Stat 561

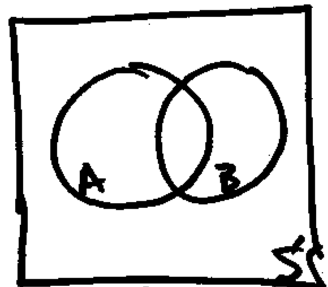
10-2-18

Conditional Probability

①

Defn: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$, provided $P(B) > 0$

"given"



Is this a valid probability function?

(1) $\forall A$, is $P(A|B) \geq 0$ yes

②

(2) $P(S|B) = \dfrac{P(S \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$ ✓

(3) Suppose $A_1 \cap A_2 = \phi$

$$P(A_1 \cup A_2 \mid B) = \frac{P\big((A_1 \cup A_2) \cap B\big)}{P(B)}$$

$$= \frac{P\big((A_1 \cap B) \cup (A_2 \cap B)\big)}{P(B)}$$

$$= \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)} = P(A_1|B) + P(A_2|B)$$ ✓

③

Defn: The events A & B are independent

if $P(A \cap B) = P(A)P(B)$

Note: Suppose $P(B) > 0$ and that A and B are independent.

Then $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A)P(B)}{P(B)}$

$= P(A)$

④

Ex: Roll 2 dice

$$S = \left\{ \begin{matrix} 11 & 12 & \cdots & 16 \\ 21 & 22 & \cdots & 26 \\ \vdots & & & \\ 61 & 62 & \cdots & 66 \end{matrix} \right\}$$

Let A be the event that the 1^{st} die $= 6$

B " " " " " 2^{nd} " $= 6$

$P(A) = \frac{6}{36} = \frac{1}{6}$ $\quad P(B) = \frac{6}{36} = \frac{1}{6}$

$P(A \cap B) = \frac{1}{36}$ $\quad$ Since $\frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6}$, A and B are independent

(5)

Ex: Draw 2 cards from a deck of 52 without replacement

let A: 1^{st} card is an ace

B: 2^{nd} " " " "

$$P(A \cap B) = \frac{4 \cdot 3}{52 \cdot 51} = \frac{1}{13} \cdot \frac{1}{17} \neq \frac{1}{13} \cdot \frac{1}{13}$$

$$P(A) = \frac{4 \cdot 51}{52 \cdot 51} = \frac{4}{52} = \frac{1}{13}$$

$\therefore$ A and B are dependent

$$P(B) = \frac{4 \cdot 3 + 48 \cdot 4}{52 \cdot 51} = \frac{4}{52} = \frac{1}{13}$$

(6)

Suppose $C_1, C_2, \ldots$ is a partition of S

From last time,

$$P(A) = \sum_{i=1}^{\infty} P(A \cap C_i)$$

$$= \sum_{i=1}^{\infty} P(C_i) P(A|C_i)$$

[exclude i if $P(C_i)=0$]

This is another form of the Law of Total Probability

(7)

Select a particular index k where $P(C_k) \neq 0$

$$P(C_k | A) = \frac{P(C_k \cap A)}{P(A)}$$

$$= \frac{P(C_k) P(A | C_k)}{\sum_{i=1}^{\infty} P(C_i) P(A|C_i)}$$

<u>Bayes' Rule</u>

(8)

If the person does not use the substance, the test results will be negative 96% of the time.

$$P(B^c | A^c) = .96$$

Question: Given that the test results are positive, what is the probability that the person uses the substance?

(9)

Ex: Let A: person uses substance

B: test result is positive

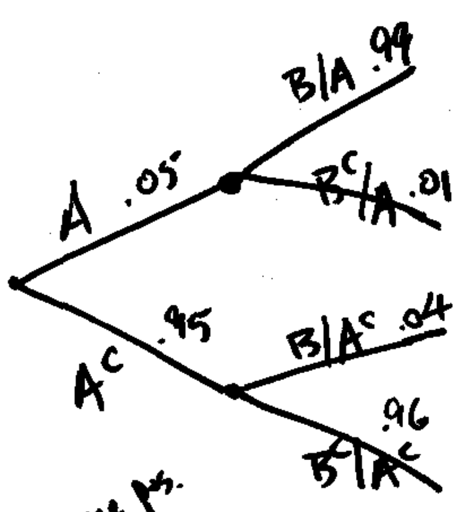
Assume: 1 in 20 persons use the substance

$P(A) = .05$

If the person uses the substance,
the test results will be positive
99% of the time.

$P(B|A) = .99$

(10)



$.0495 = P(A)P(B|A) = P(A \cap B)$

$.0005 = P(A \cap B^c)$

$.038 = P(A^c \cap B)$

$.912 = P(A^c \cap B^c)$

true pos.

false neg.

	B	B^c	
A	.0495	.0005	.0500
A^c	.0380	.9120	.9500
	.0875	.9125	1

false pos.

true neg.

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{.0495}{.0875} = .566$!!!

How does this use Bayes' Rule?

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)}$$

$$= \frac{(.05)(.99)}{(.05)(.99) + (.95)(.04)} = .566$$