Stat 561
11-15-18

(1)

The Beta distribution, from last time: $f_X(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$ $\quad 0<x<1$

And $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

$$E[X^n] = \int_0^1 \frac{x^n}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \int_0^1 \frac{x^{n+\alpha-1}}{B(\alpha,\beta)}(1-x)^{\beta-1}\,dx$$

(2)

$$= \frac{B(n+\alpha,\beta)}{B(\alpha,\beta)} \underbrace{\int_0^1 \frac{1}{B(n+\alpha,\beta)} x^{n+\alpha-1}(1-x)^{\beta-1}\,dx}_{1}$$

$$E[X^n] = \frac{B(n+\alpha,\beta)}{B(\alpha,\beta)} = \frac{\Gamma(n+\alpha)\cancel{\Gamma(\beta)}}{\Gamma(n+\alpha+\beta)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cancel{\Gamma(\beta)}}$$

So $E[X] = \frac{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\alpha+\beta+1)}$

$$= \frac{\alpha\cancel{\Gamma(\alpha)}\,\cancel{\Gamma(\alpha+\beta)}}{\cancel{\Gamma(\alpha)}\,(\alpha+\beta)\cancel{\Gamma(\alpha+\beta)}}$$

$$= \frac{\alpha}{\alpha+\beta}$$

$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$

③

$$E[X^2] = \frac{\Gamma(\alpha+2)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+2)\Gamma(\alpha)}$$

$$= \frac{(\alpha+1)\alpha\Gamma\cancel{(\alpha)}\ \Gamma\cancel{(\alpha+\beta)}}{\Gamma\cancel{(\alpha)}\ (\alpha+\beta+1)(\alpha+\beta)\Gamma\cancel{(\alpha+\beta)}}$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$\sigma^2 = E[X^2] - \mu^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

④

The Cauchy distribution

$$f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+(x-\theta)^2} \qquad -\infty < x < \infty$$

Recall: μ and σ^2 do not exist

θ is the median

The uniform distribution

$$f_X(x) = \frac{1}{b-a} \qquad a < x < b$$

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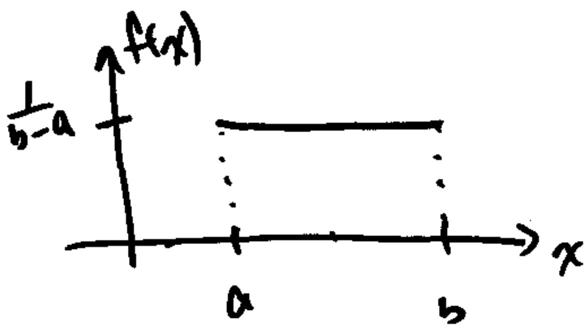

$$E[X] = \int_a^b x \frac{1}{b-a}\,dx = \frac{1}{b-a}\left.\frac{x^2}{2}\right|_a^b$$

$$= \frac{1}{b-a}\cdot\frac{1}{2}\left(b^2-a^2\right) = \frac{b+a}{2}$$

$$E[X^2] = \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{1}{b-a}\left.\frac{x^3}{3}\right|_a^b$$

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$$= \frac{1}{b-a}\cdot\frac{1}{3}\left(b^3-a^3\right)$$

$$= \frac{1}{\cancel{b-a}}\cdot\frac{1}{3}\left(\cancel{b-a}\right)\left(b^2+ab+a^2\right)$$

$$\sigma^2 = \frac{b^2+ab+a^2}{3} - \frac{(b+a)^2}{4}$$

$$= \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12}$$

$$= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}$$

⑦

$$M_X(t) = \int_a^b e^{tx} \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \left. \frac{e^{tx}}{t} \right|_a^b$$

$$= \frac{1}{b-a} \left[\frac{e^{tb}}{t} - \frac{e^{ta}}{t} \right]$$

$$= \frac{1}{b-a} \frac{e^{tb} - e^{ta}}{t}$$

Note: $M_X(t)$ is undefined at $t=0$, but the limit as $t \to 0$ is 1

⑧

Let Y have a normal distribution $N(\mu, \sigma^2)$

Let $X = e^Y$

$y = \ln x$

$\frac{dy}{dx} = \frac{1}{x}$

$$f_X(x) = g(y) \left| \frac{dy}{dx} \right|$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} \cdot \frac{1}{x}$$

(9)

$$\boxed{f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot \frac{1}{x} e^{-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2} \quad x > 0}$$

This is the <u>lognormal</u> <u>density</u>

Find $E[X] = E[e^Y] = E[e^{tY}]$ at $t=1$

$$= M_Y(t) \text{ at } t=1$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \text{ at } t=1$$

$$= e^{\mu + \frac{1}{2}\sigma^2}$$

(10)

$$E[X^2] = E[(e^Y)^2] = E[e^{2Y}]$$

$$= E[e^{tY}] \text{ at } t=2$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \text{ at } t=2$$

$$= e^{2\mu + 2\sigma^2}$$

$$V[X] = E[X^2] - (E[X])^2$$

$$= e^{2\mu + 2\sigma^2} - \left(e^{\mu + \frac{1}{2}\sigma^2}\right)^2$$

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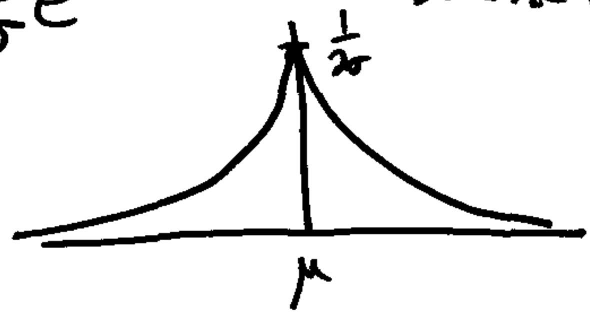
$$= e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2}$$

$$= e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$$

The double exponential (Laplace) distribution

$$f_X(x) = \frac{1}{2\sigma} e^{-\left|\frac{x-\mu}{\sigma}\right|} \qquad -\infty < x < \infty$$



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$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} \frac{1}{2\sigma} e^{-\left|\frac{x-\mu}{\sigma}\right|} dx$$

$$= \int_{-\infty}^{\infty} e^{t(\mu+\sigma y)} \frac{1}{2} e^{-|y|} dy$$

Let $y = \frac{x-\mu}{\sigma}$

$dy = \frac{1}{\sigma} dx$

$x = \mu + \sigma y$

$$= \int_{-\infty}^{0} e^{t(\mu+\sigma y)} \cdot \frac{1}{2} e^{y} dy$$

$$+ \int_{0}^{\infty} e^{t(\mu+\sigma y)} \cdot \frac{1}{2} e^{-y} dy$$

(13)

$$= \frac{1}{2}e^{t\mu}\left[\int_{-\infty}^{0} e^{(\sigma t+1)y}\,dy + \int_{0}^{\infty} e^{(\sigma t-1)y}\,dy\right]$$

$$= \frac{1}{2}e^{t\mu}\left[\frac{e^{(\sigma t+1)y}}{\sigma t+1}\Bigg|_{y=-\infty}^{0} + \frac{e^{(\sigma t-1)y}}{\sigma t-1}\Bigg|_{y=0}^{\infty}\right]$$

$$= \frac{1}{2}e^{\mu t}\left[\frac{1}{\sigma t+1} - 0 + 0 - \frac{1}{\sigma t-1}\right]$$

true if $\sigma t+1>0$, $t > \frac{-1}{\sigma}$

true if $\sigma t-1<0$, $t < \frac{1}{\sigma}$

(14)

$$= \frac{1}{2}e^{\mu t}\left[\frac{\sigma t-1-(\sigma t+1)}{(\sigma t+1)(\sigma t-1)}\right]$$

$$= \frac{1}{2}e^{\mu t}\;\frac{-2}{\sigma^2 t^2-1}$$

$$M_X(t) = \frac{e^{\mu t}}{1-\sigma^2 t^2} \qquad M_X(0)=1 \checkmark$$

$$M_X'(t) = \frac{(1-\sigma^2 t^2)\mu e^{\mu t} - e^{\mu t}(-2\sigma^2 t)}{(1-\sigma^2 t^2)^2}$$

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$$M_X'(t) = \frac{e^{\mu t}(\mu - \mu\sigma^2 t^2 + 2\sigma^2 t)}{(1-\sigma^2 t^2)^2}$$

$$M_X'(0) = \mu = E[X]$$

You can find $M_X''(t)$ + get $E[X^2]$

+ show $V[X] = 2\sigma^2$

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HW #7 p.132 #28, 33, 38

due 11/27 (Tuesday)

Final exam: material since the midterm

handed out on 11/29,

due 12/4

Stat 561 HW 7

3.28 Show that each of the following families is an exponential family.

(a) normal family with either parameter μ or σ known
(b) gamma family with either parameter α or β known or both unknown
(c) beta family with either parameter α or β known or both unknown
(d) Poisson family
(e) negative binomial family with r known, $0 < p < 1$

3.33 For each of the following families:

(i) Verify that it is an exponential family.
(ii) Describe the curve on which the $\boldsymbol{\theta}$ parameter vector lies.
(iii) Sketch a graph of the curved parameter space.

(a) $\mathrm{n}(\theta, \theta)$
(b) $\mathrm{n}(\theta, a\theta^2)$, a known
(c) $\mathrm{gamma}(\alpha, 1/\alpha)$
(d) $f(x|\theta) = C\exp\left(-(x-\theta)^4\right)$, C a normalizing constant

3.38 Let Z be a random variable with pdf $f(z)$. Define z_α to be a number that satisfies this relationship:

$$\alpha = P(Z > z_\alpha) = \int_{z_\alpha}^{\infty} f(z)dz.$$

Show that if X is a random variable with pdf $(1/\sigma)f((x-\mu)/\sigma)$ and $x_\alpha = \sigma z_\alpha + \mu$, then $P(X > x_\alpha) = \alpha$. (Thus if a table of z_α values were available, then values of x_α could be easily computed for any member of the location–scale family.)