}$$

Let $C = \{(x,y) : x \in A \text{ and } y \in B\}$

$$I_C(x,y) = \begin{cases} 1 & (x,y) \in C \\ 0 & \text{otherwise} \end{cases}$$

(13)

Then $I_C(x,y) = I_A(x)\, I_B(y)$

$$E[I_A(X)] = 1 \cdot P(I_A(X)=1) + 0 \cdot P(I_A(X)=0)$$
$$= P(X \in A)$$

Similarly, $E[I_B(Y)] = P(Y \in B)$

and $E[I_C(X,Y)] = P[(X,Y) \in C]$

$$= P[X \in A \cap Y \in B]$$
$$= P[X \in A]\, P[Y \in B] \quad \text{by independence}$$

(14)

So $P[g(X) \le u \cap h(Y) \le v]$

$$= P[g(X) \le u]\, P[h(Y) \le v]$$

$$\therefore \; F(u,v) = F_U(u)\, F_V(v)$$

$$f(u,v) = \frac{\partial^2}{\partial u\, \partial v} F(u,v) = \frac{\partial}{\partial u}\left(\frac{\partial}{\partial v} F_U(u) F_V(v) \right)$$

$$= \frac{\partial}{\partial u}\left(F_U(u)\, f_V(v) \right) = f_U(u) f_V(v)$$

$\therefore$ U & V are independent