Another transformation example:

Stat 561
10-16-18

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Let X have cdf $F_X(x)$ and
assume that F is continuous, and
monotone increasing

Define $Y = F(X)$

Find the pdf for Y.

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

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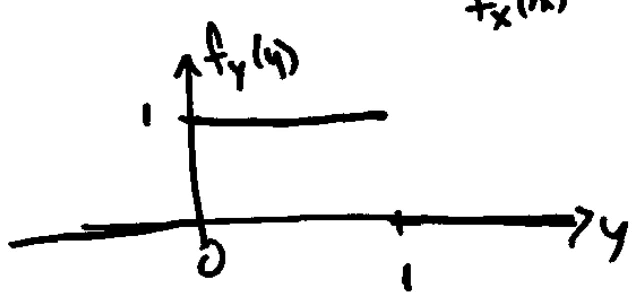
$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{f_X(x)}$$

So $f_Y(y) = f_X(x) \dfrac{1}{f_X(x)} = 1$ for $0 \le y \le 1$



Uniform density

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Expectation operator

Defn: If X is a discrete r.v. and

$g(X)$ is any function of X,

then $$E[g(X)] = \sum_{\text{all } x} g(x)\, p_X(x)$$

(provided that $\sum_{\text{all } x} |g(x)|\, p_X(x) < \infty$)

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If X is a continuous r.v.

and $g(X)$ is any function of X,

then $$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

(provided that $\int_{-\infty}^{\infty} |g(x)|\, f_X(x)\, dx < \infty$)

Special cases: $$E[X] = \begin{cases} \sum_{\text{all } x} x\, p_X(x) \\ \int_{-\infty}^{\infty} x\, f_X(x)\, dx \end{cases}$$

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$$E[X^n] = \begin{cases} \sum_{\text{all } x} x^n \, p_X(x) \\ \int_{-\infty}^{\infty} x^n f_X(x)\,dx \end{cases}$$

Defn: $E[X^n]$ is the n^{th} moment of the random variable X.

Notation: $\mu_n' = E[X^n]$

$(\text{So } \mu_1' = E[X] = \mu)$

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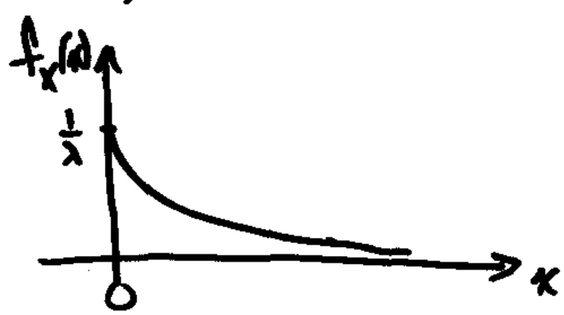
Example: Let $f_X(x) = \frac{1}{\lambda} e^{-x/\lambda}, \quad x > 0$

Find $E[X]$



$$\mu = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^{\infty} x \frac{1}{\lambda} e^{-x/\lambda}\,dx$$

let $u = x$
$dv = \frac{1}{\lambda} e^{-x/\lambda} dx$

$$= \int_0^{\infty} u\,dv = uv\Big|_0^{\infty} - \int_0^{\infty} v\,du$$

$$= -x e^{-x/\lambda} \Big|_0^\infty + \int_0^\infty e^{-x/\lambda} dx \qquad \Bigg| \begin{array}{l} du = dx \\ v = -e^{-x/\lambda} \end{array}$$ 7

$$= \frac{-x}{e^{x/\lambda}} \Big|_0^\infty - \lambda e^{-x/\lambda} \Big|_0^\infty$$

$$= 0 - (0) - [0 - \lambda] = \lambda$$

by L'Hôpital

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<u>Example</u>: let $p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$

$x = 0, 1, 2, \ldots, n$

$$\left[\text{Binomial theorem: } (a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i} \right]$$

Note: $$\sum_{\text{all } x} p_X(x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= [p + (1-p)]^n = 1$$

Find $E[X]$ 9

$$\mu = \sum_{\text{all } x} x\, p_X(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x}$$

$(q = 1 - p)$

$$= \sum_{x=1}^{n} x \binom{n}{x} p^x q^{n-x}$$

Let $y = x - 1$, $x = y+1$

$$= \sum_{y=0}^{n-1} (y+1) \binom{n}{y+1} p^{y+1} q^{n-(y+1)}$$

$$= \sum_{y=0}^{n-1} (y+1) \frac{n!}{(y+1)!\,(n-y-1)!} p^{y+1} q^{n-y-1}$$

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$$= \sum_{y=0}^{n-1} \frac{n!}{y!\,(n-y-1)!} p^{y+1} q^{n-y-1}$$

Let $m = n-1$, $n = m+1$

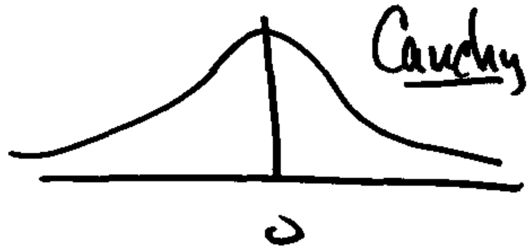
$$= \sum_{y=0}^{m} \frac{(m+1)!}{y!\,(m-y)!} p^{y+1} q^{m-y}$$

$$= (m+1)\, p \underbrace{\sum_{y=0}^{m} \frac{m!}{y!\,(m-y)!} p^{y} q^{m-y}}_{1}$$

$$= np$$

(11)

Example: $f_X(x) = \frac{1}{\pi} \frac{1}{1+x^2} \quad -\infty < x < \infty$



Cauchy distribution

Find $E[X]$. Check the extra condition

$$\int_{-\infty}^{\infty} |x| \frac{1}{\pi} \frac{1}{1+x^2}\, dx < \infty \; ?$$

$$= 2\int_0^{\infty} \frac{1}{\pi} \frac{x}{1+x^2}\, dx$$

Let $u = 1+x^2$

$du = 2x\, dx$

(12)

$$= \frac{1}{\pi} \int_1^{\infty} \frac{du}{u} = \frac{1}{\pi} \ln u \Big|_1^{\infty} = \infty$$

$\therefore$ $E[X]$ will be undefined

Properties of the expectation operator:

(1) $E(k) = \int_{-\infty}^{\infty} k\, f_X(x)\, dx = k \int_{-\infty}^{\infty} f_X(x)\, dx$

$= k$

(similar in discrete case)

(13)

② $E[k\,g(X)] = \int_{-\infty}^{\infty} k\,g(x) f_X(x)\,dx$

$= k \int_{-\infty}^{\infty} g(x) f_X(x)\,dx = k\,E[g(X)]$

③ $E[g(X) + h(X)] = \int_{-\infty}^{\infty} [g(x) + h(x)] f_X(x)\,dx$

$= \int_{-\infty}^{\infty} g(x) f_X(x)\,dx + \int_{-\infty}^{\infty} h(x) f_X(x)\,dx$

$= E[g(X)] + E[h(X)]$

(14)

Note: operators with properties ② and ③ are called <u>linear operators</u>

<u>Defn</u>: The <u>variance</u> of a random variable is $\sigma^2 = V[X] = Var[X]$

$= E[(X-\mu)^2]$

Alternate form: $\sigma^2 = E[X^2 - 2\mu X + \mu^2]$

$= E[X^2] - 2\mu E[X] + \mu^2$

$$= E[X^2] - 2\mu^2 + \mu^2$$

$$\boxed{\begin{aligned} \sigma^2 &= E[X^2] - \mu^2 \\ &= E[X^2] - (E[X])^2 \end{aligned}}$$