Chebyshev's Theorem (from last time):

$$P[g(X) \geq r] \leq \frac{E[g(X)]}{r}$$

<u>Chebyshev's Inequality</u>

Use the theorem with $g(X) = (X-\mu)^2$
and $r = t^2\sigma^2$ with $t > 0$

$$P[(X-\mu)^2 \geq t^2\sigma^2] \leq \frac{E[(X-\mu)^2]}{t^2\sigma^2} = \frac{\sigma^2}{t^2\sigma^2} = \frac{1}{t^2}$$

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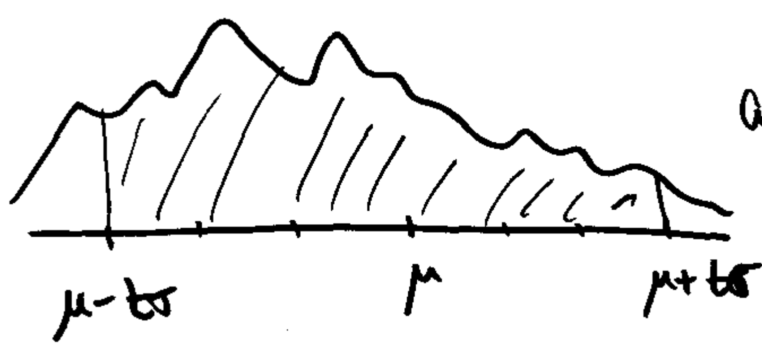
$$P[|X-\mu| \geq t\sigma] \leq \frac{1}{t^2}$$

$$\boxed{P[|X-\mu| < t\sigma] \geq 1 - \frac{1}{t^2}}$$

$P[A] \leq c$

$P[A^c] \geq 1-c$

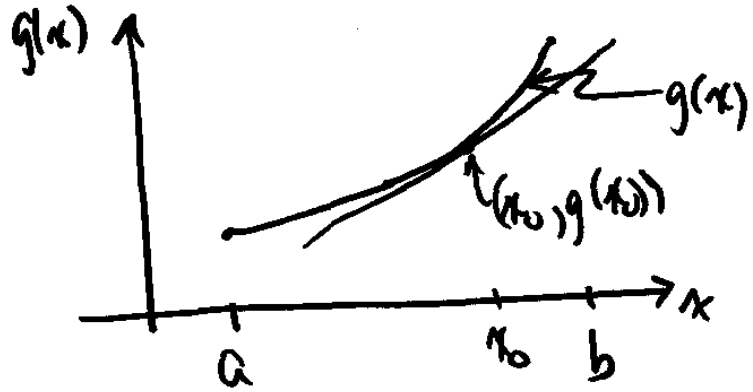


area $\geq 1 - \frac{1}{t^2}$

③

<u>Defn</u>: A function $g(x)$ defined on (a,b) is <u>convex</u> (strictly convex) if $\forall\ 0<\gamma<1$, $a<x_1<x_2<b$

$$g(\gamma x_1+(1-\gamma)x_2) \leq \gamma g(x_1)+(1-\gamma)g(x_2)$$
$$(<)$$



④

Find the equation of the tangent line at x_0

The slope will be $g'(x_0)$

$$y-g(x_0)=g'(x_0)(x-x_0)$$

$$\underline{y=g(x_0)+g'(x_0)(x-x_0)}$$

Let's show that the curve lies above the tangent line.

(5)

$$g(\gamma x_1 + (1-\gamma)x_0) \le \gamma g(x_1) + (1-\gamma)g(x_0)$$
$$\le \gamma\left[g(x_1) - g(x_0)\right] + g(x_0)$$

$$\frac{g(\gamma x_1 + (1-\gamma)x_0) - g(x_0)}{\gamma} \le g(x_1) - g(x_0)$$

$$(x_1 - x_0)\frac{g(\gamma x_1 + (1-\gamma)x_0) - g(x_0)}{\gamma(x_1 - x_0)} \le g(x_1) - g(x_0)$$

Let $h = \gamma(x_1 - x_0)$ (6)

$$(x_1 - x_0)\frac{g(h + x_0) - g(x_0)}{h} \le g(x_1) - g(x_0)$$

Take limit as $h \to 0$

$$(x_1 - x_0)\, g'(x_0) \le g(x_1) - g(x_0)$$
$$g(x_1) \ge g(x_0) + g'(x_0)(x_1 - x_0)$$

Consequence: Suppose $g(x)$ is convex on (a,b)

$$E[g(X)] = \int_a^b g(x) f_X(x)\, dx$$

$$\geq \int_a^b [g(x_0) + g'(x_0)(x-x_0)] f_X(x)\, dx$$

$$= \int_a^b g(x_0) f_X(x)\, dx + \int_a^b g'(x_0)(x-x_0) f_X(x)\, dx$$

$$= g(x_0) + g'(x_0)\big[E[X] - x_0\big]$$

$\forall\, x_0 \in (a,b)$

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In particular, this inequality holds for $x_0 = \mu$

$$\therefore \quad E[g(X)] \geq g(\mu) + 0$$

That is, $E[g(X)] \geq g(E[X])$

<u>Jensen's Inequality</u>

<u>Example</u>: $g(x) = x^2$ is convex

So $E[X^2] \geq (E[X])^2$

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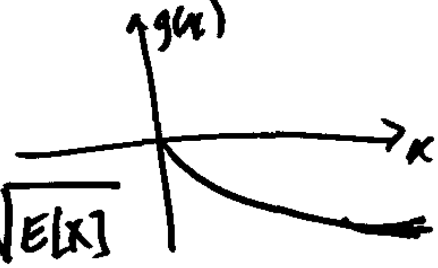
Example: $g(x) = -\sqrt{x}$ $\quad x > 0$

is convex



So $E[-\sqrt{X}] \geq -\sqrt{E[X]}$

$$E[\sqrt{X}] \leq \sqrt{E[X]}$$

Example: $g(x) = -\ln x$ $\quad x > 0$

is convex

$$E[-\ln X] \geq -\ln E[X]$$

$$E[\ln X] \leq \ln E[X]$$

(10)

Let X be a r.v. that takes on the values $a_1, a_2, \ldots, a_n$ with probability $\frac{1}{n}$ each.

$$E[X] = \sum_{i=1}^{n} x_i \, p(x_i) = \sum_{i=1}^{n} a_i \frac{1}{n} = \bar{a},$$

the arithmetic mean

$$E[\ln X] = \sum_{i=1}^{n} \ln(x_i) \, p(x_i) = \frac{1}{n} \sum_{i=1}^{n} \ln(a_i)$$

$$= \frac{1}{n} \ln\left(\prod_{i=1}^{n} a_i\right) = \ln \sqrt[n]{\prod_{i=1}^{n} a_i}$$

(11)

$$E[\ln X] \le \ln E[X]$$

$$\ln \sqrt[n]{\prod_{i=1}^{n} a_i} \le \ln(\overline{a})$$

$$\sqrt[n]{\prod_{i=1}^{n} a_i} \le \overline{a}$$

geometric mean ↑   ↑ arithmetic mean

(12)

<u>Defn</u>: The <u>discrete</u> <u>uniform</u> <u>distribution</u>

X takes on the values $\{1, 2, \dots, N\}$ with probabilities $\frac{1}{N}$ each.

$$E[X] = \sum_{i=1}^{N} i \cdot \frac{1}{N} = \frac{1}{N} \sum_{i=1}^{N} i = \frac{1}{N} \cdot \frac{N(N+1)}{2} = \frac{N+1}{2}$$

$$E[X^2] = \sum_{i=1}^{N} i^2 \cdot \frac{1}{N} = \frac{1}{N} \frac{N(N+1)(2N+1)}{6} = \frac{(N+1)(2N+1)}{6}$$

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$$V[X] = \frac{(N+1)(2N+1)}{6} - \left(\frac{N+1}{2}\right)^2$$
$$= \frac{N^2-1}{12}$$

$$M_X(t) = E[e^{tX}] = \sum_{i=1}^{N} e^{ti} \cdot \frac{1}{N}$$
$$= \frac{1}{N} \sum_{i=1}^{N} e^{ti}$$

Def^{n}: The <u>Bernoulli</u> <u>distribution</u>

x	$p(x)$
0	$1-p$
1	p

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$$E[X] = 0(1-p) + 1 \cdot p = p$$
$$E[X^2] = 0^2(1-p) + 1^2 \cdot p = p$$
$$V[X] = p - p^2 = p(1-p) = pq \quad (q = 1-p)$$

$$M_X(t) = E[e^{tX}] = e^{t \cdot 0} \cdot q + e^{t \cdot 1} \cdot p$$
$$= pe^t + q$$

(15)

Binomial experiment

- Sequence of n independent trials
- each trial results in one of 2 possible outcomes (0,1)
- the probability of a "1" is the same on each trial (p)
- X counts the number of 1s

(16)

Binomial distribution

x	$p(x)$
0	q^n
1	npq^{n-1}
2	$\binom{n}{2}p^2q^{n-2}$
$\vdots$	
n	p^n

$$p(x) = \binom{n}{x} p^x q^{n-x}$$

We know from past examples that $\mu = np$

$$\sigma^2 = npq$$

$$M_X(t) = (pe^t + q)^n$$

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Midterm 1 week from today : 10/30

- Given a density function (or prob. mass) with an unknown constant, find the constant

  Find the mean, variance, median, mode, moment-generating function, probabilities

- Given a transformation, find the new density

- Given a mgf, find μ, σ^2

- Chebyshev's & Jensen's inequalities in application

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- Using basic principles of counting, find some probabilities

- verify if something is a Borel field
- verify if " " " probability function

Bring 1 page of notes (front & back)

+ calculator