Stat 561
9-25-18

(1)

Defn: An *experiment* is a process that leads to one of several possible outcomes.

Defn: The *sample space* (S) is the set of all possible outcomes of an experiment.

Defn: An *event* is a subset of S

$\subset$: subset	A^c : complement of A
$\cap$: intersection	$\phi = \{\}$: empty set
$\cup$: union	

(2)

Defn. Two events A and B are *disjoint* or *mutually exclusive* if $A \cap B = \phi$

Defn. Let $\mathcal{B}$ be a subset of all possible subsets of S. Then $\mathcal{B}$ is a *Borel field* or a *σ-algebra* if

(1) $\phi \in \mathcal{B}$

(2) If $A \in \mathcal{B}$ then $A^c \in \mathcal{B}$

(3) If $\{A_1, A_2, \ldots\} \subset \mathcal{B}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}$

③

<u>Defn</u>: Let $\mathcal{B}$ be a Borel field. Then
P is a <u>probability set function</u> if

(1) $\forall\ A \in \mathcal{B}$, $P(A) \geq 0$

(2) $P(S) = 1$

(3) If $A_1, A_2, A_3, \dots$ are pairwise disjoint,
then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

Kolmogorov properties

④

<u>Example</u>: Roll a 6-sided die
$S = \{1,2,3,4,5,6\}$

Let $\mathcal{B} = \{\phi, S, \{1,2\}, \{3,4,5,6\}\}$
This is a well-defined Borel field

Define $P_1(A) = \begin{cases} 1 & \text{if } 2 \in A \\ 0 & \text{otherwise} \end{cases}$

This is a well-defined probability function

Define $P_2(A) = \dfrac{\#\text{elements in } A}{6}$

This is also a well-defined probability function

Theorem: $\forall A \in \mathcal{B}$, $P(A^c) = 1 - P(A)$ 5

Pf: $A \cap A^c = \emptyset$

$A \cup A^c = S$

$$\begin{aligned} P(A \cup A^c) &= P(A) + P(A^c) && \text{[by K-3]} \\ \| \\ P(S) &= 1 && \text{[by K-2]} \end{aligned}$$

Theorem: $P(\emptyset) = 0$

Pf: $\emptyset \cap S = \emptyset$

$\emptyset \cup S = S$

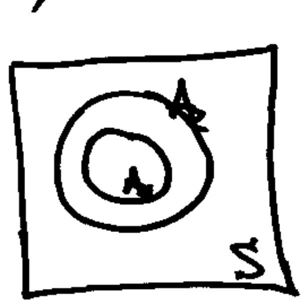
$$\begin{aligned} P(\emptyset \cup S) &= P(\emptyset) + \underbrace{P(S)}_{1} \\ \| \\ P(S) &= 1 \end{aligned}$$

Theorem: Let A_1 and A_2 be elements of $\mathcal{B}$ with $A_1 \subset A_2$. Then $P(A_1) \le P(A_2)$ 6

Pf: Let $A_1 \subset A_2$

Write $A_2 = A_1 \cup (A_2 \cap A_1^c)$

Also, $A_1 \cap (A_2 \cap A_1^c) = \emptyset$



So $P(A_2) = P(A_1) + \underbrace{P(A_2 \cap A_1^c)}_{\ge 0}$

$\therefore P(A_2) \ge P(A_1)$

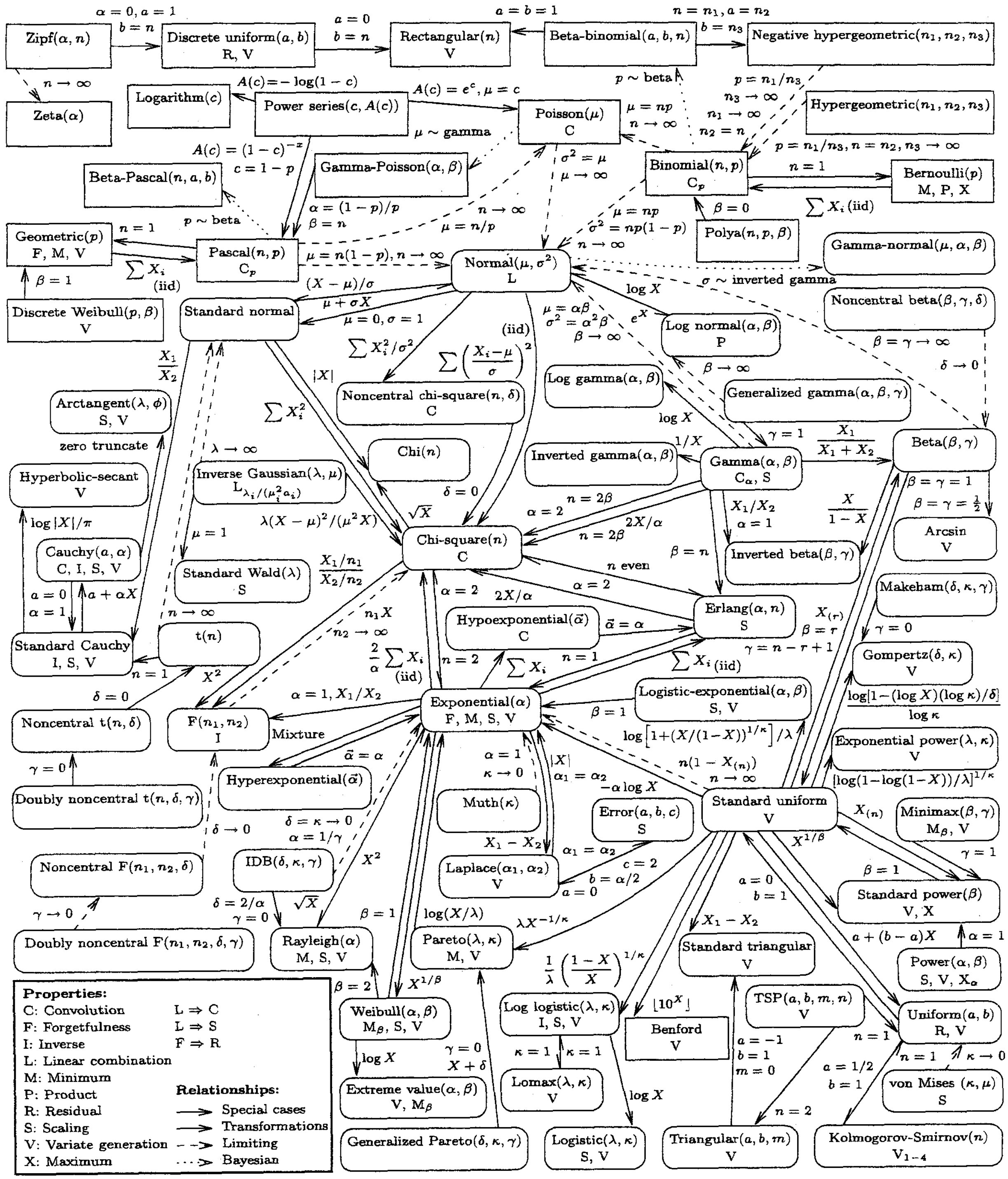


Figure 1. Univariate distribution relationships.