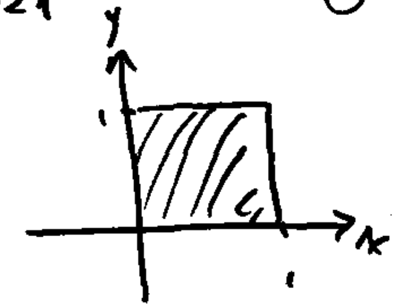
Stat 561
11-29-18

New example:

① $f(x,y) = 10xy^4 \quad 0<x<1,\ 0<y<1$



$$\int_0^1\int_0^1 10xy^4\,dx\,dy$$

$$= \int_0^1 10y^4 \left.\frac{x^2}{2}\right|_{x=0}^{1} dy$$

$$= \int_0^1 10y^4 \cdot \frac{1}{2}\,dy = 5\left.\frac{y^5}{5}\right|_0^1 = 1 \ \checkmark$$

Find $f(y|x)$

② $$f(y|x) = \frac{f(x,y)}{f_X(x)}$$

$$f_X(x) = \int_0^1 f(x,y)\,dy = \int_0^1 10xy^4\,dy$$

$$= 10x\left.\frac{y^5}{5}\right|_0^1 = 2x$$

$f_X(x) = 2x, \quad 0<x<1$

Now, $f(y|x) = \dfrac{10xy^4}{2x} = 5y^4, \quad \begin{matrix} 0<x<1 \\ 0<y<1 \end{matrix}$

③

Defn.: The random variables X and Y are *independent* if $f(x,y) = f_X(x) f_Y(y)$ $\forall (x,y)$

Factorization Theorem: The random variables X & Y are independent if and only if there exist functions $g(x)$ and $h(y)$ such that
$$f(x,y) = g(x)h(y) \quad \forall (x,y).$$

④

Proof: $\Rightarrow$: trivial, based on definition

$\Leftarrow$: Suppose $\exists\ g(x)$ & $h(y)$ such that
$$f(x,y) = g(x)h(y)$$

Then
$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-\infty}^{\infty} g(x)h(y)\,dy = g(x)\underbrace{\int_{-\infty}^{\infty} h(y)\,dy}_{k}$$

Now $f_X(x) = k\,g(x)$

⑤

Also, $1 = \int_{-\infty}^{\infty} f_X(x)\,dx$

$$= \int_{-\infty}^{\infty} k\,g(x)\,dx = k\int_{-\infty}^{\infty} g(x)\,dx$$

so $\int_{-\infty}^{\infty} g(x)\,dx = \frac{1}{k}$

And $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{-\infty}^{\infty} g(x)h(y)\,dx$

$$= h(y)\int_{-\infty}^{\infty} g(x)\,dx = \frac{1}{k}h(y)$$

⑥

$\therefore\ f(x,y) = g(x)h(y)$

$$= \underline{k\,g(x)}\ \underline{\tfrac{1}{k}h(y)}$$

$= f_X(x)\,f_Y(y)$, so X & Y are independent

Examples:

$f(x,y) = c(x+y) \quad 0<x<1 \quad 0<y<1$

Not independent

7

$$f(x,y) = ce^{-(x+y)} \qquad x>0,\ y>0$$

Independent

$$f(x,y) = c\,xy \qquad 0<x<y<1$$

$$= c\,xy\, I_{\{0<x<y<1\}}(x,y)$$

Not independent