Stat 562
1-17-19

1

<u>Theorem</u>: Let $X_1, X_2, \ldots, X_n$ be iid random variables.

Then $E\left[\sum_{i=1}^{n} g(X_i)\right] = n E[g(X_1)]$

and $V\left[\sum_{i=1}^{n} g(X_i)\right] = n V[g(X_1)]$.

<u>Proof</u>: $g(X_1), g(X_2), \ldots, g(X_n)$ are independent by the previous theorem.

2

$$E\left[\sum_{i=1}^{n} g(X_i)\right] = \sum_{i=1}^{n} E[g(X_i)] = n E[g(X_1)]$$

$$V\left[\sum_{i=1}^{n} g(X_i)\right] = \sum_{i=1}^{n} V[g(X_i)] + \cancelto{0}{\sum\sum_{i \neq j} Cov(g(X_i), g(X_j))}$$

$$= n V[g(X_1)]$$

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Assume $X_1, \ldots, X_n$ are iid with mean $\mu$ and variance $\sigma^2$

Let $T = \bar{X}$

3

Then $E(T) = E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$

$$= \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}\sum_{i=1}^{n}\mu = \mu$$

This means that $T = \bar{X}$ is an <u>unbiased</u> <u>estimator</u> of $\mu$.

And $V(T) = V(\bar{X}) = V\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$

$$= \frac{1}{n^2}\sum_{i=1}^{n} V(X_i) = \frac{1}{n^2}\, n\sigma^2 = \frac{\sigma^2}{n}$$

4

Recall $S^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right]$

$$E(S^2) = \frac{1}{n-1}\left[\sum_{i=1}^{n} E(X_i^2) - nE(\bar{X}^2)\right]$$

$$= \frac{1}{n-1}\left[n(\sigma^2+\mu^2) - n\left(\frac{\sigma^2}{n}+\mu^2\right)\right]$$

$V(X) = E(X^2) - (E(X))^2$

So $E(X^2) = V(X) + (E(X))^2$

$$= \frac{1}{n-1}\left[\sigma^2(n-1)\right]$$

$= \sigma^2$ $\quad \therefore S^2$ is an unbiased estimator of $\sigma^2$

(5)

**Theorem**: Assume $X_1, \ldots, X_n$ are iid

with moment generating function $M_X(t)$.

Then $M_{\bar{X}}(t) = [M_X(t/n)]^n$.

**Proof**: $M_{\sum_{i=1}^n X_i}(t) = E[e^{t\sum X_i}] = E[\prod_{i=1}^n e^{tX_i}]$

$$= \prod_{i=1}^n E[e^{tX_i}] = [M_X(t)]^n$$

Also let $W = aY$

(6)

Then $M_W(t) = E[e^{tW}] = E[e^{atY}]$

$$= M_Y(at)$$

$$\therefore M_{\bar{X}}(t) = [M_X(t/n)]^n$$

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**Example**: let $X_1, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$

Find the distribution of $\bar{X}$.

$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

$$M_{\bar{X}}(t) = [M_X(t/n)]^n$$

(7)

$$= \left[ e^{\mu t/n + \frac{\sigma^2 t^2}{2n^2}} \right]^n$$

$$M_{\bar{X}}(t) = e^{\mu t + \frac{\sigma^2 t^2}{2n}}$$

So $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

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Example: Assume $X_1, \ldots, X_n \sim$ iid Gamma $(\alpha, \beta)$

Find the distribution of $\bar{X}$.

$(1-\beta t)^{-\alpha}$

$\mu = \alpha\beta$

$\sigma^2 = \alpha\beta^2$

$$M_{\bar{X}}(t) = \left[ \left(1 - \beta t/n\right)^{-\alpha} \right]^n$$

(8)

$$M_{\bar{X}}(t) = \left(1 - \frac{\beta}{n} t\right)^{-n\alpha}$$

So $\bar{X} \sim \text{Gamma}(n\alpha, \beta/n)$

$$\mu_{\bar{X}} = n\alpha \beta/n = \alpha\beta = \mu$$

$$\sigma^2_{\bar{X}} = n\alpha (\beta/n)^2 = \frac{\alpha\beta^2}{n} = \frac{\sigma^2}{n}$$

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Theorem: Let X and Y be independent continuous random variables. Let $Z = X + Y$.

(9)

Then $g_Z(z) = \int_{-\infty}^{\infty} f_X(w) f_Y(z-w)\,dw$.

This is the convolution formula.

Proof: $Z = X+Y$

Let $W = X$

$X = W$

$Y = Z - W$

$$J = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$$

$g(z,w) = f(x,y)\,|J|$

(10)

$g(z,w) = f_X(x)\, f_Y(y) \cdot 1$

$\therefore\ g_Z(z) = \int_{-\infty}^{\infty} g(z,w)\,dw$

$= \int_{-\infty}^{\infty} f_X(w)\, f_Y(z-w)\,dw$

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Example: Let $X \sim \text{Cauchy}(0,\sigma)$ and $Y \sim \text{Cauchy}(0,\tau)$ and they are independent.

Let $Z = X+Y$. Find $g_Z(z)$

(11)

$$f_X(x) = \frac{1}{\pi\sigma} \cdot \frac{1}{1+(\frac{x}{\sigma})^2}$$

$$f_Y(y) = \frac{1}{\pi\tau} \cdot \frac{1}{1+(y/\tau)^2}$$

$$g_Z(z) = \int_{-\infty}^{\infty} \frac{1}{\pi\sigma} \frac{1}{1+(\frac{w}{\sigma})^2} \frac{1}{\pi\tau} \frac{1}{1+(\frac{z-w}{\tau})^2} \, dw$$

The rest is part of homework problem #7

The result should be Cauchy$(0, \sigma+\tau)$

(12)

If $X_1, \dots, X_n \sim$ iid Cauchy$(0, \sigma)$

then $\sum_{i=1}^{n} X_i \sim$ Cauchy$(0, n\sigma)$

and $\bar{X} \sim$ Cauchy$(0, \sigma)$

HW#2 due 1/24

Chp5 # 7, 10 c, 13, 15

p. 256

**5.7** In Example 5.2.10, a partial fraction decomposition is needed to derive the distribution of the sum of two independent Cauchy random variables. This exercise provides the details that are skipped in that example.

(a) Find the constants $A$, $B$, $C$, and $D$ that satisfy

$$\frac{1}{1+(w/\sigma)^2}\,\frac{1}{1+((z-w)/\tau)^2} = \frac{Aw}{1+(w/\sigma)^2}+\frac{B}{1+(w/\sigma)^2}-\frac{Cw}{1+((z-w)/\tau)^2}-\frac{D}{1+((z-w)/\tau)^2},$$

where $A$, $B$, $C$, and $D$ may depend on $z$ but not on $w$.

(b) Using the facts that

$$\int \frac{t}{1+t^2}\,dt = \frac{1}{2}\log(1+t^2)+\textit{constant} \quad \text{and} \quad \int \frac{1}{1+t^2}\,dt = \arctan(t)+\textit{constant},$$

evaluate (5.2.4) and hence verify (5.2.5).

(Note that the integration in part ($b$) is quite delicate. Since the mean of a Cauchy does not exist, the integrals $\int_{-\infty}^{\infty}\frac{Aw}{1+(w/\sigma)^2}\,dw$ and $\int_{-\infty}^{\infty}\frac{Cw}{1+((z-w)/\tau)^2}\,dw$ do not exist. However, the integral of the difference *does exist*, which is all that is needed.)

**5.10** Let $X_1,\ldots,X_n$ be a random sample from a $\mathrm{n}(\mu,\sigma^2)$ population.

(c) Calculate $\operatorname{Var} S^2$ a completely different (and easier) way: Use the fact that $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$.

**5.13** Let $X_1,\ldots,X_n$ be iid $\mathrm{n}(\mu,\sigma^2)$. Find a function of $S^2$, the sample variance, say $g(S^2)$, that satisfies $\mathrm{E}g(S^2)=\sigma$. (*Hint:* Try $g(S^2)=c\sqrt{S^2}$, where $c$ is a constant.)

**5.15** Establish the following recursion relations for means and variances. Let $\bar{X}_n$ and $S_n^2$ be the mean and variance, respectively, of $X_1,\ldots,X_n$. Then suppose another observation, $X_{n+1}$, becomes available. Show that

(a) $\bar{X}_{n+1} = \frac{X_{n+1}+n\bar{X}_n}{n+1}$.

(b) $nS^2_{n+1} = (n-1)S_n^2 + \left(\frac{n}{n+1}\right)(X_{n+1}-\bar{X}_n)^2$.