3-30

452

Stat 452/552

Chap 9 Sect 14

Maximum Likelihood Estimation

Example: Watch the customers entering a store for a 4-hour period.

X = # customers

Poisson distribution

$$p(x) = \frac{\mu^x e^{-\mu}}{x!} \quad x = 0, 1, 2, 3, \ldots$$

Observe the random variable X, n times.

Collect the values

$$X_1, X_2, \ldots, X_n$$

Based on these observations, estimate μ.

① Write the joint probability distribution of the random variables $X_1, \ldots, X_n$

$$p(x_1, x_2, \ldots, x_n) = p(x_1) \cdot p(x_2) \cdots p(x_n)$$

(assumes independence)

In our example,

$$p(x_1, \ldots, x_n) = \frac{\mu^{x_1} e^{-\mu}}{x_1!} \cdot \frac{\mu^{x_2} e^{-\mu}}{x_2!} \cdots \frac{\mu^{x_n} e^{-\mu}}{x_n!}$$

$$= e^{-n\mu} \mu^{\Sigma x_i} \Big/ \prod_{i=1}^{n} x_i!$$

(3)

(2) Write this same function, calling it the likelihood function of the unknown parameter(s)

$$L(\mu) = \frac{e^{-n\mu}\mu^{\Sigma x_i}}{\prod x_i!}$$

(3) Maximize $L(\mu)$ with respect to μ.

First, take the natural log of $L(\mu)$

$$\ell(\mu) = \ln L(\mu)$$

$$= -n\mu + \Sigma x_i \ln\mu$$

$$- \ln\left(\prod_{i=1}^{n}(x_i)!\right)$$

(4)

$$\ell'(\mu) = -n + \frac{\Sigma x_i}{\mu} + 0$$

$$\overset{set}{=} 0 \quad \text{+ solve for } \mu$$

$$-n + \frac{\Sigma x_i}{\mu} = 0$$

$$\frac{\Sigma x_i}{\mu} = n$$

$$\hat{\mu} = \frac{\Sigma x_i}{n} = \bar{x}$$

Result: for the Poisson distribution, the maximum likelihood estimator of μ is $\bar{x}$

Another example: (5)

Assume that the waiting time T for a particular event follows an exponential distribution

$$f(t) = \lambda e^{-\lambda t}$$

Observe T a total of n times.

Estimate λ

① $f(t_1, \ldots, t_n)$

$$= \lambda e^{-\lambda t_1} \cdot \lambda e^{-\lambda t_2} \cdots \lambda e^{-\lambda t_n}$$

$$= \lambda^n e^{-\lambda \Sigma t_i}$$

(6)

② $L(\lambda) = \lambda^n e^{-\lambda \Sigma t_i}$

③ $\ell(\lambda) = \ln L(\lambda)$

$$= n \ln \lambda - \lambda \Sigma t_i$$

$$\ell'(\lambda) = \frac{n}{\lambda} - \Sigma t_i \overset{\text{set}}{=} 0$$

$$\frac{n}{\lambda} = \Sigma t_i$$

$$\hat{\lambda} = \frac{n}{\Sigma t_i} = \frac{1}{\bar{t}}$$

The maximum likelihood estimator of λ for the exponential distribution is $\frac{1}{\bar{t}}$

(7)

A third example:

The number of successes in n independent trials follows a binomial distribution with parameters n and p.

Run the experiment 1 time (n trials) and estimate p.

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$L(p) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$\ell(p) = \ln\binom{n}{x} + x \ln p + (n-x)\ln(1-p)$$

(8)

$$\ell'(p) = \frac{x}{p} + \frac{(n-x)}{1-p}(-1) \overset{set}{=} 0$$

$$\frac{x}{p} = \frac{n-x}{1-p}$$

$$x(1-p) = (n-x)p$$

$$x - \cancel{xp} = np - \cancel{xp}$$

$$\hat{p} = \frac{x}{n}$$

A fourth example

Suppose that the weight (X) of an item follows a normal distribution with mean μ and st.dev. σ.

(9)

Observe $X_1, \dots, X_n$

Estimate μ and σ.

$$f(x_1,\dots,x_n) =$$

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_1-\mu}{\sigma}\right)^2} \cdots\cdots \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_n-\mu}{\sigma}\right)^2}$$

$$= \sigma^{-n}(2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2} = L(\mu,\sigma)$$

$$\ell(\mu,\sigma) = \ln L(\mu,\sigma)$$

$$= -n\ln\sigma - \frac{n}{2}\ln(2\pi) - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2$$

$$\frac{\partial \ell}{\partial \mu} = -\frac{1}{2\sigma^2}\sum 2(x_i-\mu)(-1)$$

(10)

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) \overset{set}{=} 0$$

$$\sum_{i=1}^{n} x_i - n\mu = 0$$

$$\hat{\mu} = \frac{\sum x_i}{n} = \bar{x}$$

$$\frac{\partial \ell}{\partial \sigma} = -\frac{n}{\sigma} - \frac{1}{2}\sum(x_i-\mu)^2(-2)\sigma^{-3} \overset{set}{=} 0$$

$$-\frac{n}{\sigma} + \frac{\sum(x_i-\mu)^2}{\sigma^3} = 0$$

$$-n\sigma^2 + \sum(x_i-\mu)^2 = 0$$

$$\sigma^2 = \frac{\sum(x_i-\mu)^2}{n}$$

$$\hat{\sigma} = \sqrt{\frac{\sum(x_i-\bar{x})^2}{n}}$$

11

HW #1 p.315

9.84

9.86

due next Tuesday, April 6