<u>More estimation examples</u>

Stat 562
2-26-19

①

<u>Example</u>: $f(x|\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$$L(\mu,\sigma^2) = \prod_{i=1}^{n} f(x_i|\mu,\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}$$

$$= (\sigma^2)^{-n/2}(2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2}$$

$$l(\mu,\sigma^2) = \ln L(\mu,\sigma^2)$$
$$= -\frac{n}{2}\ln(\sigma^2) - \frac{n}{2}\ln(2\pi) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

②

$$\frac{\partial l}{\partial \mu} = -\frac{1}{\cancel{2}\sigma^2}\sum_{i=1}^{n}\cancel{2}(x_i-\mu)\cancel{(-1)} \stackrel{set}{=} 0$$

$$\sum x_i - n\mu = 0$$

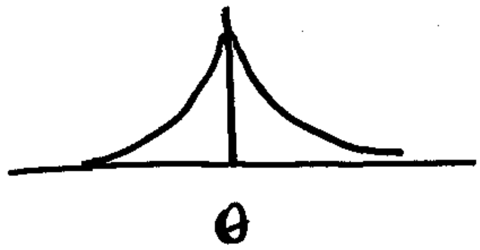
$$\hat{\mu} = \frac{\sum x_i}{n} = \bar{X}$$

$$\frac{\partial l}{\partial \sigma^2} = -\frac{n}{2}\frac{1}{\sigma^2} - \frac{1}{2}\sum(x_i-\mu)^2(-1)\frac{1}{\sigma^4} \stackrel{set}{=} 0$$

$$-n\sigma^2 + \sum(x_i-\mu)^2 = 0$$

$$\hat{\sigma}^2 = \frac{\sum(x_i-\bar{x})^2}{n} = S_n^2$$

Example: $f(x|\theta) = \frac{1}{2} e^{-|x-\theta|}$ $\quad -\infty < x < \infty$



Laplace or Double Exponential

MoM: Set $\bar{x} = \mu = \theta$ $\quad \therefore \hat{\theta}_{MoM} = \bar{X}$

MLE: $L(\theta) = \prod_{i=1}^{n} \frac{1}{2} e^{-|x_i-\theta|} = \frac{1}{2^n} e^{-\sum |x_i-\theta|}$

$$\ell(\theta) = -n\ln 2 - \sum_{i=1}^{n} |x_i - \theta|$$

$$\ell'(\theta) = -\sum_{i=1}^{n} \begin{Bmatrix} -1 & \text{if } x_i > \theta \\ 0 & \text{if } x_i = \theta \\ 1 & \text{if } x_i < \theta \end{Bmatrix}$$

$$= \sum_{i=1}^{n} \operatorname{sgn}(x_i - \theta) \overset{set}{=} 0$$

$$\hat{\theta}_{MLE} = \text{median}\{X_i\}$$

$$= \begin{cases} X_{(\frac{n+1}{2})} & \text{if } n \text{ is odd} \\ \text{any } \theta \text{ between } X_{(\frac{n}{2})} \text{ \& } X_{(\frac{n}{2}+1)} & \text{if } n \text{ is even} \end{cases}$$

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<u>Theorem:</u> If $\hat{\theta}$ is the MLE of θ,

then $\tau(\hat{\theta})$ is the MLE of $\tau(\theta)$,

for any function τ.

<u>Proof</u>: Let $\eta = \tau(\theta)$

Let $L^*(\eta) = \sup_{\{\theta : \tau(\theta) = \eta\}} L(\theta)$ $\left.\right\}$ We call this the <u>induced</u> likelihood function for $\tau(\theta)$

Let $\hat{\eta}$ be the value of η that maximizes $L^*(\eta)$

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$$L^*(\hat{\eta}) = \sup_{\eta} L^*(\eta) = \sup_{\eta} \sup_{\{\theta : \tau(\theta) = \eta\}} L(\theta)$$

$$= \sup_{\theta} L(\theta) = L(\hat{\theta})$$

Also, $L(\hat{\theta}) = \sup_{\{\theta : \tau(\theta) = \tau(\hat{\theta})\}} L(\theta)$ Why?

Consider A, B, C : $A = \{\hat{\theta}\}$

$B = \{\theta : \tau(\theta) = \tau(\hat{\theta})\}$

$C = \Omega$ (the parameter space)

$A \subset B \subset C$

⑦

$$\sup_A L(\theta) \le \sup_B L(\theta) \le \sup_C L(\theta)$$

$$\sup_A L(\theta) = L(\hat{\theta}) = \sup_C L(\theta)$$

$$\therefore \sup_B L(\theta) = L(\hat{\theta})$$

Now $L^*(\hat{\eta}) = L(\hat{\theta}) = \sup_{\{\theta : \tau(\theta) = \tau(\hat{\theta})\}} L(\theta)$

$$= L^*(\tau(\hat{\theta}))$$

$\therefore$ The MLE of $\tau(\theta)$ is $\tau(\hat{\theta})$

⑧

<u>Bayes Estimation</u>

$X_1, \ldots, X_n \sim iid\ f(x \mid \theta)$

Assume that θ is a random variable, independent of $X_1, \ldots, X_n$. Assume that θ has a probability distribution $\pi(\theta)$. We call $\pi(\theta)$ the <u>prior distribution</u> of θ.

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The joint distribution of $(X_1, \dots, X_n, \theta)$

is $g(\vec{x}, \theta) = \pi(\theta) \prod_{i=1}^{n} f(x_i | \theta)$

the marginal distribution of $(X_1, \dots, X_n)$

is $m(\vec{x}) = \int_{-\infty}^{\infty} g(\vec{x}, \theta)\, d\theta = \int_{-\infty}^{\infty} \pi(\theta) L(\theta)\, d\theta$

Then the posterior distribution of $\theta | \vec{x}$

is $\pi(\theta | \vec{x}) = \frac{g(\vec{x}, \theta)}{m(\vec{x})}$

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Let $\mathcal{L}_\theta(\hat{\theta})$ be a loss function

$\mathcal{L}_\theta(\theta) = 0$

$\mathcal{L}_\theta(\hat{\theta}) \geq 0$

Defn: The Bayes estimator of θ is

the value that minimizes $\underbrace{E\left[\mathcal{L}_\theta(\hat{\theta}) \middle| \vec{x}\right]}_{\text{posterior risk}}$

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Example: let $X_1, \dots, X_n \sim$ iid Poisson(θ)

$$\theta \sim \text{Gamma}(\underbrace{\alpha, \beta}_{\text{known}})$$

$$\pi(\theta) = \frac{\theta^{\alpha-1} e^{-\theta/\beta}}{\Gamma(\alpha)\beta^\alpha}, \quad \theta > 0$$

$$g(\vec{x}, \theta) = \frac{\theta^{\alpha-1} e^{-\theta/\beta}}{\Gamma(\alpha)\beta^\alpha} \cdot \prod_{i=1}^{n} \frac{\theta^{x_i} e^{-\theta}}{x_i!}$$

$$= \frac{\theta^{\alpha-1} e^{-\theta/\beta}}{\Gamma(\alpha)\beta^\alpha} \frac{\theta^{\Sigma x_i} e^{-n\theta}}{\prod_{i=1}^{n}(x_i!)}$$

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$$m(\vec{x}) = \int g(\vec{x}, \theta)\, d\theta$$

$$= \frac{\Gamma(\alpha+\Sigma x_i)\left(\frac{1}{\frac{1}{\beta}+n}\right)^{\alpha+\Sigma x_i}}{\Gamma(\alpha)\beta^\alpha \prod_{i=1}^{n}(x_i!)} \underbrace{\int_0^\infty \frac{\theta^{\alpha+\Sigma x_i - 1} e^{-\theta\left(\frac{1}{\beta}+n\right)}}{\Gamma(\alpha+\Sigma x_i)\left(\frac{1}{\frac{1}{\beta}+n}\right)^{\alpha+\Sigma x_i}}\, d\theta}_{1}$$

$$\pi(\theta \mid \vec{x}) = \frac{g(\vec{x}, \theta)}{m(\vec{x})}$$

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$$\pi(\theta|\vec{x}) = \frac{\left[\dfrac{\theta^{\alpha-1}e^{-\theta/\beta}}{\Gamma(\alpha)\cancel{\beta^{\alpha}}} \dfrac{\theta^{\Sigma x_i}e^{-n\theta}}{\cancel{\prod_{i=1}^{n}(x_i!)}}\right]}{\left[\dfrac{\Gamma(\alpha+\Sigma x_i)\left(\dfrac{\beta}{1+n\beta}\right)^{\alpha+\Sigma x_i}}{\cancel{\Gamma(\alpha)\beta^{\alpha}}\cancel{\prod_{i=1}^{n}(x_i!)}}\right]}$$

$$\sim \text{Gamma}\left(\alpha+\Sigma x_i, \frac{\beta}{n\beta+1}\right)$$