Stat 561
10-9-18
(1)

Theorem: A function $F(x)$ is a cumulative distribution function for some random variable X if and only if

(1) $\lim_{x \to -\infty} F(x) = 0$ & $\lim_{x \to \infty} F(x) = 1$

(2) $F(x)$ is nondecreasing

(3) $F(x)$ is right-continuous

Proof of the part that says a cdf must be right-continuous:

(2)

Let $C_1, C_2, \ldots$ be a sequence of events such that $C_1 \subset C_2 \subset \ldots$

Let $A_1 = C_1$ and for $n \geq 2$, $A_n = C_n \setminus C_{n-1}$

then $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} C_i$

$$P[\lim_{n\to\infty} C_n] = P[\lim_{n\to\infty} \bigcup_{i=1}^{n} C_i] = P[\bigcup_{i=1}^{\infty} C_i]$$

$$= P[\bigcup_{i=1}^{\infty} A_i] = \sum_{i=1}^{\infty} P(A_i) = \lim_{n\to\infty} \sum_{i=1}^{n} P(A_i)$$

$$= \lim_{n\to\infty}\left[P(A_1) + \sum_{i=2}^{n} P(A_i)\right]$$

$$= \lim_{n\to\infty}\left[P(C_1) + \sum_{i=2}^{n}\left[P(C_i) - P(C_i \cap C_{i-1})\right]\right]$$

$$= \lim_{n\to\infty}\left[P(C_1) + \sum_{i=2}^{n}\left[P(C_i) - P(C_{i-1})\right]\right]$$

$$= \lim_{n\to\infty} P(C_n)$$

So $P(\lim_{n\to\infty} C_n) = \lim_{n\to\infty} P(C_n)$

Now let $B_1, B_2, \dots$ be a sequence of events such that $B_1 \supset B_2 \supset \dots$

4

Then $B_1^c \subset B_2^c \subset \dots$

So $P[\lim_{n\to\infty} B_n^c] = \lim_{n\to\infty} P[B_n^c]$

$$P[\lim_{n\to\infty} B_n^c] = P\left[\lim_{n\to\infty} \bigcup_{i=1}^{n} B_i^c\right] = P\left[\bigcup_{i=1}^{\infty} B_i^c\right] = P\left[\left(\bigcap_{i=1}^{\infty} B_i\right)^c\right]$$

$$\lim_{n\to\infty} P[B_n^c] = \lim_{n\to\infty}\left(1 - P(B_n)\right) = 1 - \lim_{n\to\infty} P(B_n)$$

5

$$= 1 - P\left[\bigcap_{i=1}^{\infty} B_i\right]$$

$$= 1 - P\left[\lim_{n\to\infty} \bigcap_{i=1}^{n} B_i\right]$$

$$= 1 - P\left[\lim_{n\to\infty} B_n\right] \qquad \text{So } \lim_{n\to\infty} P(B_n) = P\left[\lim_{n\to\infty} B_n\right]$$

Let $B_n = \left(-\infty, x + \frac{1}{n}\right]$, $n = 1, 2, 3, \ldots$

Note: $B_1 \supset B_2 \supset \ldots$

6

$$\lim_{n\to\infty} B_n = (-\infty, x]$$

$$P\left[\lim_{n\to\infty} B_n\right] = P\left[(-\infty, x]\right] = F_X(x)$$

$$= \lim_{n\to\infty} P(B_n) = \lim_{n\to\infty} P\left[\left(-\infty, x + \tfrac{1}{n}\right]\right]$$

$$= \lim_{n\to\infty} F_X\left(x + \tfrac{1}{n}\right)$$

$$\therefore \lim_{n\to\infty} F_X\left(x + \tfrac{1}{n}\right) = F_X(x)$$

⑦

<u>Defn</u>: If there exists a function $f_X(x)$ that satisfies $\int_{-\infty}^{x} f_X(t)\,dt = F_X(x)$,

then $f_X(x)$ is the <u>probability</u> <u>density</u> <u>function</u> (pdf) for the random variable X.

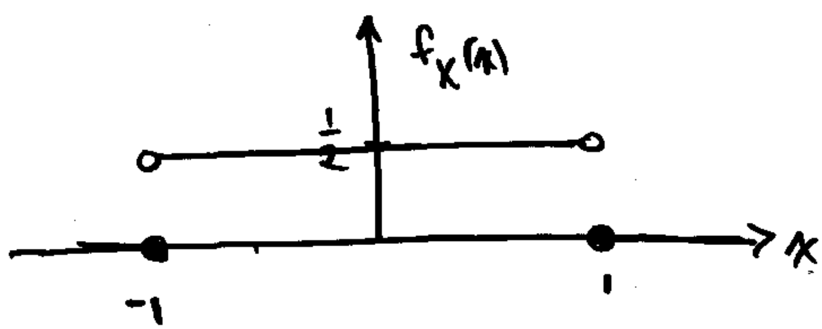
Note: If $F_X(x)$ has a derivative $\forall x$,

then $\frac{d}{dx}F_X(x)$ will be the pdf,

by the Fundamental Theorem of Calculus.

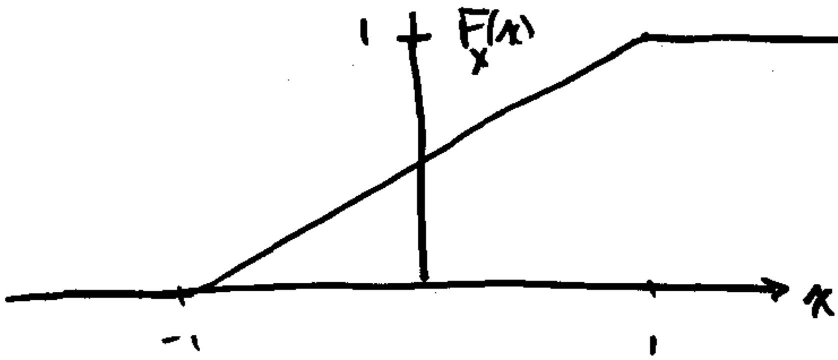
⑧

<u>Example</u>: Let $f_X(x) = \begin{cases} 1/2 & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$



Let $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} 0 & x \le -1 \\ (x+1)/2 & -1 \le x \le 1 \\ 1 & x \ge 1 \end{cases}$

9



This satisfies the 3 conditions of the theorem, so $F_X(x)$ is a valid cdf.

$P(X \le x)$

Note: Suppose $F_X(x)$ is continuous

What happens if you try to evaluate $P(X=x)$

Consider $P(X \le x) - P(X \le x - \frac{1}{n})$ + take lim as $n \to \infty$

$$\lim_{n \to \infty} \left(F_X(x) - F\left(x - \frac{1}{n}\right) \right) = 0 \qquad (10)$$