Stat 562
3-12-19


Task ① Use the delta method to estimate the expected value & variance of $S$ [Assume $X_1,\dots,X_n \sim$ iid $N(\mu,\sigma^2)$]

② Find the actual expected value & variance

Use: $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \sim \text{Gamma}(\alpha = \frac{n-1}{2}, \beta = 2)$

If $Y \sim \text{Gamma}(\alpha, \beta)$ then $E[Y^k] = \frac{\beta^k \Gamma(\alpha+k)}{\Gamma(\alpha)}$

Also, $E[S^2] = \sigma^2$ and $V[S^2] = \frac{2\sigma^4}{n-1}$



① Let $g(y) = \sqrt{y}$, so $g'(y) = \frac{1}{2} y^{-\frac{1}{2}}$, $g''(y) = -\frac{1}{4} y^{-3/2}$

$$\sqrt{y} \approx \sqrt{y_0} + \frac{1}{2} y_0^{-\frac{1}{2}}(y - y_0) - \frac{1}{4} y_0^{-3/2} \frac{(y-y_0)^2}{2}$$

with $y = S^2$ and $y_0 = \sigma^2$,

$$S \approx \sigma + \frac{1}{2\sigma}(S^2 - \sigma^2) - \frac{1}{8\sigma^3}(S^2 - \sigma^2)^2$$

$$E[S] \approx \sigma - \frac{1}{8\sigma^3} V(S^2) = \sigma - \frac{1}{8\sigma^3} \frac{2\sigma^4}{n-1}$$

$$= \sigma\left(1 - \frac{1}{4(n-1)}\right)$$

$$V[S] \approx \frac{1}{4\sigma^2} V[S^2] = \frac{1}{4\sigma^2} \frac{2\sigma^4}{n-1} = \frac{\sigma^2}{2(n-1)}$$

2 Let $X = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \sim \text{Gamma}\left(\frac{n-1}{2}, 2\right)$

$$E[X^{\frac{1}{2}}] = \frac{2^{1/2}\,\Gamma\left(\frac{n-1}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

$$\parallel$$

$$\frac{\sqrt{n-1}}{\sigma}E[S]$$

So $$E[S] = \frac{\sigma\sqrt{2}}{\sqrt{n-1}}\,\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

$$E[S^2] = \sigma^2$$

So $$V[S] = \sigma^2 - \frac{\sigma^2 2}{n-1}\left[\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\right]^2 = \sigma^2\left[1 - \frac{2}{n-1}\left(\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\right)^2\right]$$