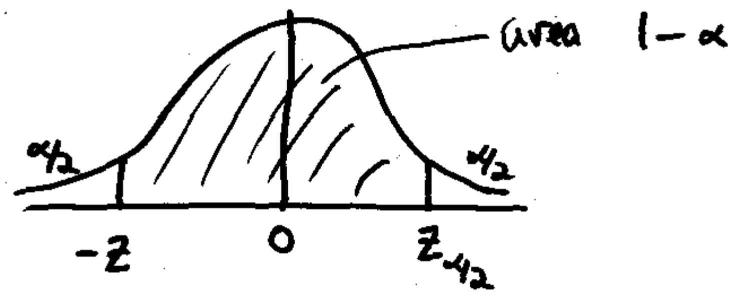
① Stat 452 4-6

From last time:

Confidence interval for μ was

$$\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}} \quad \text{for 95\% confidence}$$



$$z_{.025} = 1.96$$

In general, a $(1-\alpha)\times 100\%$ confidence interval for μ is

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

②

Consider the case where σ is unknown.

From Chapter 8:

$\dfrac{(n-1)s^2}{\sigma^2}$ is a random variable

having a chi-squared (χ^2) distribution with $n-1$ degrees of freedom, provided that the original population is normally distributed.

Also, s^2 and $\bar{x}$ are independent random variables.

And if Z has a standard normal distribution and

(3)

If W has a χ^2_{n-1} and if Z and W are independent,

then $\dfrac{Z}{\sqrt{\frac{W}{n-1}}}$ will have a Student's t distribution with $n-1$ df.

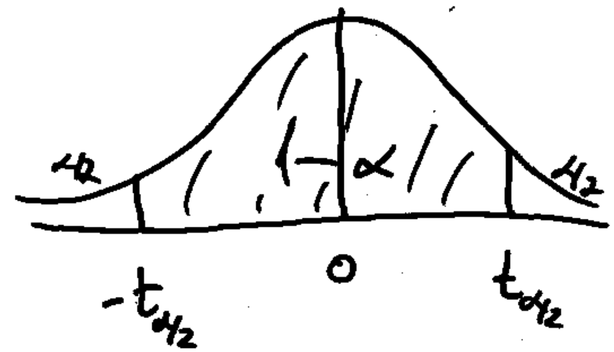
Let $Z = \dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

and $W = \dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

$$\frac{Z}{\sqrt{\frac{W}{n-1}}} = \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{S^2}{\sigma^2}}} = \frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$$

(4)

Following the same steps as before:



$$P\left(-t_{\alpha/2} < \frac{\bar{X}-\mu}{S/\sqrt{n}} < t_{\alpha/2}\right) = 1-\alpha$$

$\bar{X} - t_{\alpha/2}\dfrac{S}{\sqrt{n}} < \mu < \bar{X} + t_{\alpha/2}\dfrac{S}{\sqrt{n}}$ is

a $(1-\alpha)\times 100\%$ conf. int. for μ

(5)

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

$$\bar{x} \pm t_{\alpha/2} \frac{s}{\sqrt{n}}$$

Let e be the desired margin of error.

How large must n be?

Set $z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = e$

$$n = \left(\frac{z_{\alpha/2}\sigma}{e}\right)^2$$

(6)

Set $t_{\alpha/2} \frac{s}{\sqrt{n}} = e$

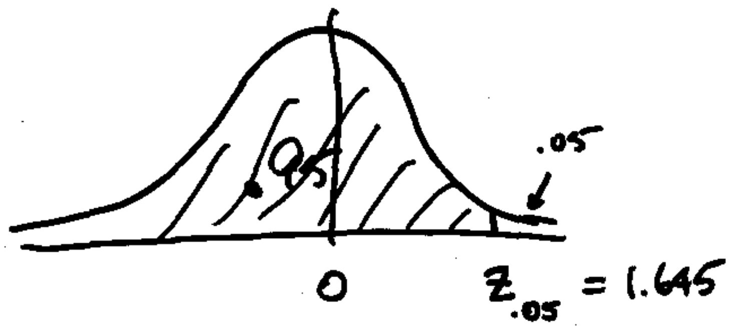
$$n = \left(\frac{t_{\alpha/2} s}{e}\right)^2$$

Because the final df are unknown, we use $z_{\alpha/2}$ instead of $t_{\alpha/2}$ in this formula

Result: $n = \left[\frac{z_{\alpha/2} \cdot (s \text{ or } \sigma)}{e}\right]^2$

and round up to the nearest integer.

One-sided confidence intervals



$$P\left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} < 1.645 \right) = .95$$

$\mu > \bar{x} - 1.645\frac{\sigma}{\sqrt{n}}$ with 95% confidence

$\bar{x} - z_\alpha \frac{\sigma}{\sqrt{n}}$ 1-sided lower C.I.

$\bar{x} + z_\alpha \frac{\sigma}{\sqrt{n}}$ 1-sided upper C.I.

Prediction interval

Observe n values $X_1, \ldots, X_n$

Compute $\bar{X}$. Assume σ known.

Predict a new value X and give its margin of error.

Let $W = X - \bar{X}$

Then $E(W) = E(X - \bar{X})$

$$= E(X) - E(\bar{X})$$

$$= \mu - \mu$$

$$= 0$$

And $V(W) = V(X - \bar{X})$

$$= V(X) + V(\bar{X})$$
$$= \sigma^2 + \frac{\sigma^2}{n}$$
$$= \sigma^2\left(1 + \frac{1}{n}\right)$$

Assume that the original population was normally distributed.

Then $W \sim N\left(0, \sigma_W^2 = \sigma^2\left(1+\frac{1}{n}\right)\right)$

$$\frac{W - 0}{\sigma\sqrt{1+\frac{1}{n}}} \sim N(0,1)$$

$$P\left(-z_{\alpha/2} < \frac{W}{\sigma\sqrt{1+\frac{1}{n}}} < z_{\alpha/2}\right) = 1-\alpha$$

$$-z_{\alpha/2} < \frac{X - \bar{X}}{\sigma\sqrt{1+\frac{1}{n}}} < z_{\alpha/2}$$

$$\bar{X} - z_{\alpha/2}\,\sigma\sqrt{1+\frac{1}{n}} < X < \bar{X} + z_{\alpha/2}\,\sigma\sqrt{1+\frac{1}{n}}$$

with $(1-\alpha)\times 100\%$ confidence

HW#2 due Tuesday April 13.

p.285 9.4
9.8
9.16
9.26