Stat 561
11-13-18

(1)

Let X have a standard normal distribution and $Y = X^2$. Find the density of Y.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$-\infty < x < \infty$$

$$G_Y(y) = P(Y \le y) = P(X^2 \le y)$$

$$= (-\sqrt{y} \le X \le \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$g_Y(y) = \frac{d}{dy} G_Y(y) = \frac{d}{dy}\left[F_X(\sqrt{y}) - F_X(-\sqrt{y})\right]$$

(2)

$$= f_X(\sqrt{y}) \cdot \frac{1}{2} y^{-1/2} - f_X(-\sqrt{y})\left(-\frac{1}{2} y^{-1/2}\right)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \frac{1}{2\sqrt{y}}$$

$$g_Y(y) = \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{1}{2}y} \qquad y > 0$$

So $Y \sim \text{Gamma}(\alpha = \frac{1}{2}, \beta = 2)$

$\sim \chi^2(k = 1)$

③

Let T have a gamma distribution with parameters α & β, where α is an integer.

$$P(T>t) = \int_t^\infty \frac{1}{\underbrace{\Gamma(\alpha)}_{(\alpha-1)!}\beta^\alpha} y^{\alpha-1} e^{-y/\beta} dy$$

$$= \frac{1}{(\alpha-1)!\,\beta^\alpha}\left[-\beta y^{\alpha-1} e^{-y/\beta}\Big|_t^\infty + \int_t^\infty \beta(\alpha-1) y^{\alpha-2} e^{-y/\beta} dy \right]$$

Let $u = y^{\alpha-1}$ $\quad dv = e^{-y/\beta} dy$

$du = (\alpha-1) y^{\alpha-2} dy$ $\quad v = -\beta e^{-y/\beta}$

④

$$= \frac{1}{(\alpha-1)!\,\beta^\alpha}\left[\beta t^{\alpha-1} e^{-t/\beta} + \beta(\alpha-1)\int_t^\infty y^{\alpha-2} e^{-y/\beta} dy \right]$$

$$= \frac{t^{\alpha-1} e^{-t/\beta}}{(\alpha-1)!\,\beta^{\alpha-1}} + \frac{1}{(\alpha-2)!\,\beta^{\alpha-1}} \underbrace{\int_t^\infty y^{\alpha-2} e^{-y/\beta} dy}$$

$u = y^{\alpha-2}$ $\quad dv = e^{-y/\beta} dy$

$du = (\alpha-2) y^{\alpha-3} dy$ $\quad v = -\beta e^{-y/\beta}$

$$= \frac{t^{\alpha-1} e^{-t/\beta}}{(\alpha-1)!\,\beta^{\alpha-1}} + \frac{1}{(\alpha-2)!\,\beta^{\alpha-1}}\left[-\beta y^{\alpha-2} e^{-y/\beta}\Big|_t^\infty + \int_t^\infty \beta(\alpha-2) y^{\alpha-3} e^{-y/\beta} dy \right]$$

⑤

$$= \frac{t^{\alpha-1} e^{-t/\beta}}{(\alpha-1)!\, \beta^{\alpha-1}} + \frac{\beta t^{\alpha-2} e^{-t/\beta}}{(\alpha-2)!\, \beta^{\alpha-1}} + \cdots$$

$$= \frac{\left(\frac{t}{\beta}\right)^{\alpha-1} e^{-t/\beta}}{(\alpha-1)!} + \frac{\left(\frac{t}{\beta}\right)^{\alpha-2} e^{-t/\beta}}{(\alpha-2)!} + \frac{\left(\frac{t}{\beta}\right)^{\alpha-3} e^{-t/\beta}}{(\alpha-3)!} + \cdots$$

$$= P(X=\alpha-1) + P(X=\alpha-2) + P(X=\alpha-3) + \cdots$$

where $X \sim \text{Poisson}(\lambda = t/\beta)$

⑥

$$P(T>t) = P(X \le \alpha-1)$$

Where $T \sim \text{Gamma}(\alpha, \beta)$ & $X \sim \text{Poisson}(\lambda = t/\beta)$

That is, T is measuring the waiting time for the α^{th} occurrence in a Poisson process.

<u>Example</u>: Suppose a pizza place gets 1 order every 3 minutes, following a Poisson process.

Assume that the driver is engaged when the 5^{th} order arrives.

Find the probability that this happens in 10 minutes or less.

Let T = waiting time for 5 orders

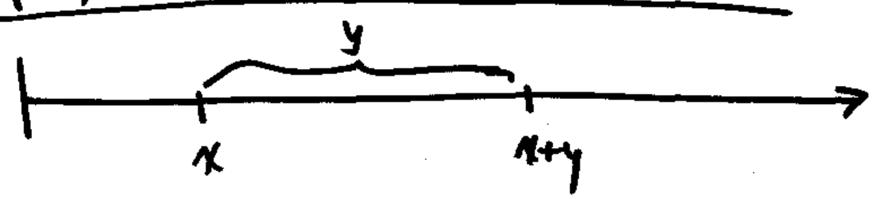
$T \sim \text{Gamma}(\alpha = 5, \underbrace{\beta = 3}_{})$

expected time between occurrences

$$P(T \leq 10) = 1 - P(T > 10)$$
$$= 1 - P(X \leq 4) \quad \text{where } X \sim \text{Poisson}\left(\lambda = \frac{t}{\beta} = \frac{10}{3}\right)$$

$$= 1 - .7565 \quad \text{(using poissoncdf on TI-84)}$$
$$= .2435$$

A property of the exponential distribution



$$P(X > x+y \mid X > x) = \frac{P(X > x+y \cap X > x)}{P(X > x)}$$

waiting time until 1^{st} occurrence

$$= \frac{P(X > x+y)}{P(X > x)}$$

$$= \frac{1 - F_X(x+y)}{1 - F_X(x)}$$

$f_X(x) = \lambda e^{-\lambda x}$

$F_X(x) = 1 - e^{-\lambda x}$

$$= \frac{e^{-\lambda(x+y)}}{e^{-\lambda x}}$$

$$= e^{-\lambda y} = 1 - F_X(y)$$

$$= P(X > y)$$

"memoryless" or "forgetfullness"

The Beta distribution

The Beta function is

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt$$

The Beta density function is

$$f_X(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \qquad 0 < x < 1$$

How is this related to the gamma function?

$$\Gamma(\alpha)\Gamma(\beta) = \left(\int_0^\infty s^{\alpha-1} e^{-s} ds\right)\left(\int_0^\infty t^{\beta-1} e^{-t} dt\right)$$

$$= \int_0^\infty \int_0^\infty s^{\alpha-1} t^{\beta-1} e^{-(s+t)} ds\, dt$$

let $s = uv$, $t = (1-u)v$

$u = \frac{s}{v}$ $\quad t = (1-\frac{s}{v})v$

$= v - s$

so $v = s+t$

$u = s/s+t$

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$$J = \begin{vmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{vmatrix} = \begin{vmatrix} \frac{(s+t)-s}{(s+t)^2} & s(-1)(s+t)^{-2} \\ 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{t}{(s+t)^2} & \frac{-s}{(s+t)^2} \\ 1 & 1 \end{vmatrix}$$

$$= \frac{t}{(s+t)^2} - \frac{-s}{(s+t)^2} = \frac{s+t}{(s+t)^2} = \frac{1}{s+t} = \frac{1}{v}$$

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty \int_0^1 (uv)^{\alpha-1} [(1-u)v]^{\beta-1} e^{-v}\, v \, du\, dv$$

$$= \int_0^\infty \int_0^1 u^{\alpha-1}(1-u)^{\beta-1} v^{\alpha+\beta-1} e^{-v}\, du\, dv$$

$$= \int_0^1 u^{\alpha-1}(1-u)^{\beta-1} du \int_0^\infty v^{\alpha+\beta-1} e^{-v} dv$$

$$= B(\alpha,\beta)\ \Gamma(\alpha+\beta)$$

$$\therefore\ B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$