Stat 561
11-20-18

①

Defn: Let $f(x)$ be a density function with parameters $\vec{\theta} = (\theta_1, \theta_2, \ldots, \theta_d)$. Then $f(x|\vec{\theta})$ [or $f_{\vec{\theta}}(x)$] is an <u>exponential</u> <u>family</u> of densities if you can write

$$f(x|\vec{\theta}) = h(x)\, c(\vec{\theta})\, e^{\sum_{i=1}^{k} w_i(\vec{\theta}) t_i(x)}, \quad -\infty < x < \infty.$$

If $d = k$, then the family is full.

If $d < k$, then the family is curved.

②

Example: $\text{Bino}(n, p)$ with n known

$$f(x|p) \ [\text{or } f_p(x)] = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n$$

Let $I(x) = \begin{cases} 1 & \text{if } x = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$

$$f(x|p) = \binom{n}{x} I(x) (1-p)^n e^{[x \ln p - x \ln(1-p)]} \quad -\infty < x < \infty$$

$h(x) = \binom{n}{x} I(x)$ $\quad w_1(\vec{\theta}) = \ln p$ $\quad t_1(x) = x$

$c(\vec{\theta}) = (1-p)^n$ $\quad w_2(\vec{\theta}) = \ln(1-p)$ $\quad t_2(x) = -x$

$$f(x|p) = \binom{n}{x} I(x) (1-p)^n e^{\left[x \cdot \ln\left(\frac{p}{1-p}\right)\right]}$$

$-\infty < x < \infty$

$h(x) = \binom{n}{x} I(x)$ $\quad w_1(\vec{\theta}) = \ln\left(\frac{p}{1-p}\right)$

$c(\vec{\theta}) = (1-p)^n$ $\quad t_1(x) = x$

$\therefore$ Bino(n, p) with n known is a full exponential family.

<u>Example</u>: $N(\mu, \sigma^2)$

$$f(x|\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < x < \infty$$

$$f(x|\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x^2 - 2\mu x + \mu^2)} \qquad -\infty < x < \infty$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} e^{\frac{\mu x}{\sigma^2}} e^{-\frac{\mu^2}{2\sigma^2}}$$

$h(x) = \frac{1}{\sqrt{2\pi}}$ $\quad c(\vec{\theta}) = \frac{1}{\sigma} e^{-\frac{\mu^2}{2\sigma^2}}$

$w_1(\vec{\theta}) = \frac{-1}{2\sigma^2}$ $\quad t_1(x) = x^2$

$w_2(\vec{\theta}) = \frac{\mu}{\sigma^2}$ $\quad t_2(x) = x$

$\therefore$ $N(\mu, \sigma^2)$ is a full exponential family

⑤

Example: Unif (a,b)

$$f(x \mid a,b) = \frac{1}{b-a} \qquad a<x<b$$

$$= \frac{1}{b-a} I(x) \qquad -\infty < x < \infty$$

$$\text{where } I(x) = \begin{cases} 1 & a<x<b \\ 0 & \text{otherwise} \end{cases}$$

This is not an exponential family.

⑥

Theorem: Let $f(x)$ be a pdf.

Let μ be any real constant.

Let σ be any positive real constant.

Then $\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) = g(x \mid \mu,\sigma)$

is also a valid pdf.

Proof: $g(x \mid \mu,\sigma) \geq 0$ ✓

$$\int_{-\infty}^{\infty} g(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) dx$$

Let $u = \frac{x-\mu}{\sigma}$

$du = \frac{1}{\sigma} dx$

⑦

$$= \int_{-\infty}^{\infty} f(u)\,du = 1 \quad ✓$$

Def^n. Let $f(x)$ be a pdf.

Let $g(x|\mu) = f(x-\mu)$. Then $g(x|\mu)$ is a location family of densities. And μ is a location parameter

Let $g(x|\sigma) = \frac{1}{\sigma} f\left(\frac{x}{\sigma}\right)$, for $\sigma > 0$.

Then $g(x|\sigma)$ is a scale family.

σ is a scale parameter

⑧

Let $g(x|\mu,\sigma)$

$$= \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) \text{ for } \sigma > 0.$$

Then $g(x|\mu,\sigma)$ is a location-scale family.

Example: $N(\mu,\sigma^2)$

$$g(x|\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$= \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \quad \text{where } z = \frac{x-\mu}{\sigma}$$

$$= \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) \quad \text{where } f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

So $N(\mu,\sigma^2)$ is a location-scale family

(9)

Example: Gamma(α, β)

$$g(x \mid \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}$$

$$= \frac{1}{\beta} \frac{1}{\Gamma(\alpha)} \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-x/\beta}$$

$$= \frac{1}{\beta} f\left(\frac{x}{\beta}\right) \quad \text{where } f(z) = \frac{1}{\Gamma(\alpha)} z^{\alpha-1} e^{-z}$$

So β is a scale parameter

(10)

Example: Cauchy(θ)

$$g(x \mid \theta) = \frac{1}{\pi} \frac{1}{1+(x-\theta)^2} \qquad -\infty < x < \infty$$

$$= \frac{1}{\pi} \frac{1}{1+z^2} \quad \text{where } z = x - \theta$$

$$= f(x-\theta) \quad \text{where } f(z) = \frac{1}{\pi} \frac{1}{1+z^2}$$

So θ is a location parameter

(11)

<u>Theorem</u>: Let $f(x)$ be a p.d.f.

Let μ & σ be constants, $\sigma > 0$

Then X is a random variable with pdf

$$g(x|\mu,\sigma) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$$

<u>iff</u> $\exists$ a random variable Z with pdf $f(z)$ and

$$X = \sigma Z + \mu.$$

<u>Proof</u>: $\Leftarrow$: $g(x) = f(z) \left|\frac{dz}{dx}\right|$

$x = \sigma z + \mu$

$z = \frac{x-\mu}{\sigma}$

$\frac{dz}{dx} = \frac{1}{\sigma}$

(12)

$$g(x) = f(z) \cdot \frac{1}{\sigma}$$
$$= \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$$

$\Rightarrow$: Let $Z = \frac{X-\mu}{\sigma}$

$$f(z) = g(x) \left|\frac{dx}{dz}\right|$$

$z = \frac{x-\mu}{\sigma}$

$x = \sigma z + \mu$

$\frac{dx}{dz} = \sigma$

$$= g(x) \cdot \sigma$$
$$= \sigma g(\sigma z + \mu)$$
$$= \sigma \cdot \frac{1}{\sigma} f\left(\frac{\sigma z + \mu - \mu}{\sigma}\right)$$
$$= f(z)$$