Latin Square design

Stat 566

4-10-24

(1)

		column		
		1	2	3
row	1	A	B	C
	2	B	C	A
	3	C	A	B

Every row has all letters

" column " " "

Model: $y_{ijk} = \mu + \underbrace{\tau_i}_{\text{treatment}} + \underbrace{\alpha_j + \beta_k}_{\text{blocking factors}} + \varepsilon_{ijk}$

$i, j, k = 1, \dots, p$

(2)

Matrix version $(p = 3)$

$$\begin{bmatrix} y_{111} \\ y_{123} \\ y_{132} \\ y_{212} \\ y_{221} \\ y_{233} \\ y_{313} \\ y_{322} \\ y_{331} \end{bmatrix}_{9\times 1} = \left[\begin{array}{c|cc|cc|cc} 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & -1 & -1 \\ 1 & 1 & 0 & -1 & -1 & 0 & 1 \\ \hline 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 & -1 & -1 & -1 \\ \hline 1 & -1 & -1 & 1 & 0 & -1 & -1 \\ 1 & -1 & -1 & 0 & 1 & 0 & 1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 0 \end{array}\right]_{9\times 7} \begin{bmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{bmatrix}_{7\times 1} + \vec{\varepsilon}$$

Notice that orthogonality was maintained

③

We find the parameter estimates using least squares

$$\hat{\mu} = \bar{y}_{\cdots}, \quad \hat{\tau}_i = \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots}, \quad \hat{\alpha}_j = \bar{y}_{\cdot j \cdot} - \bar{y}_{\cdots},$$

$$\hat{\beta}_k = \bar{y}_{\cdot\cdot k} - \bar{y}_{\cdots}$$

Then $\hat{y}_{ijk} = \bar{y}_{\cdots} + (\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots}) + (\bar{y}_{\cdot j\cdot} - \bar{y}_{\cdots}) + (\bar{y}_{\cdot\cdot k} - \bar{y}_{\cdots})$

$$= \bar{y}_{i\cdot\cdot} + \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdot\cdot k} - 2\bar{y}_{\cdots}$$

Decompose $SS_{TOT} = \sum_i \sum_j \sum_k \left(y_{ijk} - \hat{y}_{ijk} + \hat{y}_{ijk} - \bar{y}_{\cdots} \right)^2$

④

$$SS_{TOT} = SS_E + SS_{TRT} + SS_{ROW} + SS_{COL}$$

$$SS_{TRT} = \sum_{i=1}^{p} \frac{y_{i\cdot\cdot}^2}{p} - \frac{y_{\cdots}^2}{N}$$

$$SS_{ROW} = \sum_{j=1}^{p} \frac{y_{\cdot j\cdot}^2}{p} - \frac{y_{\cdots}^2}{N}$$

$$SS_{COL} = \sum_{k=1}^{p} \frac{y_{\cdot\cdot k}^2}{p} - \frac{y_{\cdots}^2}{N}$$

SS_E is found by subtraction

5

ANOVA table

Source	SS	df	MS	F
TRT	SS_{TRT}	$p-1$		MS_{TRT}/MS_E
ROW	SS_{ROW}	$p-1$	$\frac{SS}{df}$	
COL	SS_{COL}	$p-1$		
ERR	SS_E	$(p-1)(p-2)$		
TOT	SS_T	$N-1$		

$$df_E = p^2 - 1 - 3(p-1) = p^2 - 3p + 2$$
$$= (p-1)(p-2)$$

6

Replication in the Latin Square Design

Case 1: Use the original design, but collect n observations in each cell, where $n<p$

Source	df
TRT	$p-1$
ROW	$p-1$
COL	$p-1$
REP	$n-1$
ERR	$np^2-3p-n+3$
TOT	$N-1$

$$df_E = np^2 - 1 - 3(p-1) - (n-1)$$

⑦

Case 2: Use same batches of material (row) but a new set of p operators (col) on each replication

Source	df
TRT	$p-1$
ROW	$p-1$
COL	$n(p-1)$
REP	$n-1$
ERR	$(p-1)(np-2)$
TOT	$N-1 = np^2-1$

$np^2-1-2(p-1)-n(p-1)-(n-1)$

⑧

Case 3: Use new batches & new operators on each replication

Source	df
TRT	$p-1$
ROW	$n(p-1)$
COL	$n(p-1)$
REP	$n-1$
ERR	$(p-1)[n(p-1)-1]$ by subtraction
TOT	$N-1 = np^2-1$

Graeco-Latin Square Design

2 treatment factors + 2 blocking factors

Each of the 4 factors has p levels

$p=4$ example

row \ col	1	2	3	4
1	$A\alpha$	$B\beta$	$C\gamma$	$D\delta$
2	$B\delta$	$A\gamma$	$D\beta$	$C\alpha$
3	$C\beta$	$D\alpha$	$A\delta$	$B\gamma$
4	$D\gamma$	$C\delta$	$B\alpha$	$A\beta$

Necessary Conditions:

- The Roman letters must form a Latin square
- The Greek letters must form a " "
- Each Roman/Greek letter combination appears exactly once

Model: $y_{ijkl} = \mu + \overset{\text{1st trt}}{\tau_i} + \overset{\text{2nd trt}}{\gamma_j} + \overset{\text{row}}{\alpha_k} + \overset{\text{col}}{\beta_l} + \varepsilon_{ijkl}$

Note: orthogonality has to be preserved

Parameter estimates & SS decomp follow the same pattern as in the Latin Square

Source	df
TRT1	$p-1$
TRT2	$p-1$
BLK1	$p-1$
BLK2	$p-1$
ERR	$(p-1)(p-3)$
TOT	$N-1 = p^2-1$

$$df_E = p^2 - 1 - 4(p-1)$$
$$= p^2 - 4p + 3$$
$$= (p-1)(p-3)$$

5.31. An article in *Quality Progress* (May 2011, pp. 42–48) describes the use of factorial experiments to improve a silver powder production process. This product is used in conductive pastes to manufacture a wide variety of products ranging from silicon wafers to elastic membrane switches. Powder density (g/cm^2) and surface area (cm^2/g) are the two critical characteristics of this product. The experiments involved three factors—reaction temperature, ammonium percent, and stirring rate. Each of these factors had two levels and the design was replicated twice. The design is shown below.

(**a**) Analyze the density response. Are any interactions significant? Draw appropriate conclusions about the effects of the significant factors on the response.

Ammonium (%)	Stir Rate (RPM)	Temperature (°C)	Density	Surface Area
2	100	8	14.68	0.40
2	100	8	15.18	0.43
30	100	8	15.12	0.42
30	100	8	17.48	0.41
2	150	8	7.54	0.69
2	150	8	6.66	0.67
30	150	8	12.46	0.52
30	150	8	12.62	0.36
2	100	40	10.95	0.58
2	100	40	17.68	0.43
30	100	40	12.65	0.57
30	100	40	15.96	0.54
2	150	40	8.03	0.68
2	150	40	8.84	0.75
30	150	40	14.96	0.41
30	150	40	14.96	0.41

6.21. I am always interested in improving my golf scores. Since a typical golfer uses the putter for about 35–45 percent of his or her strokes, it seems reasonable that improving one's putting is a logical and perhaps simple way to improve a golf score ("The man who can putt is a match for any man."—Willie Parks, 1864–1925, two time winner of the British Open). An experiment was conducted to study the effects of four factors on putting accuracy. The design factors are length of putt, type of putter, breaking putt versus straight putt, and level versus downhill putt. The response variable is distance from the ball to the center of the cup after the ball comes to rest. One golfer performs the experiment, a 2^4 factorial design with seven replicates was used, and all putts are made in random order. The results are shown in Table P6.4.

(a) Analyze the data from this experiment. Which factors significantly affect putting performance?

■ **TABLE P6.4**
The Putting Experiment from Problem 6.21

Design Factors				Distance from Cup (replicates)						
Length of putt (ft)	**Type of putter**	**Break of putt**	**Slope of putt**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
10	Mallet	Straight	Level	10.0	18.0	14.0	12.5	19.0	16.0	18.5
30	Mallet	Straight	Level	0.0	16.5	4.5	17.5	20.5	17.5	33.0
10	Cavity back	Straight	Level	4.0	6.0	1.0	14.5	12.0	14.0	5.0
30	Cavity back	Straight	Level	0.0	10.0	34.0	11.0	25.5	21.5	0.0
10	Mallet	Breaking	Level	0.0	0.0	18.5	19.5	16.0	15.0	11.0
30	Mallet	Breaking	Level	5.0	20.5	18.0	20.0	29.5	19.0	10.0
10	Cavity back	Breaking	Level	6.5	18.5	7.5	6.0	0.0	10.0	0.0
30	Cavity back	Breaking	Level	16.5	4.5	0.0	23.5	8.0	8.0	8.0
10	Mallet	Straight	Downhill	4.5	18.0	14.5	10.0	0.0	17.5	6.0
30	Mallet	Straight	Downhill	19.5	18.0	16.0	5.5	10.0	7.0	36.0
10	Cavity back	Straight	Downhill	15.0	16.0	8.5	0.0	0.5	9.0	3.0
30	Cavity back	Straight	Downhill	41.5	39.0	6.5	3.5	7.0	8.5	36.0
10	Mallet	Breaking	Downhill	8.0	4.5	6.5	10.0	13.0	41.0	14.0
30	Mallet	Breaking	Downhill	21.5	10.5	6.5	0.0	15.5	24.0	16.0
10	Cavity back	Breaking	Downhill	0.0	0.0	0.0	4.5	1.0	4.0	6.5
30	Cavity back	Breaking	Downhill	18.0	5.0	7.0	10.0	32.5	18.5	8.0

7.21. Consider the 2^6 design in eight blocks of eight runs each with *ABCD, ACE*, and *ABEF* as the independent effects chosen to be confounded with blocks. Generate the design. Find the other effects confounded with blocks.