Stat 563
4-22-24

(1)

Additional comment on the Bayesian hypothesis test from last time.

Instead of rejecting H_0 when $P(H_1) \geq \frac{1}{2}$,
consider this:

Suppose the cost of a Type I error is C_1
" " " " " " II " " C_2

These are the <u>losses</u>

Recall that the <u>risk</u> is the expected loss.

$$\text{Risk} = C_1 P(\text{Type I}) + C_2 P(\text{Type II})$$

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$$\text{Risk} = C_1 P(H_0 \text{ is true} \cap \text{Reject } H_0)$$
$$+ C_2 P(H_1 \text{ is true} \cap \text{Fail to reject } H_0)$$

$$= C_1 P(H_0 \text{ is true}) P(\text{Reject } H_0 \mid H_0 \text{ is true})$$
$$+ C_2 P(H_1 \text{ is true}) P(\text{Fail to reject } H_0 \mid H_1 \text{ is true})$$

Then the Bayesian rule will be adjusted to minimize this risk.

Power & p-values

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Recall: $\alpha = \text{Prob}(\text{Rej } H_0 \mid H_0 \text{ true})$

$\beta = \text{Prob}(\text{Fail to rej. } H_0 \mid H_1 \text{ is true})$

The power of the test is $1-\beta =$

$\text{Prob}(\text{rej } H_0 \mid H_1 \text{ is true})$

Example: $H_0: \mu = \mu_0$ $\quad$ $X_1, \ldots, X_n \sim \text{iid } N(\mu, \sigma^2)$

$H_1: \mu > \mu_0$ $\quad$ (σ^2) known

L.R.T. said to reject H_0 when $\bar{x} > c$

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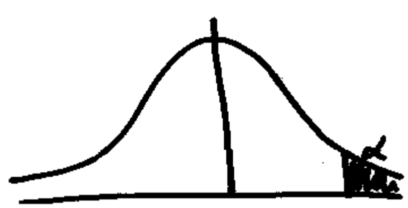
Given that H_0 is true,

$$\bar{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)$$

$$\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$$

$\bar{X} > c$ is equivalent to $\dfrac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > k$



$0 \qquad k = z_\alpha$

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Our rejection rule is $\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > z_\alpha$

Select a specific $\mu_1 > \mu_0$

$$\begin{aligned}
\text{Power} &= P(\text{Reject } H_0 \mid \mu = \mu_1) \\
&= P\left(\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > z_\alpha \,\middle|\, \mu = \mu_1 \right) \\
&= P\left(\bar{X} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \,\middle|\, \mu = \mu_1 \right) \\
&= P\left(\frac{\bar{X}-\mu_1}{\sigma/\sqrt{n}} > \frac{\mu_0-\mu_1+z_\alpha \sigma/\sqrt{n}}{\sigma/\sqrt{n}} \,\middle|\, \mu = \mu_1 \right)
\end{aligned}$$

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$$= P\left(\frac{\bar{X}-\mu_1}{\sigma/\sqrt{n}} > z_\alpha - \frac{(\mu_1-\mu_0)\sqrt{n}}{\sigma} \,\middle|\, \mu = \mu_1 \right)$$

$$\text{Power} = P\left(Z > z_\alpha - \frac{\delta\sqrt{n}}{\sigma} \right) \quad \text{where } \delta = \mu_1 - \mu_0$$

Note: as $n \uparrow$ Power $\uparrow$

as $\mu_1 \uparrow$ Power $\uparrow$

as $\alpha \downarrow$ Power $\downarrow$

If σ is large, power will be small

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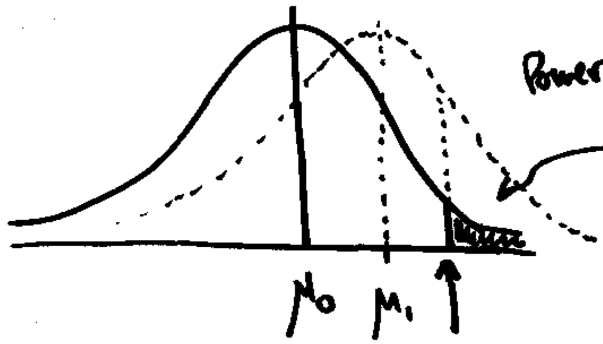

Power = area under dotted curve

$\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$

Defn: A test whose power never drops below α is called <u>unbiased</u>

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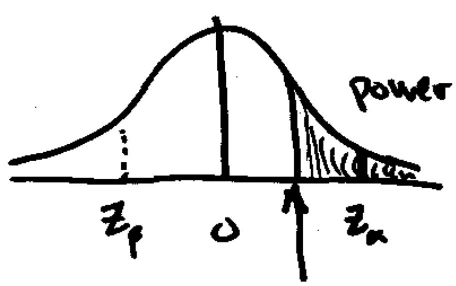
Problem: For a specific δ, find the sample size necessary to achieve a given power.

$$\text{Power} = P\left(Z > z_\alpha - \frac{\delta\sqrt{n}}{\sigma}\right)$$



power = $1-\beta$ = shaded area

$$z_\alpha - \frac{\delta\sqrt{n}}{\sigma} = z_{1-\beta} = -z_\beta$$

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So set $Z_\alpha - \frac{\delta\sqrt{n}}{\sigma} = -Z_\beta$ + solve for n

$$Z_\alpha + Z_\beta = \frac{\delta\sqrt{n}}{\sigma}$$

$$n = \left[\frac{(Z_\alpha + Z_\beta)\sigma}{\delta}\right]^2$$

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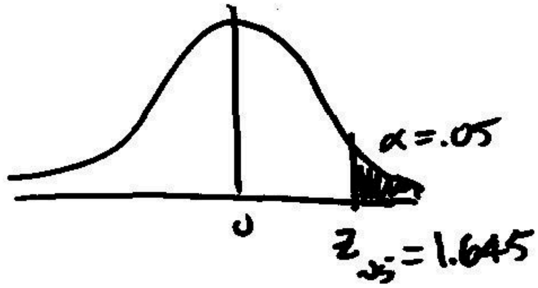
Defn: The p-value is the probability that the test statistic $W(\vec{X})$ is more extreme than the observed value $w(\vec{x})$, given H_0.

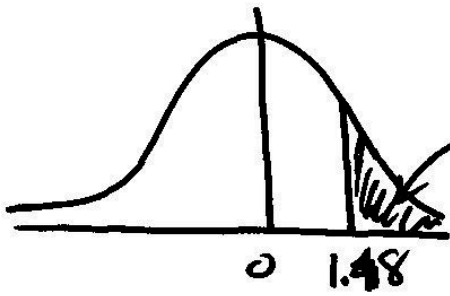
Example, continued $H_0: \mu = 6$ $\alpha = .05$
$H_1: \mu > 6$

Suppose we observe $\bar{x} = 6.7$, $n = 18$, $\sigma = 2$



$$\text{Test stat} = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{6.7 - 6}{2/\sqrt{18}} = 1.48$$

we fail to reject H_0, since $T.S. < z_\alpha$



$$\text{p-value} = P(Z > 1.48) = .069$$

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Alternative way of expressing the rejection rule for <u>any</u> hypothesis test:

Reject H_0 if p-value $< \alpha$.