Stat 561
9-27-18

<u>Defn</u>: $A \setminus B = A \cap B^c$

<u>Theorem</u>: $P(A \setminus B) = P(A) - P(A \cap B)$ ①

<u>Pf</u>: $A = (A \cap B) \cup (A \cap B^c)$

and $(A \cap B) \cap (A \cap B^c) = \phi$

By K-3, $P(A) = P(A \cap B) + P(A \cap B^c)$

$\therefore P(A \setminus B) = P(A) - P(A \cap B)$

<u>Defn</u>: $C_1, C_2, C_3, \ldots$ form a <u>partition</u> ②
of S if

(1) $\bigcup_{i=1}^{\infty} C_i = S$

and (2) $C_i \cap C_j = \phi \quad \forall i \neq j$ (pairwise disjoint)

<u>Theorem</u>: If $C_1, C_2, C_3, \ldots$ form a partition of S, then

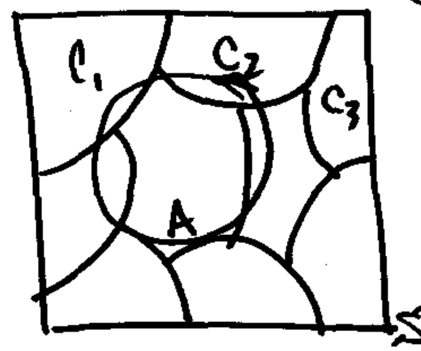
$$P(A) = \sum_{i=1}^{\infty} P(A \cap C_i)$$ [Law of Total Probability]

③

Pf: $A = A \cap S$

$= A \cap \left(\bigcup_{i=1}^{\infty} C_i \right)$

$= \bigcup_{i=1}^{\infty} (A \cap C_i)$



$\therefore P(A) = \sum_{i=1}^{\infty} P(A \cap C_i)$

since $A \cap C_i$ and $A \cap C_j$ are disjoint $\forall i \neq j$

④

<u>Theorem</u>: Let $A_1, A_2, A_3, \ldots$ be events.

Then $P\left(\bigcup_{i=1}^{\infty} A_i \right) \leq \sum_{i=1}^{\infty} P(A_i)$

[Boole's Inequality]

Pf: Let $A_1^* = A_1$

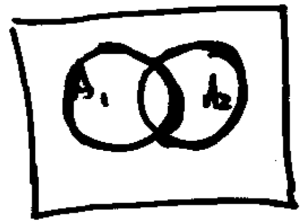
$A_2^* = A_2 \setminus A_1 = A_2 \cap A_1^c$

$A_3^* = A_3 \setminus (A_1 \cup A_2)$

$\vdots$

$A_i^* = A_i \setminus \left(\bigcup_{j=1}^{i-1} A_j \right)$

$\vdots$



(5)

Note: $A_i^* \cap A_j^* = \phi \quad \forall\, i \neq j$

and $\bigcup_{i=1}^{\infty} A_i^* = \bigcup_{i=1}^{\infty} A_i$ and $A_i^* \subset A_i$

So $P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i^*\right)$

$$= \sum_{i=1}^{\infty} P(A_i^*)$$

$$\leq \sum_{i=1}^{\infty} P(A_i) \quad \text{by subset theorem}$$

(6)

Suppose we have events $A_1, A_2, \ldots, A_n$

with $p_i = P(A_i)$

$$P\left[(A_1 \cap A_2 \cap \ldots \cap A_n)^c\right] = 1 - P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

|| by DeMorgan by complement theorem

$$P\left[A_1^c \cup A_2^c \cup \ldots \cup A_n^c\right] \leq P(A_1^c) + P(A_2^c) + \ldots + P(A_n^c)$$

$$= 1 - P(A_1) + \ldots + 1 - P(A_n)$$

$$= n - \sum_{i=1}^{n} P(A_i)$$

(7)

$$1 - P(A_1 \cap A_2 \cdots \cap A_n) \le n - \sum_{i=1}^{n} p_i$$

$$\boxed{P(A_1 \cap A_2 \cap \cdots \cap A_n) \ge \sum_{i=1}^{n} p_i - (n-1)}$$

<u>Bonferroni's</u> <u>Inequality</u>

Example of its usage: Assume $p(A_i) = p \ \forall i$

What must p be, in order to guarantee that $P(A_1 \cap A_2 \cap \cdots \cap A_n) \ge .95$

(8)

Set $\sum_{i=1}^{n} p_i - (n-1) = .95$

$$np - (n-1) = .95$$

$$np = .95 + n - 1$$

$$= n - .05$$

$$p = 1 - \frac{.05}{n}$$

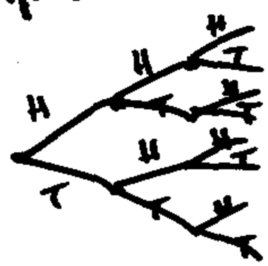
Interpretation: If you do 10 simultaneous confidence intervals, each at 99.5% confidence, you will have joint confidence of at least 95%.

9

Counting Rules

(1) Multiplication rule: If you perform a sequence of procedures, then the total # of possible outcomes is the <u>product</u> of the # of possible outcomes at each step.

Ex: Flip 3 coins in sequence. $2 \times 2 \times 2 = 8$



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(2) Factorial rule: The # of ways of arranging or ordering n objects is $n!$

$$\underline{n} \times \underline{n-1} \times \underline{n-2} \times \underline{\quad} \cdots \times \underline{1}$$

(3) Permutation rule: Start with n objects. The # of ways of selecting and ordering r of them is

$$P^n_r = \frac{n!}{(n-r)!}$$

$$\underbrace{\underline{\;n\;} \times \underline{\;n-1\;} \times \underline{\;n-2\;} \times \underline{\quad} \cdots \times \underline{\;n-(r-1)\;}}_{r \text{ items}}$$

Note: $$\frac{n(n-1)(n-2)\cdots(n-(r-1))\cdot(n-r)(n-r-1)\cdots 1}{(n-r)(n-r-1)\cdots 1}$$

$$= \frac{n!}{(n-r)!}$$

4 Combination rule: The number of ways of selecting r objects from n objects, without regard to order, is

$$C_r^n = \binom{n}{r} = \text{"}n \text{ choose } r\text{"}$$

$$= \frac{P_r^n}{r!} = \frac{n!}{r!(n-r)!}$$

5 Permutations of like objects:

If there are n objects of k distinct types, then the # of ways that they can be ordered is

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}$$, where n_i is the # of items of type i

and $n_1 + \dots + n_k = n$

Example: How many different ways can the letters in "STATISTICS" be arranged?

Type 1:	S	$n_1 = 3$	
2:	T	$n_2 = 3$	$n = 10$
3:	A	$n_3 = 1$	
4:	I	$n_4 = 2$	
5:	C	$n_5 = 1$	

$$\frac{10!}{3!\,3!\,1!\,2!\,1!} = \binom{10}{3\ 3\ 1\ 2\ 1} = 50{,}400$$

HW#1 due Oct 4, beginning of class

p. 37 # 13, 19, 20, 27, 38

(scanned pages to follow)

Stat 561 HW 1

1.13 If $P(A) = \frac{1}{3}$ and $P(B^c) = \frac{1}{4}$, can A and B be disjoint? Explain.

1.19 If a multivariate function has continuous partial derivatives, the order in which the derivatives are calculated does not matter. Thus, for example, the function $f(x, y)$ of two variables has equal third partials

$$\frac{\partial^3}{\partial x^2 \partial y} f(x, y) = \frac{\partial^3}{\partial y \partial x^2} f(x, y).$$

(a) How many fourth partial derivatives does a function of three variables have?

(b) Prove that a function of n variables has $\binom{n+r-1}{r}$ rth partial derivatives.

1.20 My telephone rings 12 times each week, the calls being randomly distributed among the 7 days. What is the probability that I get at least one call each day?

1.27 Verify the following identities for $n \geq 2$.

(a) $\sum_{k=0}^{n}(-1)^k \binom{n}{k} = 0$

(b) $\sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}$

(c) $\sum_{k=1}^{n}(-1)^{k+1} k \binom{n}{k} = 0$

1.38 Prove each of the following statements. (Assume that any conditioning event has positive probability.)

(a) If $P(B) = 1$, then $P(A|B) = P(A)$ for any A.

(b) If $A \subset B$, then $P(B|A) = 1$ and $P(A|B) = P(A)/P(B)$.

(c) If A and B are mutually exclusive, then

$$P(A|A \cup B) = \frac{P(A)}{P(A) + P(B)}.$$

(d) $P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C)$.