Stat 561
11-27-18

1

Random vectors

A random vector $\vec{X}$ is a function from the sample space S into $\mathbb{R}^n$.

(i.e. it is a vector whose components are random variables)

A random vector is discrete if its range in $\mathbb{R}^n$ is finite or countably infinite.

2

For $A \subset \mathbb{R}^n$,

$$P(A) = P[\vec{X} \in A]$$

Defn: For a discrete random vector, the joint probability mass function is

$$p(x_1, x_2, \dots, x_n) = P[X_1 = x_1 \cap X_2 = x_2 \cap \dots \cap X_n = x_n]$$

So $$P(A) = \underbrace{\sum \sum \cdots \sum}_{(x_1, \dots, x_n) \in A} p(x_1, x_2, \dots, x_n)$$

③

Example: 4 slips of paper, numbered 1, 2, 3, 4

Draw 2 w.o.r.

Let $X_1 = \min$ $\quad X_2 = \max$

$$S = \{12, 13, 14, 21, 23, 24, 31, 32, 34, 41, 42, 43\}$$

$p(x_1, x_2)$		X_2: 2	3	4	
X_1	1	1/6	1/6	1/6	3/6
	2	0	1/6	1/6	2/6
	3	0	0	1/6	1/6
		1/6	2/6	3/6	1

④

Theorem: $$p_1(x_1) = \sum_{\text{all } x_2} p(x_1, x_2)$$

Proof. $$p_1(x_1) = P[X_1 = x_1]$$

$$= P[X_1 = x_1 \cap -\infty < X_2 < \infty]$$

$$= \sum_{\text{all } x_2} P[X_1 = x_1 \cap X_2 = x_2]$$

$$= \sum_{\text{all } x_2} p(x_1, x_2)$$

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Continuous Case: Assume every component of the random vector is continuous

Defn: $f(x_1, \dots, x_n)$ is a joint probability density function if, $\forall A \subset \mathbb{R}^n$

$$P[\vec{X} \in A] = \int\int_A \cdots \int f(x_1, \dots, x_n)\, dx_1 \cdots dx_n$$

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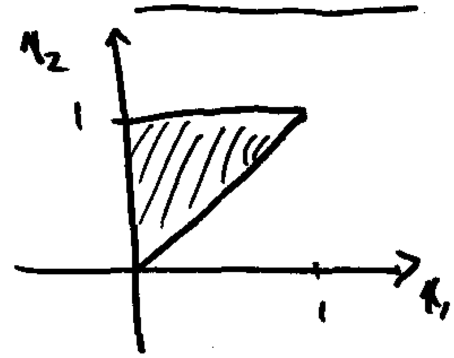
Example: $f(x_1, x_2) = 8 x_1 x_2 \quad \underline{0 < x_1 < x_2 < 1}$



Check to see if $f(x_1, x_2)$ integrates to 1.

$$\int_0^1 \int_0^{x_2} 8 x_1 x_2 \, dx_1\, dx_2 \quad \text{OR} \quad \int_0^1 \int_{x_1}^1 8 x_1 x_2 \, dx_2\, dx_1$$

$$8 \int_0^1 x_2 \frac{x_1^2}{2}\Big|_{x_1=0}^{x_2} dx_2 = 8 \int_0^1 \frac{x_2^3}{2}\, dx_2 = x_2^4 \Big|_0^1 = 1 \checkmark$$

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The marginal prob. density fn for X_1 is

$$f_1(x_1) = \int_{x_1}^{1} f(x_1, x_2)\, dx_2$$

$$= \int_{x_1}^{1} 8 x_1 x_2\, dx_2 = 8 x_1 \left. \frac{x_2^2}{2} \right|_{x_2 = x_1}^{1}$$

$$= 4x_1 - 4x_1^3$$

$$f_1(x_1) = 4x_1 - 4x_1^3, \quad 0 < x_1 < 1$$

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The marginal prob. density fn. for X_2 is

$$f_2(x_2) = \int_{0}^{x_2} 8 x_1 x_2\, dx_1$$

$$= 8 x_2 \left. \frac{x_1^2}{2} \right|_{x_1 = 0}^{x_2} = 4x_2^3$$

$$f_2(x_2) = 4x_2^3 \qquad 0 < x_2 < 1$$

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Let $g(X_1, X_2, \ldots, X_n)$ be a new random variable.

Defn. $E[g(\vec{X})] = \begin{cases} \sum\limits_{\text{all } \vec{x}} \cdots \sum g(\vec{x})p(\vec{x}) \\ \int \cdots \int_{\mathbb{R}^n} g(\vec{x})f(\vec{x})\,d\vec{x} \end{cases}$

In our example, find $E[X_1 + X_2]$

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Method 1: $E[X_1 + X_2] = \int_0^1 \int_0^{x_2} (x_1 + x_2)\, 8x_1 x_2 \, dx_1 \, dx_2$

$$= \int_0^1 \int_0^{x_2} \left(8x_1^2 x_2 + 8x_1 x_2^2\right) dx_1 \, dx_2$$

$$= \int_0^1 \left(\frac{8x_1^3 x_2}{3} + \frac{8x_1^2 x_2^2}{2}\right)\Bigg|_{x_1=0}^{x_2} dx_2$$

$$= \int_0^1 \frac{8x_2^4}{3} + \frac{8x_2^4}{2}\, dx_2 = \frac{20}{3}\int_0^1 x_2^4 \, dx_2$$

$$= \frac{20}{3} \frac{x_2^5}{5}\Big|_0^1 = \frac{4}{3}.$$

Method 2: $E[X_1 + X_2] = E[X_1] + E[X_2]$

$$E[X_1] = \int_0^1 x_1 (4x_1 - 4x_1^3)\,dx_1 = 8/15$$

$$E[X_2] = \int_0^1 x_2\, 4x_2^3\, dx_2 = \frac{4}{5}$$

So $E[X_1 + X_2] = 8/15 + 4/5 = \frac{4}{3}$.

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Conditional distributions

Defn: $$\begin{cases} f(x_2 \mid x_1) = \dfrac{f(x_1, x_2)}{f_1(x_1)} \\[2ex] p(x_2 \mid x_1) = \dfrac{p(x_1, x_2)}{p_1(x_1)} \end{cases}$$

Note: $$\int_{\text{all } x_2} f(x_2 \mid x_1)\, dx_2 = \int_{\text{all } x_2} \frac{f(x_1, x_2)}{f_1(x_1)}\, dx_2$$

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$$= \frac{1}{f_1(x_1)} \underbrace{\int_{\text{all } x_2} f(x_1, x_2)\, dx_2}_{f_1(x_1)} = 1 \checkmark$$

Let $g(X_2)$ be a function of X_2.

Defn. $E[g(X_2) | x_1] = \begin{cases} \sum_{\text{all } x_2} g(x_2)\, p(x_2|x_1) \\ \int_{\text{all } x_2} g(x_2)\, f(x_2|x_1)\, dx_2 \end{cases}$

Note: $E[g(X_2)] = $ number

$E[g(X_2)|x_1] = $ function of x_1

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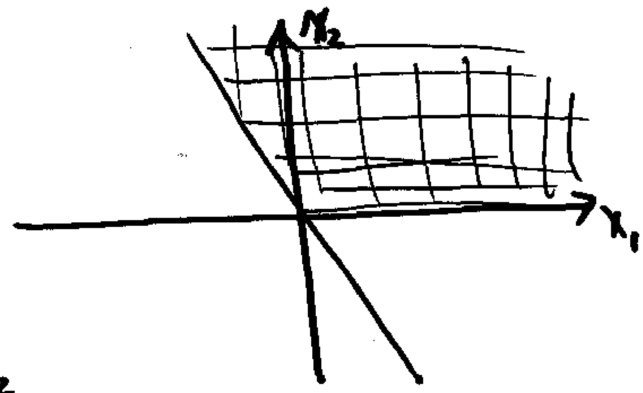
Example. $f(x_1, x_2) = \frac{1}{2} e^{-(x_1+x_2)}$, $\quad x_2 > 0$

$\qquad x_2 > -2x_1$

Find $f(x_2|x_1)$



$$f_1(x_1) = \int_{\text{all } x_2} f(x_1, x_2)\, dx_2$$

$$= \begin{cases} \int_0^\infty \frac{1}{2} e^{-(x_1+x_2)}\, dx_2 & x_1 \geq 0 \\ \int_{-2x_1}^\infty \frac{1}{2} e^{-(x_1+x_2)}\, dx_2 & x_1 < 0 \end{cases}$$

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$$= \begin{cases} \frac{1}{2}e^{-x_1}\left[-e^{-x_2}\right]_{x_2=0}^{\infty} & x_1 \geq 0 \\ = 0 - \left(-\frac{1}{2}e^{-x_1}\right) = \frac{1}{2}e^{-x_1} & \\ \hline \frac{1}{2}e^{-x_1}\left[-e^{-x_2}\right]_{x_2=-2x_1}^{\infty} & x_1 < 0 \\ = 0 + \frac{1}{2}e^{-x_1}e^{2x_1} = \frac{1}{2}e^{x_1} & \end{cases}$$

$$= \frac{1}{2}e^{-|x_1|} \qquad -\infty < x_1 < \infty$$

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$$f(x_2|x_1) = \frac{f(x_1,x_2)}{f_1(x_1)}$$

$$= \frac{\frac{1}{2}e^{-(x_1+x_2)}}{\frac{1}{2}e^{-|x_1|}}$$

$$= e^{|x_1| - x_1 - x_2} \qquad \begin{matrix} x_2 > 0 \\ x_2 > -2x_1 \end{matrix}$$