Stat 562

1-22-19

(1)

Recall $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$

$$= \frac{1}{n-1}\left[(X_1 - \bar{X})^2 + \sum_{i=2}^{n} (X_i - \bar{X})^2 \right]$$

Also $\sum_{i=1}^{n} (X_i - \bar{X}) = \sum_{i=1}^{n} X_i - n\bar{X} = 0$

So $X_1 - \bar{X} = -\sum_{i=2}^{n} (X_i - \bar{X})$

So $S^2 = \frac{1}{n-1}\left[\left(\sum_{i=2}^{n} (X_i - \bar{X}) \right)^2 + \sum_{i=2}^{n} (X_i - \bar{X})^2 \right]$

(2)

That is, S^2 is a function of

$$X_2 - \bar{X},\ X_3 - \bar{X}, \ldots, X_n - \bar{X}$$

Assume $X_1, \ldots, X_n \sim$ iid $N(0,1)$

Let
$$Y_1 = \bar{X} \qquad X_1 = Y_1 - \sum_{2}^{n} Y_i$$
$$Y_2 = X_2 - \bar{X} \qquad X_2 = Y_1 + Y_2$$
$$\vdots \qquad \vdots$$
$$Y_n = X_n - \bar{X} \qquad X_n = Y_1 + Y_n$$

$Y_1 = \frac{1}{n} \sum X_i$

$nY_1 = \sum X_i$

$X_1 = nY_1 - \sum_{2}^{n} X_i$

(3)

$$J = \begin{vmatrix} 1 & -1 & \cdots & \cdots & -1 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ 1 & 0 & \cdots & 0 & 1 \end{vmatrix} = n$$

$$\begin{aligned} g(y_1, \dots, y_n) &= f(x_1, \dots, x_n)\, |J| \\ &= n \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x_i^2} \right) \\ &= \frac{n}{(2\pi)^{n/2}} e^{-\frac{1}{2} \sum_{i=1}^{n} x_i^2} \end{aligned}$$

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$$\begin{aligned} \sum_{i=1}^{n} x_i^2 &= \left(y_1 - \sum_{i=2}^{n} y_i \right)^2 + \sum_{i=2}^{n} (y_1 + y_i)^2 \\ &= y_1^2 + \left(\sum_{i=2}^{n} y_i \right)^2 - \cancel{2 y_1 \sum_{i=2}^{n} y_i} \\ &\quad + (n-1) y_1^2 + \sum_{i=2}^{n} y_i^2 + \cancel{2 y_1 \sum_{i=2}^{n} y_i} \\ &= n y_1^2 + \left(\sum_{i=2}^{n} y_i \right)^2 + \sum_{i=2}^{n} y_i^2 \end{aligned}$$

(5)

Now $g(y_1,\dots,y_n) = \frac{n}{(2\pi)^{n/2}} e^{-\frac{1}{2}\left(ny_1^2 + \left(\sum_2^n y_i\right)^2 + \sum_2^n y_i^2\right)}$

$= h(y_1)\; k(y_2,\dots,y_n)$

By our factorization theorem,

Y_1 is independent of $Y_2,\dots,Y_n$

$\therefore\ \bar{X}$ is independent of S^2

(6)

Things we know about the χ^2 distribution

(1) If $Z \sim N(0,1)$ then $Z^2 \sim \chi^2_1$

(2) $\chi^2_r \sim \text{Gamma}\left(\alpha = \frac{r}{2}, \beta = 2\right)$

(3) $W_1,\dots,W_n \sim$ indep $\chi^2_{r_i} \Rightarrow \sum_{i=1}^n W_i \sim \chi^2_{\sum r_i}$

Let $Z_1,\dots,Z_n \sim$ iid $N(0,1)$

Fact: $(n-1)S^2 = (n-2)\underbrace{S^2_{n-1}} + \frac{n-1}{n}\left(Z_n - \underbrace{\bar{Z}_{n-1}}\right)^2$ | This is HW #5

based on $Z_1,\dots,Z_{n-1}$

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<u>Theorem</u>. If $Z_1, \ldots, Z_n \sim$ iid $N(0,1)$,

then $(n-1)S^2 \sim \chi^2_{n-1}$

<u>Proof</u>: Start with $n=2$.

$$S^2 = \frac{1}{2}(Z_2 - Z_1)^2$$

$$Z_1 \sim N(0,1), \; Z_2 \sim N(0,1)$$

$$Z_2 - Z_1 \sim N(0,2)$$

$$\frac{Z_2 - Z_1}{\sqrt{2}} \sim N(0,1) \quad \therefore \frac{(Z_2-Z_1)^2}{2} \sim \chi^2_1$$

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Induction step: Assume true for n, show true for $n+1$

Our fact states that

$$n S^2 = \underbrace{(n-1)S_n^2}_{\sim \chi^2_{n-1}} + \underbrace{\frac{n}{n+1}(Z_{n+1} - \bar{Z}_n)^2}_{\chi^2_1}$$

$Z_{n+1} \sim N(0,1)$, indep of $Z_1, \ldots, Z_n$, so it is indep of $\bar{Z}_n$

So $Z_{n+1} - \bar{Z}_n \sim N\left(0, 1 + \frac{1}{n}\right)$

(9)

$$\frac{Z_{n+1} - \bar{Z}_n}{\sqrt{1 + \frac{1}{n}}} \sim N(0,1)$$

$$\frac{(Z_{n+1} - \bar{Z}_n)^2}{1 + \frac{1}{n}} = \frac{n(Z_{n+1} - \bar{Z}_n)^2}{n+1} \sim \chi^2_1$$

Also S^2_n is indep of Z_{n+1} and $\bar{Z}_n$, so

$n\, S^2 \sim \chi^2_n$

(10)

Generalization:

Assume $X_1, \dots, X_n \sim$ iid $N(\mu, \sigma^2)$

Then if $Z_i = \frac{X_i - \mu}{\sigma}$,

$$Z_1, \dots, Z_n \sim \text{iid } N(0,1)$$

Our theorem $\Rightarrow (n-1) S^2_Z \sim \chi^2_{n-1}$

But $S^2_Z = \frac{1}{n-1}\left(\sum_1^n Z_i^2 - n\bar{Z}^2\right)$

$$= \frac{1}{n-1} \sum_{i=1}^n (Z_i - \bar{Z})^2$$

(11)

$$S_Z^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma} - \frac{\bar{X}-\mu}{\sigma}\right)^2$$

$$= \frac{1}{n-1}\frac{\sum_1^n (X_i-\bar{X})^2}{\sigma^2} = \frac{S^2}{\sigma^2}$$

So $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

Review: With the normality assumption of $X_1, \dots, X_n \sim \text{iid } N(\mu, \sigma^2)$,

(12)

① $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

② $\bar{X}$ & S^2 are independent

③ $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

Without the normality assumption,

① $E(\bar{X}) = \mu$

② $V(\bar{X}) = \sigma^2/n$

③ $E[S^2] = \sigma^2$