Stat 561
11-6-18

from last time:

$$\frac{\partial f(x,w)}{\partial w} = -\lambda f(x,w) + \lambda f(x-1,w) \quad \text{for } x \ge 1$$

(1)

$x=1$ case:

$$\frac{\partial f(1,w)}{\partial w} = -\lambda f(1,w) + \lambda \underbrace{f(0,w)}_{e^{-\lambda w}}$$

$$e^{\lambda w}\frac{\partial f(1,w)}{\partial w} = -\lambda e^{+\lambda w} f(1,w) + \lambda$$

$$e^{\lambda w}\frac{\partial f(1,w)}{\partial w} + \lambda e^{\lambda w} f(1,w) = \lambda$$

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$$\frac{d}{dw}\left[e^{\lambda w} f(1,w)\right] = \lambda$$

$$e^{\lambda w} f(1,w) = \lambda w + c$$

$$f(1,w) = \lambda w e^{-\lambda w} + c e^{-\lambda w}$$

Use $f(1,0) = 0$

$$\therefore \underbrace{f(1,0)}_{0} = 0 + c \qquad \underline{c=0}$$

So $\underline{f(1,w) = \lambda w e^{-\lambda w}}$

$k=2$ case:

$$\frac{\partial f(2,w)}{\partial w} = -\lambda f(2,w) + \lambda \underbrace{f(1,w)}_{\lambda w e^{-\lambda w}}$$

$$e^{\lambda w}\frac{\partial f(2,w)}{\partial w} + \lambda e^{\lambda w} f(2,w) = \lambda^2 w$$

$$\frac{d}{dw}\left[e^{\lambda w} f(2,w)\right] = \lambda^2 w$$

$$e^{\lambda w} f(2,w) = \lambda^2 \frac{w^2}{2} + c$$

$$f(2,w) = \frac{\lambda^2 w^2}{2} e^{-\lambda w} + c e^{-\lambda w}$$

Use $f(2,0)=0$

$$0 = f(2,0) = 0 + c \qquad \therefore c=0$$

$$f(2,w) = \frac{\lambda^2 w^2}{2} e^{-\lambda w}$$

Summary:

$$f(0,w) = e^{-\lambda w}$$
$$f(1,w) = \lambda w e^{-\lambda w}$$
$$f(2,w) = \frac{(\lambda w)^2}{2} e^{-\lambda w}$$
$$\vdots$$
$$f(k,w) = \frac{(\lambda w)^k}{k!} e^{-\lambda w}$$

You usually see this as

$$f_X(x) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}$$

$$\text{or} \quad f_X(x) = \frac{\mu^x e^{-\mu}}{x!} \qquad x = 0, 1, 2, \ldots$$

This is the Poisson Distribution

Check $\displaystyle 1 \stackrel{?}{=} \sum_{x=0}^{\infty} \frac{\mu^x e^{-\mu}}{x!} = e^{-\mu} \underbrace{\left(1 + \mu + \frac{\mu^2}{2!} + \cdots\right)}_{e^{\mu}}$

$$= 1 \checkmark$$

$$M_X(t) = E[e^{tX}]$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{\mu^x e^{-\mu}}{x!}$$

$$= e^{-\mu} \sum_{x=0}^{\infty} \frac{(\mu e^t)^x}{x!}$$

$$= e^{-\mu} e^{\mu e^t} = e^{\mu(e^t - 1)}$$

$$M_X'(t) = e^{\mu(e^t - 1)} \mu e^t$$

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$$M_X''(t) = \mu\left[e^{\mu(e^t-1)}e^t + e^t e^{\mu(e^t-1)}\mu e^t\right]$$

$$E[X] = M_X'(0) = \mu$$

$$E[X^2] = M_X''(0) = \mu[1+\mu] = \mu + \mu^2$$

$$\sigma^2 = \mu + \mu^2 - \mu^2 = \mu$$

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Start with Bino(n,p)

let $n \to \infty$

$p \to 0$ (hold $\underset{\mu}{np}$ constant)

$$p(x) = \binom{n}{x} p^x q^{n-x} \qquad p = \frac{\mu}{n}$$

$$= \frac{n!}{x!(n-x)!} \frac{\mu^x}{n^x} \frac{\left(1-\frac{\mu}{n}\right)^n}{\left(1-\frac{\mu}{n}\right)^x}$$

$$= \frac{\mu^x}{x!} \frac{n(n-1)\cdots(n-x+1)}{n \cdot n \cdots n} \frac{\left(1-\frac{\mu}{n}\right)^n}{\left(1-\frac{\mu}{n}\right)^x}$$

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$$\lim_{n\to\infty} p(x) = \lim_{n\to\infty} \frac{\mu^x}{x!}\, 1\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\frac{\left(1-\frac{\mu}{n}\right)^n}{\left(1-\frac{\mu}{n}\right)^x}$$

$$= \frac{\mu^x}{x!} e^{-\mu}$$

Let $y = \left(1-\frac{\mu}{n}\right)^n$

$$\ln y = n \ln\left(1-\frac{\mu}{n}\right) = \frac{\ln\left(1-\frac{\mu}{n}\right)}{\frac{1}{n}}$$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\frac{1}{(1-\mu/n)}\cdot\frac{\mu}{n^2}}{-\frac{1}{n^2}} = -\mu$$

$$\therefore \lim_{n\to\infty} y = e^{-\mu}$$

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$$\begin{array}{ccccc} & & \text{Bernoulli} & & \\ & & | & & \\ \text{Hypergeom} & \xrightarrow[\substack{N\to\infty \\ \frac{M}{N}\ \text{const.}}]{} & \text{Bino} & \xrightarrow[\substack{n\to\infty \\ np\ \text{const.}}]{} & \text{Poisson} \end{array}$$

$$\begin{array}{ccccc} \text{Neg Hyper} & \longrightarrow & \text{Neg Bino} & \longrightarrow & ? \\ & & | & & \\ & & \text{geometric} & & \end{array}$$

$\left.\right\}$ Not in our book

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<u>Numerical example</u>: Population of 10,000 people

Take a sample of 1000, without replacement

In the population, 500 have blue eyes.

Find the prob. of getting exactly 45 people in the sample with blue eyes.

Hyper$(N = 10000, M = 500, K = 1000)$

$$P(X=45) = \frac{\binom{500}{45}\binom{9500}{955}}{\binom{10000}{1000}} = .047136$$

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Do the Binomial approximation

Bin$(n = 1000, p = .05)$

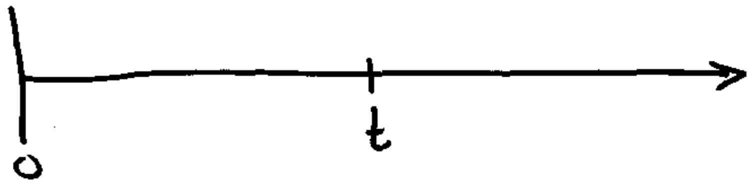
$$P(X=45) = \binom{1000}{45}(.05)^{45}(.95)^{955} = .046281$$

And the Poisson approximation

Poisson$(\mu = 50)$

$$P(X=45) = \frac{50^{45}}{45!}e^{-50} = .045826$$

Suppose we have a Poisson process



Let T be the waiting time until the r^{th} occurrence.

Find $P(T > t) = P\begin{pmatrix}\text{there were at most } r-1 \text{ occurrences} \\ \text{in } (0,t)\end{pmatrix}$

$= P(X \leq r-1),$

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where X has a Poisson distribution

with $\mu = \lambda t$

λ: expected # occurrences per unit time

Special case of $r=1$:

T is the waiting time until the 1^{st} occurrence

$$P(T > t) = P(X \leq 0) = P(X=0)$$

$$= \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t}$$

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$$F_T(t) = P(T \le t) = 1 - P(T > t)$$
$$= 1 - e^{-\lambda t}$$

$$f_T(t) = \frac{d}{dt} F_T(t) = \lambda e^{-\lambda t} \qquad t \geq 0$$

This is the exponential density function

Define the gamma function:

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$$

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Properties:

$$\Gamma(1) = \int_0^\infty t^0 e^{-t}\, dt = -e^{-t}\Big|_0^\infty$$
$$= 0 - (-1)$$
$$= 1$$

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$$
$$= -t^{\alpha-1} e^{-t}\Big|_0^\infty + \int_0^\infty (\alpha-1) t^{\alpha-2} e^{-t}\, dt$$

Let $u = t^{\alpha-1}$ $\quad dv = e^{-t} dt$

$du = (\alpha-1) t^{\alpha-2} dt$ $\quad v = -e^{-t}$

$$= 0 + (\alpha-1)\int_0^\infty t^{\alpha-2}e^{-t}\,dt$$

by L'H, for $\alpha \geq 1$

$$\therefore\ \Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1)$$

So $\Gamma(1) = 1$

$\Gamma(2) = 1\cdot\Gamma(1) = 1$

$\Gamma(3) = 2\Gamma(2) = 2$

$\Gamma(4) = 3\Gamma(3) = 6$

Pattern:

$\Gamma(n) = (n-1)!$