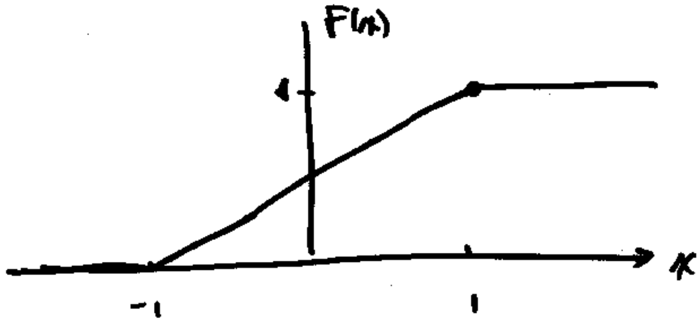
Transformations of random variables

Stat 561
10-11-18

①

Example from Tuesday: $F_X(x) = \begin{cases} 0 & x \le -1 \\ \frac{1}{2}(x+1) & -1 \le x \le 1 \\ 1 & x \ge 1 \end{cases}$



Let $Y = X^2$ Find its cdf.

②

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(X^2 \le y) \\ &= P(-\sqrt{y} \le X \le \sqrt{y}) \\ &= P(X \le \sqrt{y}) - P(X < -\sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned}$$

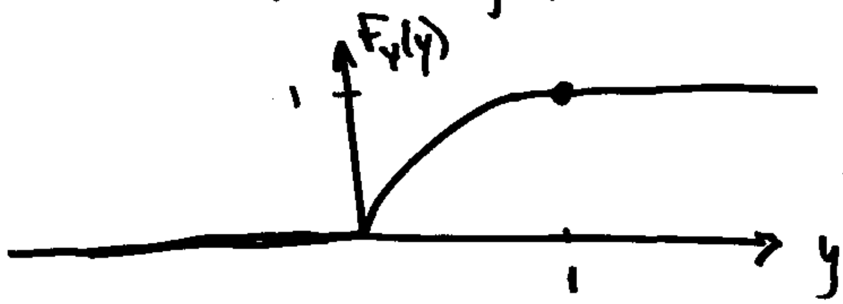
$(X \le \sqrt{y}) \cap (X < -\sqrt{y})^c$

Note: remember that our F was continuous

③

$$= \begin{cases} 0 & y<0 \\ \frac{1}{2}(\sqrt{y}+1) - \frac{1}{2}(-\sqrt{y}+1) & 0 \le y \le 1 \\ 1 & y > 1 \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & y<0 \\ \sqrt{y} & 0 \le y \le 1 \\ 1 & y>1 \end{cases}$$



④

Find the pdf for Y

Since $F_Y(y)$ was continuous, find $f_Y(y) = \frac{d}{dy} F_Y(y)$

$$= \begin{cases} \frac{1}{2\sqrt{y}} & 0 < y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

<u>Special</u> <u>cases</u>: Assume the cdf is continuous

Let $Y = g(X)$ be a tranformation of X.

Suppose $g \uparrow$ (monotone increasing)

(5)

$$F_Y(y) = P(Y \le y)$$
$$= P(g(X) \le y)$$
$$= P(X \le g^{-1}(y))$$
$$= F_X(g^{-1}(y))$$

Now $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(g^{-1}(y))$

$$= f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

(6)

$f_Y(y) = f_X(x) \cdot \frac{dx}{dy}$ But $y = g(x)$ so $x = g^{-1}(y)$

Suppose $g \downarrow$

$$F_Y(y) = P(Y \le y)$$
$$= P(g(X) \le y)$$
$$= P(X \ge g^{-1}(y))$$
$$= 1 - P(X < g^{-1}(y))$$
$$= 1 - F_X(g^{-1}(y))$$ works since F was cont.

(7)

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(1 - F_X(g^{-1}(y))\right)$$
$$= -f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$
$$= -f_X(x) \frac{dx}{dy}$$

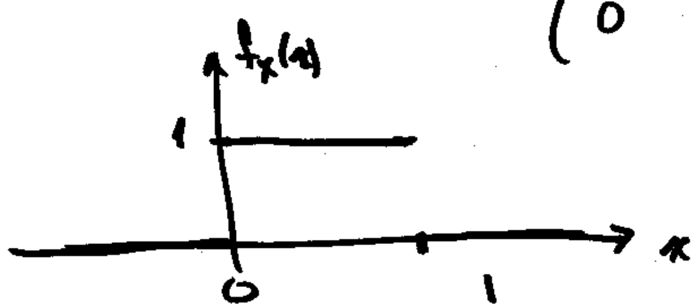
Summary:

Case 1 $g\uparrow$	Case 2 $g\downarrow$
$f_Y(y) = f_X(x) \frac{dx}{dy}$	$f_Y(y) = -f_X(x) \frac{dx}{dy}$

$$\boxed{f_Y(y) = f_X(x) \left|\frac{dx}{dy}\right|}$$

(8)

<u>Example</u>: Let X be a continuous r.v. with pdf $f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$



Let $Y = -2\ln X$ Note: $\downarrow$

Find the pdf for Y

(9)

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| \qquad y = -2\ln x$$

$$= 1 \cdot \frac{1}{2}e^{-\frac{1}{2}y} \qquad \ln x = -\frac{1}{2}y$$

$$x = e^{-\frac{1}{2}y}$$

for $0 < x < 1$ $\qquad \dfrac{dx}{dy} = -\dfrac{1}{2}e^{-\frac{1}{2}y}$

$$0 < e^{-\frac{1}{2}y} < 1$$

$$-\infty < -\frac{1}{2}y < 0$$

$$\infty > y > 0$$

$$f_Y(y) = \begin{cases} \frac{1}{2}e^{-\frac{1}{2}y} & 0 < y < \infty \\ 0 & \text{elsewhere} \end{cases}$$

(10)

<u>Example</u>: $f_X(x) = \dfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2} \qquad -\infty < x < \infty$

Let $Y = X^2$ Not monotone, so <u>no shortcut</u>

$$F_Y(y) = P(Y \le y) = P(X^2 \le y)$$

$$= P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

Unfortunately, there is no closed form for $F_X(x)$

(11)

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}\left[F_X(\sqrt{y}) - F_X(-\sqrt{y})\right]$$

$$= f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_X(-\sqrt{y})\,\frac{-1}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y}\frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y}\frac{1}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y}\frac{1}{\sqrt{y}} \quad \text{for} \quad -\infty < x < \infty, \; y \underset{>}{\neq} 0$$

(12)

$$f_Y(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi y}}e^{-\frac{1}{2}y} & y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

HW #3 due 10/18

p. 43 #48

p. 76 # 2, 12, 17

1.48 Prove the necessity part of Theorem 1.5.3.

2.2 In each of the following find the pdf of Y.

(a) $Y = X^2$ and $f_X(x) = 1,\ 0 < x < 1$

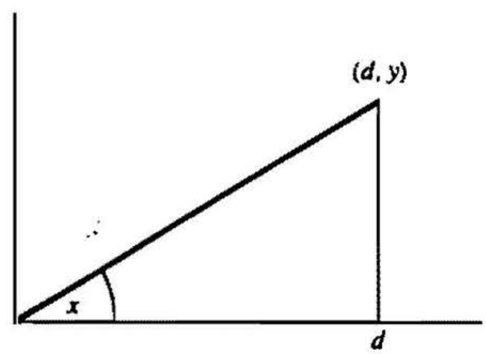
(b) $Y = -\log X$ and X has pdf

$$f_X(x) = \frac{(n+m+1)!}{n!\,m!}x^n(1-x)^m, \quad 0 < x < 1, \quad m, n \text{ positive integers}$$

(c) $Y = e^X$ and X has pdf

$$f_X(x) = \frac{1}{\sigma^2}xe^{-(x/\sigma)^2/2}, \quad 0 < x < \infty, \quad \sigma^2 \text{ a positive constant}$$

2.12 A random right triangle can be constructed in the following manner. Let X be a random angle whose distribution is uniform on $(0, \pi/2)$. For each X, construct a triangle as pictured below. Here, Y = height of the random triangle. For a fixed constant d, find the distribution of Y and $\mathrm{E}Y$.



2.17 A *median* of a distribution is a value m such that $P(X \le m) \ge \frac{1}{2}$ and $P(X \ge m) \ge \frac{1}{2}$. (If X is continuous, m satisfies $\int_{-\infty}^{m} f(x)\,dx = \int_{m}^{\infty} f(x)\,dx = \frac{1}{2}$.) Find the median of the following distributions.

(a) $f(x) = 3x^2, \quad 0 < x < 1$

(b) $f(x) = \dfrac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$