Stat 561
10-25-18

The geometric distribution

(1)

Run a sequence of independent trials,
2 possible outcomes on each,
Constant success probability p

$\left.\right\}$ Bernoulli trials

X = trial on which 1st success occurs

x	$p(x)$
1	p
2	qp
3	$q^2 p$
⋮	

$$\boxed{p_X(x) = pq^{x-1} \quad x = 1,2,3,\ldots}$$

Check: $\displaystyle\sum_{x=1}^{\infty} pq^{x-1} = p[1+q+q^2+q^3+\ldots]$
$= p\dfrac{1}{1-q} = 1$

(2)

$$M_X(t) = E[e^{tX}] = \sum_{x=1}^{\infty} e^{tx} pq^{x-1}$$

$$= pe^t \sum_{x=1}^{\infty} e^{t(x-1)} q^{x-1}$$

$$= pe^t \sum_{x=1}^{\infty} (qe^t)^{x-1}$$

$$= pe^t[1 + qe^t + (qe^t)^2 + \ldots]$$

$$= \frac{pe^t}{1-qe^t} \quad \text{(provided } qe^t < 1\text{)}$$

$$M_X(0) = \frac{p}{1-q} = 1 \checkmark$$

$$M_X'(t) = \frac{(1-qe^t)(pe^t) - pe^t(-qe^t)}{(1-qe^t)^2}$$

$$= \frac{pe^t}{(1-qe^t)^2} \qquad M_X'(0) = \frac{p}{(1-q)^2} = \frac{1}{p}$$

$$M_X''(t) = \frac{(1-qe^t)^2 pe^t - pe^t \cdot 2(1-qe^t)(-qe^t)}{(1-qe^t)^4}$$

$$M_X''(0) = \frac{(1-q)^2 p - p\,2(1-q)(-q)}{(1-q)^4}$$

$$= \frac{p^3 + 2p^2 q}{p^4}$$

$$= \frac{p+2q}{p^2} = E[X^2]$$

$$\sigma^2 = \frac{p+2q}{p^2} - \frac{1}{p^2} = \frac{1-q+2q-1}{p^2} = \frac{q}{p^2}$$

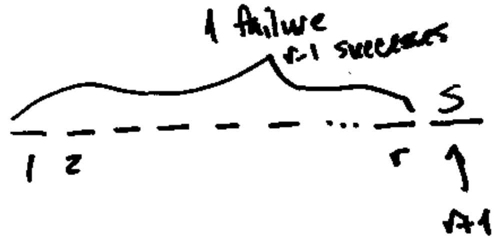
<u>Negative</u> <u>Binomial</u> Distri<u>bution</u> (<u>Pascal</u> <u>Distribution</u>)

Run a sequence of Bernoulli trials

X = trial on which the r^{th} success occurs

(5)

x	$p(x)$
r	p^r
$r+1$	$r\,p^{r-1}qp$
$r+2$	$\binom{r+1}{2}p^{r-1}q^2p$
$r+3$	$\binom{r+2}{3}p^{r-1}q^3p$
$\vdots$	



$$p_X(x) = \binom{x-1}{x-r} p^r q^{x-r}$$

$$\boxed{p_X(x) = \binom{x-1}{r-1} p^r q^{x-r}} \quad x = r, r+1, \ldots$$

Since $\binom{n}{a} = \binom{n}{n-a}$

(6)

$$M_X(t) = E[e^{tX}] = \sum_{x=r}^{\infty} e^{tx}\binom{x-1}{r-1}p^r q^{x-r}$$

$$= \sum_{x=r}^{\infty}\binom{x-1}{r-1}p^r\,\frac{e^{tx}q^x}{q^r}$$

$$= \frac{(qe^t)^r\,p^r}{(1-qe^t)^r\,q^r}\underbrace{\sum_{x=r}^{\infty}\binom{x-1}{r-1}(1-qe^t)^r\frac{(qe^t)^x}{(qe^t)^r}}_{1}$$

$$M_X(t) = \left[\frac{pe^t}{1-qe^t}\right]^r$$

(7)

Note: $X = X_1 + X_2 + \dots + X_r$

trial on which 1st success occurred

# of additional trials for 2nd success

Each $X_i \sim \text{Geom}(p)$

And they are indep.

So $E[X] = r\,E[X_i] = \frac{r}{p}$

$V[X] = r\,V[X_i] = \frac{rq}{p^2}$

(8)

Hypergeometric Distribution

Take a random sample of K objects, without replacement, from a population of N objects. In the population, M items are of a particular type.

X counts the number of items of that type in the sample.

$$p_X(x) = \frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}}$$

$x \geq 0$

$x \leq M$, $x \leq K$

$K - x \leq N - M$

$x \geq K - N + M$

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So $x \le \min(M, K)$

$x \ge \max(0, K - N + M)$

There is no closed form for $M_X(t)$ for the hypergeometric distribution.