Chapter 13

Random effects

$$y_{ij} = \mu + \tau_i + \varepsilon_{ij} \qquad i = 1,\dots,a \quad j = 1,\dots,n$$

$E[\varepsilon_{ij}] \equiv 0$ $\quad$ τ_i's are random variables

$V[\varepsilon_{ij}] = \sigma^2$ $\quad$ $E[\tau_i] \equiv 0$ $\quad$ All of the ε_{ij}'s and

$V[\tau_i] \equiv \sigma^2_\tau$ $\quad$ τ_i's are independent

$H_0: \sigma^2_\tau = 0$

$H_1: \sigma^2_\tau > 0$

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Decomposition of sum of squares

$$\underbrace{\sum_{i=1}^{a}\sum_{j=1}^{n}(y_{ij}-\bar{y}_{..})^2}_{SS_{TOT}} = \underbrace{n\sum_{i=1}^{a}(\bar{y}_{i.}-\bar{y}_{..})^2}_{SS_{TRT}} + \underbrace{\sum_i\sum_j (y_{ij}-\bar{y}_{i.})^2}_{SS_E}$$

Find $E[SS_{TRT}] = n\,E\left[\sum_{i=1}^{a}(\bar{y}_{i.}-\bar{y}_{..})^2\right]$

$$= n\,E\left[\sum_{i=1}^{a}\bar{y}_{i.}^2 - 2\bar{y}_{..}\underbrace{\sum_{i=1}^{a}\bar{y}_{i.}}_{a\bar{y}_{..}} + a\bar{y}_{..}^2\right]$$

$$= n\,E\left[\sum_{i=1}^{a}\bar{y}_{i.}^2 - a\bar{y}_{..}^2\right]$$

Let $E[\bar{y}_{i.}^2] = C_i$

Let $E[\bar{y}_{..}^2] = D$

③

C_i: $y_{ij} = \mu + \tau_i + \varepsilon_{ij}$

$$\bar{y}_{i\cdot} = \mu + \tau_i + \bar{\varepsilon}_{i\cdot}$$

$$\bar{y}_{i\cdot}^2 = \mu^2 + \tau_i^2 + \bar{\varepsilon}_{i\cdot}^2 + 2\mu\tau_i + 2\mu\bar{\varepsilon}_{i\cdot} + 2\tau_i\bar{\varepsilon}_{i\cdot}$$

$$C_i = E[\bar{y}_{i\cdot}^2] = \mu^2 + \underbrace{E[\tau_i^2]}_{\substack{= V[\tau_i] + (E[\tau_i])^2 \\ \sigma_\tau^2 + 0}} + \underbrace{E[\bar{\varepsilon}_{i\cdot}^2]}_{\substack{V[\bar{\varepsilon}_{i\cdot}] + (E[\bar{\varepsilon}_{i\cdot}])^2 \\ \frac{\sigma^2}{n} + 0}} + 0 + 0 + 0$$

$$C_i = \mu^2 + \sigma_\tau^2 + \frac{\sigma^2}{n}$$

④

D: $y_{ij} = \mu + \tau_i + \varepsilon_{ij}$

$$\bar{y}_{\cdot\cdot} = \mu + \bar{\tau}_{\cdot} + \bar{\varepsilon}_{\cdot\cdot}$$

$$\bar{y}_{\cdot\cdot}^2 = \mu^2 + \bar{\tau}_{\cdot}^2 + \bar{\varepsilon}_{\cdot\cdot}^2 + 2\mu\bar{\tau}_{\cdot} + 2\mu\bar{\varepsilon}_{\cdot\cdot} + 2\bar{\tau}_{\cdot}\bar{\varepsilon}_{\cdot\cdot}$$

$$D = E[\bar{y}_{\cdot\cdot}^2] = \mu^2 + E[\bar{\tau}_{\cdot}^2] + E[\bar{\varepsilon}_{\cdot\cdot}^2] + 0 + 0 + 0$$

$$= \mu^2 + \frac{\sigma_\tau^2}{a} + \frac{\sigma^2}{an}$$

Now $E[SS_{TRT}] = n\left[\sum_{i=1}^{a} C_i - aD\right]$

$$= n\left[a\left(\mu^2 + \sigma_\tau^2 + \frac{\sigma^2}{n}\right) - a\left(\mu^2 + \frac{\sigma_\tau^2}{a} + \frac{\sigma^2}{an}\right)\right]$$

⑤

$$= n\left[(a-1)\sigma_\tau^2 + \sigma^2\left(\frac{a}{n} - \frac{1}{n}\right)\right]$$

$$= (a-1)\left[n\sigma_\tau^2 + \sigma^2\right] = E\left[SS_{TRT}\right]$$

Define $MS_{TRT} = \dfrac{SS_{TRT}}{a-1}$

Then $E[MS_{TRT}] = n\sigma_\tau^2 + \sigma^2$

Under $H_0: \sigma_\tau^2 = 0$, $E[MS_{TRT}] = \sigma^2$

Recall: In the fixed effects model,

$$E[MS_{TRT}] = n\,\frac{\sum_{i=1}^{a}\tau_i^2}{a-1} + \sigma^2$$

⑥

Facts (require normality assumption on all τ_i's and ε_{ij}'s)

1. $\dfrac{SSE}{\sigma^2} \sim \chi^2_{N-a}$ $(N = na)$

2. $\dfrac{SS_{TRT}}{\sigma^2} \sim \chi^2_{a-1}$

3. SS_E and SS_{TRT} are independent

$\Rightarrow$ Under H_0, $$\frac{\frac{SS_{TRT}}{\sigma^2}/(a-1)}{\frac{SSE}{\sigma^2}/(N-a)} = \frac{MS_{TRT}}{MS_E} \sim F_{a-1,N-a}$$

⑦

So there is no difference in the F test for a 1-way ANOVA with fixed vs. random effects.

Suppose H_0 is rejected and you wish to estimate σ^2_τ

$$E[MS_{TRT}] = n\sigma^2_\tau + \sigma^2$$

$$E[MS_E] = \sigma^2$$

$$E\left[\frac{MS_{TRT} - MS_E}{n}\right] = \sigma^2_\tau$$

So an unbiased estimator of σ^2_τ is $\hat{\sigma}^2_\tau = \dfrac{MS_{TRT} - MS_E}{n}$

⑧

$\hat{\sigma}^2_\tau$ doesn't have a known distribution.

Consider $\dfrac{\sigma^2_\tau}{\sigma^2 + \sigma^2_\tau}$

$$y_{ij} = \mu + \tau_i + \varepsilon_{ij}$$

$$V(y_{ij}) = \sigma^2_\tau + \sigma^2$$

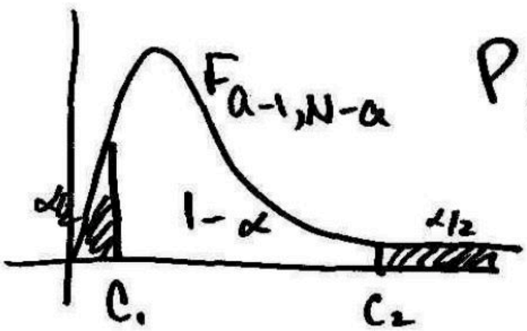
This is the proportion of the variance of y_{ij} that is explained by the factor.

We know: Under H_0, $\frac{SS_{TRT}}{\sigma^2} \sim \chi^2_{a-1}$

Under either H_0 or H_1, $\frac{SS_E}{\sigma^2} \sim \chi^2_{N-a}$

Fact: Under H_1, $\dfrac{SS_{TRT}}{\sigma^2 + n\sigma_\tau^2} \sim \chi^2_{a-1}$

$$\frac{\dfrac{SS_{TRT}}{\sigma^2 + n\sigma_\tau^2} \Big/ (a-1)}{\dfrac{SSE}{\sigma^2} \Big/ (N-a)} = \frac{MS_{TRT}}{MS_E} \frac{\sigma^2}{\sigma^2 + n\sigma_\tau^2} \sim F_{a-1, N-a}$$



$$P\left(C_1 < \frac{MS_{TRT}}{MSE} \frac{\sigma^2}{\sigma^2 + n\sigma_\tau^2} < C_2\right) = 1 - \alpha$$

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$$\underbrace{C_1 \frac{MSE}{MS_{TRT}}}_{b_1} < \frac{\sigma^2}{\sigma^2 + n\sigma_\tau^2} < \underbrace{C_2 \frac{MSE}{MS_{TRT}}}_{b_2}$$ with $1-\alpha$ confidence

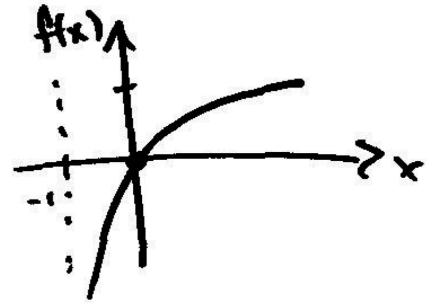
$$\frac{1}{b_1} > \frac{\sigma^2 + n\sigma_\tau^2}{\sigma^2} > \frac{1}{b_2}$$

$$\underbrace{\frac{\frac{1}{b_1} - 1}{n}}_{d_1} > \frac{\sigma_\tau^2}{\sigma^2} > \underbrace{\frac{\frac{1}{b_2} - 1}{n}}_{d_2}$$

$$d_1 > \frac{\sigma_z^2}{\sigma^2} > d_2$$

$$f(d_1) > f\left(\frac{\sigma_z^2}{\sigma^2}\right) > f(d_2) \quad \text{if } f \text{ is } \uparrow$$

Consider $f(x) = \dfrac{x}{1+x}$



$$\frac{d_1}{1+d_1} > \frac{\sigma_z^2/\sigma^2}{1+\frac{\sigma_z^2}{\sigma^2}} > \frac{d_2}{1+d_2}$$

$$\frac{d_1}{1+d_1} > \frac{\sigma_z^2}{\sigma^2+\sigma_z^2} > \frac{d_2}{1+d_2}$$