Stat 561
10-18-18

Notation note from last time:

(1)

μ_n' is the n^{th} moment, $E[X^n]$

Also, μ_n is the n^{th} central moment, $E[(X-\mu)^n]$

$E[X] = \mu = \mu_1'$

Note: $\mu_1 = E[X-\mu] = E[X]-\mu$
$= 0$

$E[X^2] = \mu_2'$

Note $\mu_2 = E[(X-\mu)^2]$
$= V[X] = \sigma^2$
$= E[X^2] - (E[X])^2$
$= \mu_2' - (\mu_1')^2$

(2)

Properties of $V[X]$

$$V[k] = E[k^2] - (E[k])^2$$
$$= k^2 - k^2 = 0$$

$$V[aX] = E[(aX)^2] - (E[aX])^2$$
$$= E[a^2X^2] - (aE[X])^2$$
$$= a^2E[X^2] - a^2(E[X])^2$$
$$= a^2[E[X^2] - (E[X])^2]$$
$$= a^2V[X]$$

③

$$V[g(X)+h(X)]$$

$$= E[(g(X)+h(X))^2] - (E[g(X)+h(X)])^2$$

$$= E[(g(X))^2+(h(X))^2+2g(X)h(X)] - (E[g(X)]+E[h(X)])^2$$

$$= \underline{E[(g(X))^2]} + \underline{E[(h(X))^2]} + 2E[\underline{g(X)h(X)}] - \{(\underline{E[g(X)]})^2 + (\underline{E[h(X)]})^2 + 2E[\underline{g(X)]E[h(X)}]\}$$

④

$$= V[g(X)] + V[h(X)] + 2[\underbrace{E[g(X)h(X)] - E[g(X)]E[h(X)]}_{\text{This is the covariance of } g(X) \text{ with } h(X)}]$$

The variance is a quadratic operator

$$V[aX+b] = V[aX]+V[b]+ 2[E[abX] - E[aX]E[b]]$$

$$= a^2V[X] + 0 + 2[\cancel{abE[X] - aE[X]b}]$$

⑤

<u>Defn</u>: $M_X(t) = E[e^{tX}]$

This is called the moment-generating function for the r.v. X.

<u>Properties</u>: $M_X(0) = E[e^0] = 1$

$$\frac{d}{dt} M_X(t) = \frac{d}{dt} E[e^{tX}]$$

$$= E\left[\frac{d}{dt} e^{tX}\right] \quad \text{Why?}$$

⑥

Because: $\displaystyle \frac{d}{dt} \int_a^b f(x,t)\,dx$

$$= \lim_{h\to 0} \frac{\int_a^b f(x,t+h)\,dx - \int_a^b f(x,t)\,dx}{h}$$

$$= \lim_{h\to 0} \frac{\int_a^b [f(x,t+h) - f(x,t)]\,dx}{h}$$

$$= \lim_{h\to 0} \int_a^b \frac{f(x,t+h) - f(x,t)}{h}\,dx$$

$$= \int_a^b \lim_{h \to 0} \frac{f(x,t+h) - f(x,t)}{h} \, dx$$ (7)

by Liebniz' Theorem

$$= \int_a^b \frac{\partial}{\partial t} f(x,t) \, dx$$

So $$\frac{d}{dt} M_X(t) = \frac{d}{dt} E[e^{tX}] = E\left[\frac{d}{dt} e^{tX}\right]$$
$$= E[Xe^{tX}]$$
$$M_X'(0) = E[X]$$

(8)

$$M_X''(t) = E[X^2 e^{tX}]$$
$$M_X''(0) = E[X^2]$$

<u>Example</u>: $f_X(x) = \frac{1}{\lambda} e^{-x/\lambda}, \quad x > 0$

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} \frac{1}{\lambda} e^{-x/\lambda} \, dx$$
$$= \frac{1}{\lambda} \int_0^\infty e^{x(t - \frac{1}{\lambda})} dx$$
$$= \frac{1}{\lambda} \left. \frac{e^{x(t-\frac{1}{\lambda})}}{t - \frac{1}{\lambda}} \right|_{x=0}^{\infty}$$

$$= \frac{1}{\lambda}\left[0 - \frac{1}{t - \frac{1}{\lambda}} \right] \quad \text{provided } t < \frac{1}{\lambda}$$

$$= \frac{-1}{t\lambda - 1}$$

$$\boxed{M_X(t) = \frac{1}{1-\lambda t}} = (1-\lambda t)^{-1} \quad [M_X(0) = 1 \checkmark]$$

$$M_X'(t) = -(1-\lambda t)^{-2}(-\lambda) = \lambda(1-\lambda t)^{-2}$$

$$M_X''(t) = \lambda(-2)(1-\lambda t)^{-3}(-\lambda) = 2\lambda^2(1-\lambda t)^{-3}$$

$$M_X'(0) = \lambda = E[X] \qquad M_X''(0) = 2\lambda^2 = E[X^2]$$

$$\sigma^2 = 2\lambda^2 - \lambda^2 = \lambda^2$$

Example: $p_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, \ldots, n$

$$M_X(t) = E[e^{tX}] = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x}$$

$(q = 1-p)$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x}$$

$$= (pe^t + q)^n \quad \text{by binomial theorem}$$

Check $M_X(0) = (p+q)^n = 1 \checkmark$

(11)

$$M_X'(t) = n(pe^t+q)^{n-1}pe^t$$
$$= npe^t(pe^t+q)^{n-1}$$

$$M_X''(t) = np\left[e^t(n-1)(pe^t+q)^{n-2}pe^t + e^t(pe^t+q)^{n-1}\right]$$

$$M_X'(0) = np = E[X]$$

$$M_X''(0) = np[(n-1)p+1] = n(n-1)p^2+np$$
$$= E[X^2]$$

$$\sigma^2 = n(n-1)p^2+np - n^2p^2$$
$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2} = np(1-p) = npq$$

(12)

Defn: The standard deviation of a r.v. X is $\sigma = \sqrt{V[X]}$

Defn: The coefficient of variation of a r.v. X is $CV = \frac{\sigma}{\mu}$

Chebyshev's Theorem: Let $g(X)$ be a non-negative function of the r.v. X and let r be a positive constant.

Then $P[g(X) \geq r] \leq \dfrac{E[g(X)]}{r}$ (13)

Proof: (continuous case)

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

Let $A = \{ x \mid g(x) \geq r \}$

Then
$$E[g(X)] = \int_A g(x) f_X(x)\,dx + \int_{A^c} g(x) f_X(x)\,dx$$
$$\geq \int_A g(x) f_X(x)\,dx$$
$$\geq \int_A r f_X(x)\,dx \quad (14)$$
$$= r \int_A f_X(x)\,dx$$
$$= r\,P(X \in A)$$

$$E[g(X)] \geq r P(g(X) \geq r)$$

$$P(g(X) \geq r) \leq \frac{E[g(X)]}{r}$$

HW #4 due (10/25) p.79 # 24, 28, 33

Stat 561 HW 4

2.24 Compute $\mathrm{E}\,X$ and $\mathrm{Var}\,X$ for each of the following probability distributions.

(a) $f_X(x) = ax^{a-1}$, $0 < x < 1$, $a > 0$

(b) $f_X(x) = \frac{1}{n}$, $x = 1, 2, \ldots, n$, $n > 0$ an integer

(c) $f_X(x) = \frac{3}{2}(x-1)^2$, $0 < x < 2$

2.28 Let μ_n denote the nth central moment of a random variable X. Two quantities of interest, in addition to the mean and variance, are

$$\alpha_3 = \frac{\mu_3}{(\mu_2)^{3/2}} \quad \text{and} \quad \alpha_4 = \frac{\mu_4}{\mu_2^2}.$$

The value α_3 is called the *skewness* and α_4 is called the *kurtosis*. The skewness measures the lack of symmetry in the pdf (see Exercise 2.26). The kurtosis, although harder to interpret, measures the peakedness or flatness of the pdf.

(a) Show that if a pdf is symmetric about a point a, then $\alpha_3 = 0$.

(b) Calculate α_3 for $f(x) = e^{-x}$, $x \geq 0$, a pdf that is *skewed to the right*.

(c) Calculate α_4 for each of the following pdfs and comment on the peakedness of each.

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \quad -\infty < x < \infty$$

$$f(x) = \frac{1}{2}, \quad -1 < x < 1$$

$$f(x) = \frac{1}{2}e^{-|x|}, \quad -\infty < x < \infty$$

2.33 In each of the following cases verify the expression given for the moment generating function, and in each case use the mgf to calculate $\mathrm{E}\,X$ and $\mathrm{Var}\,X$.

(a) $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $M_X(t) = e^{\lambda(e^t-1)}$, $x = 0, 1, \ldots;\ \lambda > 0$

(b) $P(X=x) = p(1-p)^x$, $M_X(t) = \frac{p}{1-(1-p)e^t}$, $x = 0, 1, \ldots;\ 0 < p < 1$

(c) $f_X(x) = \frac{e^{-(x-\mu)^2/(2\sigma^2)}}{\sqrt{2\pi}\sigma}$, $M_X(t) = e^{\mu t + \sigma^2 t^2/2}$, $-\infty < x < \infty;\ -\infty < \mu < \infty,\ \sigma > 0$