Stat 566

4-17-24

(1)

2-Way ANOVA, both random effects

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}$$

$E(\tau_i) \equiv 0 \equiv E(\beta_j) \equiv E(\tau\beta_{ij})$
$\equiv E(\varepsilon_{ijk})$

$i = 1, \ldots, a$
$j = 1, \ldots, b$
$k = 1, \ldots, n$
$N = abn$

(2)

$V(\tau_i) \equiv \sigma^2_\tau$, $V(\beta_j) \equiv \sigma^2_\beta$,
$V(\tau\beta_{ij}) \equiv \sigma^2_{\tau\beta}$, $V(\varepsilon_{ijk}) \equiv \sigma^2$,

all τ_i, β_j, $(\tau\beta)_{ij}$, ε_{ijk} are indep

As before, $MSA = \dfrac{nb \sum_{i=1}^{a} (\bar{y}_{i..} - \bar{y}_{...})^2}{a-1}$ ③

Find $E[MSA]$.

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}$$

$$\bar{y}_{i..} = \mu + \tau_i + \bar{\beta}_. + \overline{(\tau\beta)}_{i.} + \bar{\varepsilon}_{i..}$$

$$\bar{y}_{...} = \mu + \bar{\tau}_. + \bar{\beta}_. + \overline{(\tau\beta)}_{..} + \bar{\varepsilon}_{...}$$

$$\bar{y}_{i..} - \bar{y}_{...} = (\tau_i - \bar{\tau}_.) + \left[\overline{(\tau\beta)}_{i.} - \overline{(\tau\beta)}_{..}\right] + (\bar{\varepsilon}_{i..} - \bar{\varepsilon}_{...})$$

④

$$E(\bar{y}_{i..} - \bar{y}_{...})^2 = E(\tau_i - \bar{\tau}_.)^2 + E\left[\overline{(\tau\beta)}_{i.} - \overline{(\tau\beta)}_{..}\right]^2 + E(\bar{\varepsilon}_{i..} - \bar{\varepsilon}_{...})^2 + 0 + 0 + 0$$

$$E(MSA) = nb\left[E\left(\frac{\sum_{i=1}^{a}(\tau_i - \bar{\tau}_.)^2}{a-1}\right) + E\left(\frac{\sum_{i=1}^{a}\left(\overline{(\tau\beta)}_{i.} - \overline{(\tau\beta)}_{..}\right)^2}{a-1}\right) + E\left(\frac{\sum_{i=1}^{a}(\bar{\varepsilon}_{i..} - \bar{\varepsilon}_{...})^2}{a-1}\right)\right]$$

$$= nb\left[\sigma_\tau^2 + \frac{\sigma_{\tau\beta}^2}{b} + \frac{\sigma^2}{bn}\right]$$

$$= nb\sigma_\tau^2 + n\sigma_{\tau\beta}^2 + \sigma^2$$

(5)

Similarly, $E(MSB) = na\sigma_\beta^2 + n\sigma_{\tau\beta}^2 + \sigma^2$

and $E(MS_{AB}) = n\sigma_{\tau\beta}^2 + \sigma^2$

To test $H_0: \sigma_\tau^2 = 0$, $F = \dfrac{MSA}{MS_{AB}}$

$H_0: \sigma_\beta^2 = 0$, $F = \dfrac{MSB}{MS_{AB}}$

$H_0: \sigma_{\tau\beta}^2 = 0$, $F = \dfrac{MS_{AB}}{MSE}$

(6)

Mixed model: A is fixed, B is random

$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}$ $i = 1, \ldots, a$; $j = 1, \ldots, b$; $k = 1, \ldots, n$

$\sum_{i=1}^{a} \tau_i = 0$, $E(\beta_j) \equiv 0$, $V(\beta_j) \equiv \sigma_\beta^2$ $N = nab$

$E(\tau\beta_{ij}) \equiv 0$, $V(\tau\beta_{ij}) \equiv \sigma_{\tau\beta}^2$

$E(\varepsilon_{ijk}) \equiv 0$, $V(\varepsilon_{ijk}) \equiv \sigma^2$

And $\sum_{i=1}^{a} (\tau\beta)_{ij} = 0 \quad \forall j$ } This added assumption gives the <u>restricted model</u>

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$$E(MSA) = nb\,\frac{\sum_{i=1}^{a}\tau_i^2}{a-1} + n\sigma^2_{\tau\beta} + \sigma^2$$

$$E(MSB) = na\left[E\left[\frac{\sum(\beta_j-\bar{\beta}_.)^2}{b-1}\right] + E\left[\frac{\sum(\overset{0}{\cancel{\overline{\tau\beta}_{.j}}} - \overset{0}{\cancel{\overline{\tau\beta}_{..}}})^2}{b-1}\right] + E\left[\frac{\sum(\bar{\varepsilon}_{.j.}-\bar{\varepsilon}_{...})^2}{b-1}\right]\right]$$

$$= na\left[\sigma^2_\beta + 0 + \frac{\sigma^2}{na}\right]$$

$$= na\,\sigma^2_\beta + \sigma^2$$

$E(MS_{AB}) = n\sigma^2_{\tau\beta} + \sigma^2$ as before

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$H_0: \tau_i = 0 \;\forall i \qquad F = \dfrac{MSA}{MSAB}$

$H_0: \sigma^2_\beta = 0 \qquad F = \dfrac{MSB}{MS_E}$

$H_0: \sigma^2_{\tau\beta} = 0 \qquad F = \dfrac{MS_{AB}}{MS_E}$

Rules for finding
expected mean squares (restricted model version)

(1) Write $\varepsilon_{ijkl} = \varepsilon_{(ijk)l}$

(2) If a term has a subscript in parentheses then there will be no interactions between the factors represented by indices in ( ) and those outside.

Example: $\beta_{j(i)} \Rightarrow$ no AB interaction

(3) Subscripts outside the ( ) are <u>live</u>
" inside the ( ) are <u>dead</u>
" not present in the term are <u>absent</u>

(4) For a random effect, the variance component is of the form σ_τ

For a fixed effect, the variance component is of the form $\dfrac{\sum_{i=1}^{a} \tau_i^2}{a-1}$

EMS table (Example) (11)

Factor	F i	R j	R k	EMS
τ_i				
β_j				
$\tau\beta_{ij}$				
$\varepsilon_{(ij)k}$				

5 (a) Enter a "1" in the table for each dead subscript in that term

(b) Enter a "0" for a live subscript in an F column
& a "1" " " " " " R "

(12)

(c) For absent subscripts, enter the number of levels of that factor

Complete parts (a), (b), (c) for last example (A fixed, B random)

Factor	F i	R j	R k	EMS	F
τ_i	0	b	n	$bn \frac{\sum \tau_i^2}{a-1} + n\sigma^2_{\tau\beta} + \sigma^2$	MSA/MSAB
β_j	a	1	n	$an\,\sigma^2_\beta + \sigma^2$	MSB/MSE
$\tau\beta_{ij}$	0	1	n	$n\,\sigma^2_{\tau\beta} + \sigma^2$	MSAB/MSE
$\varepsilon_{(ij)k}$	1	1	1	σ^2	

(5) In each row, cover all of the columns corresponding to the live subscripts for that term. Then the EMS will contain terms corresponding to each row with <u>at least</u> the same subscripts, and the coefficient of the variance is the product of the visible numbers.

(6) For each H_0, find a term whose EMS matches the EMS of the numerator, under H_0. Use that term as the denominator

4.23. An industrial engineer is investigating the effect of four assembly methods (*A, B, C, D*) on the assembly time for a color television component. Four operators are selected for the study. Furthermore, the engineer knows that each assembly method produces such fatigue that the time required for the last assembly may be greater than the time required for the first, regardless of the method. That is, a trend develops in the required assembly time. To account for this source of variability, the engineer uses the Latin square design shown below. Analyze the data from this experiment ($\alpha = 0.05$) and draw appropriate conclusions.

Order of Assembly	Operator 1	2	3	4
1	$C = 10$	$D = 14$	$A = 7$	$B = 8$
2	$B = 7$	$C = 18$	$D = 11$	$A = 8$
3	$A = 5$	$B = 10$	$C = 11$	$D = 9$
4	$D = 10$	$A = 10$	$B = 12$	$C = 14$

4.35. The yield of a chemical process was measured using five batches of raw material, five acid concentrations, five standing times (A, B, C, D, E), and five catalyst concentrations (α, β, γ, δ, ϵ). The Graeco-Latin square that follows was used. Analyze the data from this experiment (use $\alpha = 0.05$) and draw conclusions.

	Acid Concentration		
Batch	**1**	**2**	**3**
1	$A\alpha = 26$	$B\beta = 16$	$C\gamma = 19$
2	$B\gamma = 18$	$C\delta = 21$	$D\epsilon = 18$
3	$C\epsilon = 20$	$D\alpha = 12$	$E\beta = 16$
4	$D\beta = 15$	$E\gamma = 15$	$A\delta = 22$
5	$E\delta = 10$	$A\epsilon = 24$	$B\alpha = 17$

	Acid Concentration	
Batch	**4**	**5**
1	$D\delta = 16$	$E\epsilon = 13$
2	$E\alpha = 11$	$A\beta = 21$
3	$A\gamma = 25$	$B\delta = 13$
4	$B\epsilon = 14$	$C\alpha = 17$
5	$C\beta = 17$	$D\gamma = 14$