Stat 562
1-10-19

Transformations

①

Suppose that X and Y have joint pdf

$$f(x,y) = \frac{1}{4} e^{-\frac{1}{2}(x+y)} \qquad \begin{matrix} 0 < x < \infty \\ 0 < y < \infty \end{matrix}$$

Find the pdf of $U = \frac{1}{2}(X-Y)$

Let $V = Y$

Solve for X & Y in terms of U & V

$\underline{Y = V}$ $\quad U = \frac{1}{2}(X - V)$ $\quad X = \underline{2U + V}$

②

$$\text{Jacobian} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$J = 2$$

$$g(u,v) = f(x,y)\,|J|$$

$$= \frac{1}{4} e^{-\frac{1}{2}(x+y)} \cdot 2$$

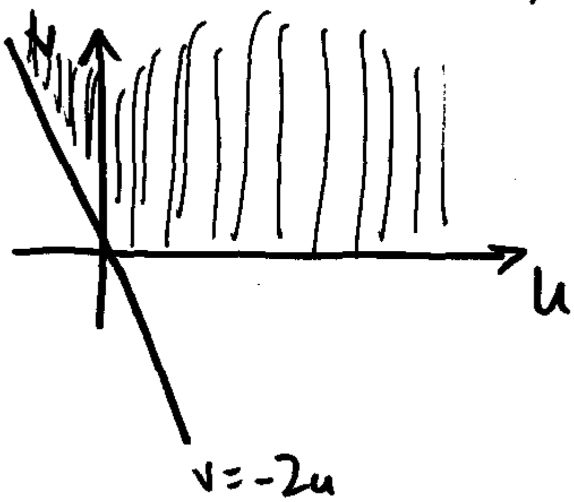
$$= \frac{1}{4} e^{-\frac{1}{2}(2u+v+v)} \cdot 2$$

$$= \frac{1}{2} e^{-(u+v)}$$ for what (u,v)?

3

$$\left.\begin{array}{l}0<x<\infty\\0<y<\infty\end{array}\right\} \Rightarrow \begin{array}{l}0<2u+v<\infty\\0<v<\infty\end{array}$$

$\hookrightarrow v>-2u$



$$g_u(u)=\begin{cases}\displaystyle\int_{-2u}^{\infty}\frac{1}{2}e^{-(u+v)}\,dv & \text{if } u<0\\[2ex] \displaystyle\int_{0}^{\infty}\frac{1}{2}e^{-(u+v)}\,dv & \text{if } u\geq 0\end{cases}$$

4

$u<0$:
$$g_u(u)=\frac{1}{2}e^{-u}\int_{-2u}^{\infty}e^{-v}\,dv$$
$$=\frac{1}{2}e^{-u}\left[-e^{-v}\right]_{-2u}^{\infty}$$
$$=\frac{1}{2}e^{-u}\left[0+e^{2u}\right]$$
$$=\frac{1}{2}e^{u}$$

$u\geq 0$:
$$g_u(u)=\frac{1}{2}e^{-u}\int_{0}^{\infty}e^{-v}\,dv$$

(5)

$$= \frac{1}{2}e^{-u}\left[-e^{-v}\right]_0^{\infty}$$

$$= \frac{1}{2}e^{-u}\left[0+1\right]$$

$$= \frac{1}{2}e^{-u}$$

$$g_u(u) = \begin{cases} \frac{1}{2}e^{u} & u<0 \\ \frac{1}{2}e^{-u} & u\geq 0 \end{cases}$$

$$= \frac{1}{2}e^{-|u|}$$ (Double Exponential or Laplace)

(6)

<u>Example</u>: Let X & Y be independent standard normal random variables

Let $U = \frac{X}{Y}$. Find the pdf of U.

Let $V = Y$

$\underline{Y = V}$ $\quad u = \frac{x}{v}$ $\quad \underline{X = uv}$

$$J = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

(7)

$$g(u,v) = f(x,y)\,|J|$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} |v|$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} |v|$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}(u^2v^2+v^2)} |v|$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}v^2(u^2+1)} |v|$$

(8)

$$\left.\begin{array}{l} -\infty < x < \infty \\ -\infty < y < \infty \end{array}\right\} \Rightarrow \left\{\begin{array}{l} -\infty < uv < \infty \\ -\infty < v < \infty \end{array}\right.$$

$$g_u(u) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}v^2(u^2+1)} |v| \, dv$$

$$= 2 \int_0^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}v^2(u^2+1)} v \, dv$$

because the integrand
was an even function

let $s = \frac{1}{2}v^2(u^2+1)$

$ds = v(u^2+1)\,dv$

$$g_u(u) = \frac{2}{2\pi}\int_0^\infty e^{-s}\,\frac{ds}{u^2+1} = \frac{1}{\pi}\frac{1}{u^2+1}\left[-e^{-s}\right]_0^\infty$$

$$= \frac{1}{\pi}\frac{1}{u^2+1}\left[0+1\right]$$

$$g_u(u) = \frac{1}{\pi}\frac{1}{u^2+1} \qquad -\infty < u < \infty$$

(Cauchy)

Recall: $\mathrm{Cov}(X,Y) = \sigma_{XY} = E\left[(X-\mu_x)(Y-\mu_y)\right]$

$$= E[XY] - \mu_x\mu_y$$

$$\mathrm{Corr}(X,Y) = \rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y}$$

<u>Example</u>: $f(x,y) = x+y \qquad 0<x<1$

$0<y<1$

Find ρ_{XY}

(11)

$$E[XY] = \int_0^1\int_0^1 xy(x+y)\,dx\,dy$$

$$= \int_0^1\int_0^1 (x^2y + xy^2)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^3y}{3} + \frac{x^2y^2}{2}\right]_{x=0}^{1} dy$$

$$= \int_0^1 \frac{y}{3} + \frac{y^2}{2}\,dy = \left[\frac{y^2}{6} + \frac{y^3}{6}\right]_0^1$$

$$= \frac{1}{3}$$

(12)

$$f_X(x) = \int_0^1 x+y\,dy$$

$$= xy + \frac{y^2}{2}\Big|_{y=0}^{1} = x + \frac{1}{2}, \quad 0<x<1$$

$$E[X] = \int_0^1 x\left(x+\frac{1}{2}\right)dx = \int_0^1 x^2 + \frac{1}{2}x\,dx$$

$$= \frac{x^3}{3} + \frac{x^2}{4}\Big|_0^1 = \frac{7}{12}$$

$$E[X^2] = \int_0^1 x^2\left(x+\frac{1}{2}\right)dx = \int_0^1 x^3 + \frac{1}{2}x^2\,dx$$

$$= \frac{x^4}{4} + \frac{x^3}{6}\Big|_0^1 = \frac{5}{12}$$

Because X and Y were interchangeable in $f(x,y)$, we can conclude

$$f_Y(y) = y + \frac{1}{2} \quad 0<y<1$$

$$E[Y] = \frac{7}{12} \quad E[Y^2] = \frac{5}{12}$$

$$\sigma_{XY} = \frac{1}{3} - \frac{7}{12}\cdot\frac{7}{12} = -\frac{1}{144}$$

$$\sigma_X = \sqrt{\frac{5}{12} - \left(\frac{7}{12}\right)^2} = \frac{\sqrt{11}}{12} = \sigma_Y$$

$$\rho_{XY} = \frac{-\frac{1}{144}}{\frac{\sqrt{11}}{12}\cdot\frac{\sqrt{11}}{12}} = -\frac{1}{11}$$

What does ρ_{XY} actually measure?

Suppose $E[Y|X]$ is a linear function of X,

i.e. suppose that

$$E[Y|X] = a + bX$$

(15)

Then $E[Y] = E[E[Y|X]]$

$= E[a + bX]$

$= a + bE[X]$

Also, $E[X E[Y|X]] = E[X(a+bX)]$

$= E[aX + bX^2] = aE[X] + bE[X^2]$

Alternately, $E[X E[Y|X]] = \int_{-\infty}^{\infty} x E[Y|x] f_X(x) dx$

(16)

$$= \int_{-\infty}^{\infty} x \left[\int_{-\infty}^{\infty} y \underbrace{f(y|x)}_{\frac{f(x,y)}{f_X(x)}} dy \right] f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x y f(x,y)\, dy\, dx = E[XY]$$

Now $\sigma_{XY} = E[XY] - \mu_X \mu_Y$

$= a\mu_X + bE[X^2] - \mu_X \mu_Y$

17

$$= \cancel{a\mu_x} + b(\sigma_x^2 + \cancel{\mu_x^2}) - \mu_x(\cancel{a} + \cancel{b\mu_x})$$

$$\sigma_{xy} = b\sigma_x^2$$

$$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y} = \frac{b\sigma_x^2}{\sigma_x \sigma_y} = b\frac{\sigma_x}{\sigma_y}$$

$\therefore$ If $E[Y|X] = a + bX$

then $\rho_{xy} = b\frac{\sigma_x}{\sigma_y}$

18

HW #1 due 1/17

p.192 # 4, 11, 16, 17, 20

4.4 A pdf is defined by

$$f(x,y) = \begin{cases} C(x+2y) & \text{if } 0 < y < 1 \text{ and } 0 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of C.
(b) Find the marginal distribution of X.
(c) Find the joint cdf of X and Y.
(d) Find the pdf of the random variable $Z = 9/(X+1)^2$.

4.11 Let U = the number of trials needed to get the first head and V = the number of trials needed to get two heads in repeated tosses of a fair coin. Are U and V independent random variables?

4.16 Let X and Y be independent random variables with the same geometric distribution.

(a) Show that U and V are independent, where U and V are defined by

$$U = \min(X, Y) \quad \text{and} \quad V = X - Y.$$

(b) Find the distribution of $Z = X/(X+Y)$, where we define $Z = 0$ if $X + Y = 0$.
(c) Find the joint pdf of X and $X + Y$.

4.17 Let X be an exponential(1) random variable, and define Y to be the integer part of $X + 1$, that is

$$Y = i + 1 \quad \text{if and only if} \quad i \leq X < i + 1, \quad i = 0, 1, 2, \ldots.$$

(a) Find the distribution of Y. What well-known distribution does Y have?
(b) Find the conditional distribution of $X - 4$ given $Y \geq 5$.

4.20 X_1 and X_2 are independent $n(0, \sigma^2)$ random variables.

(a) Find the joint distribution of Y_1 and Y_2, where

$$Y_1 = X_1^2 + X_2^2 \quad \text{and} \quad Y_2 = \frac{X_1}{\sqrt{Y_1}}.$$

(b) Show that Y_1 and Y_2 are independent, and interpret this result geometrically.