Stat 566
4-1-24

Balanced Incomplete Block Design (BIBD)

Start with the RCBD, but purposely allow some missing cells. (1)

1 treatment factor (a levels)
1 blocking factor (b levels)
} If $a=b$, the BIBD is symmetric

But $N \neq ab$

- In each block, k levels of the treatment will be tested ($k < a$)

- Each level of the treatment factor appears in r blocks ($r < b$)

- Each pair of treatment levels will appear in λ blocks. (2)

Example: 5 fertilizers ($a = 5$)
10 fields ($b = 10$)

A RCBD would require 50 observations.

Suppose that field can only accommodate 3 fertilizers

$K = 3$

$N = bK = ar$
$10 \cdot 3 = 5 \cdot r \qquad \therefore r = 6$

③

- A particular level of the treatment appears in r blocks
  - In each block that our particular treatment level appears, k levels are being tested, so our particular treatment level appears with $k-1$ other levels.
  - So our particular treatment level appears in the same block with $r(k-1)$ other treatment levels
- There are $a-1$ other treatment levels

④

$$\therefore \quad \lambda = \frac{r(k-1)}{a-1}$$

$$\lambda = \frac{6(3-1)}{5-1} = 3$$

Note that r, k, and λ must be integers

each pair of treatment levels will appear in 3 blocks

Model $\quad y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij}$

$$i = 1, \ldots, a \qquad j = 1, \ldots, b$$

$$N \neq ab \qquad N = ar = bk$$

(5)

$$\sum_{i=1}^{a} \tau_i = 0 = \sum_{j=1}^{b} \beta_j \qquad E[\varepsilon_{ij}] \equiv 0$$
$$V[\varepsilon_{ij}] \equiv \sigma^2$$

$$SSE = \sum_i \sum_j \left(y_{ij} - \mu - \tau_i - \beta_j\right)^2$$

$$\frac{\partial SSE}{\partial \mu} = \sum_i \sum_j \cancel{2}\left(y_{ij} - \mu - \tau_i - \beta_j\right)(-1) \overset{\text{set}}{=} 0$$

$$y_{..} - N\mu - r\underbrace{\sum_i \tau_i}_{0} - 0$$

$$\hat{\mu} = \frac{y_{..}}{N} = \bar{y}_{..}$$

(6)

$$\frac{\partial SSE}{\partial \tau_i} = \sum_j \cancel{2}\left(y_{ij} - \mu - \tau_i - \beta_j\right)(-1) \overset{\text{set}}{=} 0$$

$$y_{i.} - r\mu - r\tau_i - \sum_{j=1}^{b} n_{ij}\beta_j = 0$$

$$n_{ij} = \begin{cases} 1 & \text{if level } i \text{ is in block } j \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{\tau}_i = \bar{y}_{i.} - \bar{y}_{..} - \frac{1}{r}\sum_{j=1}^{b} n_{ij}\beta_j$$

Similarly,

$$\hat{\beta}_j = \bar{y}_{.j} - \bar{y}_{..} - \frac{1}{k}\sum_{i=1}^{a} n_{ij}\tau_i$$

Substitute:

$$\hat{\tau}_i = \bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot} - \frac{1}{r}\sum_{j=1}^{b} n_{ij}\left[\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot} - \frac{1}{k}\sum_{i=1}^{a} n_{ij}\hat{\tau}_i\right]$$

$$= \bar{y}_{i\cdot} - \cancel{\bar{y}_{\cdot\cdot}} - \frac{1}{r}\sum_{j=1}^{b} n_{ij}\bar{y}_{\cdot j} + \cancel{\frac{1}{r}\bar{y}_{\cdot\cdot}\underbrace{\sum_{j=1}^{b} n_{ij}}_{r}} + \frac{1}{rk}\sum_{j=1}^{b}\sum_{m=1}^{a} n_{ij}n_{mj}\hat{\tau}_m$$

$$\hat{\tau}_i = \bar{y}_{i\cdot} - \frac{1}{r}\sum_{j=1}^{b} n_{ij}\bar{y}_{\cdot j} + \left[\frac{1}{rk}\sum_{j=1}^{b}\sum_{\substack{m=1\\ m\neq i}}^{a} n_{ij}n_{mj}\hat{\tau}_m + \underbrace{\frac{1}{rk}\sum_{j=1}^{b} n_{ij}^2\hat{\tau}_i}_{r\hat{\tau}_i}\right]$$

$$\hat{\tau}_i\left(1 - \frac{1}{k}\right) = \bar{y}_{i\cdot} - \frac{1}{r}\sum_{j=1}^{b} n_{ij}\bar{y}_{\cdot j} + \frac{1}{rk}\sum_{\substack{m=1\\ m\neq i}}^{a}\sum_{j=1}^{b} n_{ij}n_{mj}\hat{\tau}_m$$

$$= \bar{y}_{i\cdot} - \frac{1}{r}\sum_{j=1}^{b} n_{ij}\bar{y}_{\cdot j} + \frac{1}{rk}\sum_{\substack{m=1\\ m\neq i}}^{a}\left[\hat{\tau}_m\underbrace{\sum_{j=1}^{b} n_{ij}n_{mj}}_{\lambda}\right]$$

$$\hat{\tau}_i\left(1 - \frac{1}{k}\right) = \bar{y}_{i\cdot} - \frac{1}{r}\sum_{j=1}^{b} n_{ij}\bar{y}_{\cdot j} + \frac{\lambda}{rk}\underbrace{\sum_{\substack{m=1\\ m\neq i}}^{a}\hat{\tau}_m}_{-\hat{\tau}_i}$$

$$\hat{\tau}_i\underbrace{\left(1 - \frac{1}{k} + \frac{\lambda}{rk}\right)}_{} = \bar{y}_{i\cdot} - \frac{1}{r}\sum_{j=1}^{b} n_{ij}\bar{y}_{\cdot j}$$

$$\frac{rk - r + \lambda}{rk} = \frac{r(k-1) + \lambda}{rk} = \frac{\lambda(a-1) + \lambda}{rk} = \frac{\lambda a}{rk}$$

(9)

$$\hat{\tau}_i = \frac{rk}{\lambda a}\left(\bar{y}_{i\cdot} - \frac{1}{r}\sum_{j=1}^{b} n_{ij}\bar{y}_{\cdot j}\right)$$

$$= \frac{k}{\lambda a}\underbrace{\left(y_{i\cdot} - \frac{1}{k}\sum_{j=1}^{b} n_{ij}\, y_{\cdot j}\right)}_{Q_i}$$

$$\hat{\tau}_i = \frac{k}{\lambda a} Q_i$$

$$SS_{TOT} = \sum_i \sum_j (y_{ij} - \bar{y}_{\cdot\cdot})^2$$

(10)

$$= \sum_i \sum_j (y_{ij} - \hat{y}_{ij} + \hat{y}_{ij} - \bar{y}_{\cdot\cdot})^2$$

$$= \sum_i \sum_j (\underbrace{y_{ij} - \hat{y}_{ij}} + \hat{\mu} + \hat{\tau} + \hat{\beta}_j - \bar{y}_{\cdot\cdot})^2$$

$$= \ldots$$

$$= SS_E + \underbrace{\ldots}$$

$$SS_{BLK} = \sum_{j=1}^{b} \frac{y_{\cdot j}^2}{k} - \frac{y_{\cdot\cdot}^2}{N} \quad \text{(unadjusted)}$$

Find SS_{TRT} by $= SS_{TOT} - SS_E - SS_{BLK}$ (adjusted for blocks)

ANOVA for BIBD

Source	SS	df	MS	F
TRT(adj)	$SS_{TRT(adj)}$	$a-1$	$\frac{SS}{df}$	$\frac{MS_{TRT(adj)}}{MS_E}$
BLK	SS_{BLK}	$b-1$		
ERR	SS_E			
TOT	SS_{TOT}	$N-1$		

$$df_E = N - 1 - (a-1) - (b-1)$$
$$= N - a - b + 1$$

$ar = bk$