Stat 563
4-17-24

①

Example: $X_1, \dots, X_n \sim \text{iid } N(\theta, 1)$

$H_0: \theta = \theta_0$

$H_1: \theta \neq \theta_0$

Let θ_1 be any value of $\theta \neq \theta_0$

$$\Lambda = \frac{L(\theta_0)}{L(\theta_1)} = \frac{\cancel{\left(\frac{1}{\sqrt{2\pi}}\right)^n} e^{-\frac{1}{2}\sum(x_i-\theta_0)^2}}{\cancel{\left(\frac{1}{\sqrt{2\pi}}\right)^n} e^{-\frac{1}{2}\sum(x_i-\theta_1)^2}} \overset{\text{set}}{\leq} c$$

$$e^{-\frac{1}{2}\left[\cancel{\sum x_i^2} - 2\theta_0 \sum x_i + n\theta_0^2 - \left(\cancel{\sum x_i^2} - 2\theta_1 \sum x_i + n\theta_1^2\right)\right]} \leq c$$

②

$$-2\theta_0 \sum x_i + n\theta_0^2 + 2\theta_1 \sum x_i - n\theta_1^2 \geq c'$$

$$2\bar{x}(\theta_1 - \theta_0) + \theta_0^2 - \theta_1^2 \geq c''$$

$$\bar{x}(\theta_1 - \theta_0) \geq c'''$$

Our reject region is $\bar{x} \geq k$ if $\theta_1 > \theta_0$

or $\bar{x} \leq k$ if $\theta_1 < \theta_0$

$\therefore$ There is no single rejection region that gives a most powerful test $\forall\, \theta_1$.

That is, there is no UMP test.

③

Def'n: $L(\theta)$ has monotone likelihood ratio (MLR) in $T = g(X_1, \ldots, X_n)$ if

$\forall \theta_1 < \theta_2$, $\dfrac{L(\theta_2)}{L(\theta_1)}$ is a monotone function of T.

Some families of distributions with this property:

Normal (σ known), Poisson, Binomial, most exponential families

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Assume that T is a sufficient statistic for θ and that $L(\theta)$ has MLR in T

④

Consider $H_0: \theta = \theta_0$

$H_1: \theta = \theta_1$ where $\theta_1 > \theta_0$

Neyman-Pearson theorem says that the MP test is the one which rejects H_0 when

$$\Lambda = \frac{L(\theta_0)}{L(\theta_1)} \leq c$$

But our additional assumption says

$$\frac{L(\theta_1)}{L(\theta_0)} = g(t) \text{, which is } \uparrow \text{ or } \downarrow$$

⑤

Case 1: $g(t) \uparrow$

Then Λ is $\downarrow$ in t

So $\Lambda \le c \iff \frac{1}{g(t)} \le c$

$$\iff g(t) \ge c''$$

$$\iff t \ge c'''$$

Case 2: $g(t) \downarrow$

Then Λ is $\uparrow$ in t

$\Lambda \le c \iff \frac{1}{g(t)} \le c$

⑥

$$g(t) \ge c''$$

$$t \le c'''$$

This is the Karlin-Rubin theorem, which is a corollary to the Neyman-Pearson theorem.

Summary: If $L(\theta)$ has MLR in T (sufficient), then the MP test of $H_0: \theta = \theta_0$
$H_1: \theta = \theta_1$

will be of the form $T \le c$ or $T \ge c$
$(g(t) \downarrow)$ $(g(t) \uparrow)$

(7)

Bayesian Hypothesis Tests

We now assume that θ is a random variable whose distribution is known, and θ is independent of $X_1, \ldots, X_n$.

Example (normal-normal)

known

$$X_1, \ldots, X_n \sim \text{iid } N(\theta, \sigma^2)$$

$$\theta \sim N(\mu, \tau^2)$$

both known

(8)

Prior distribution

$$h(\theta) = \frac{1}{\tau\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\theta-\mu}{\tau}\right)^2}$$

$$L(\theta) = f(x_1, \ldots, x_n \mid \theta)$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\theta}{\sigma}\right)^2}$$

the joint distr. of $(X_1, \ldots, X_n, \theta)$ is $L(\theta)h(\theta)$

Recall that the posterior distribution of $\theta \mid \vec{x}$ is proportional to $L(\theta)h(\theta)$

(9)

$$L(\theta)h(\theta) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \frac{1}{\tau\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\Sigma(x_i-\theta)^2} e^{-\frac{1}{2\tau^2}(\theta-\mu)^2}$$

This is proportional to a normal density

with mean $\frac{n\tau^2\bar{x}+\sigma^2\mu}{n\tau^2+\sigma^2}$ and

variance $\frac{\sigma^2\tau^2}{n\tau^2+\sigma^2}$

Decision rule: Reject H_0 if

$$P(\theta \in \Omega_0^c \mid \vec{X}) \geq P(\theta \in \Omega_0 \mid \vec{X})$$

(10)

Suppose that we were testing

$$H_0: \theta \leq \theta_0 \quad \text{vs} \quad H_1: \theta > \theta_0$$

Rule: reject H_0 if

$$P(\theta > \theta_0 \mid \vec{X}) \geq .5$$

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$$P\left(\frac{\theta - \frac{n\tau^2\bar{x}+\sigma^2\mu}{n\tau^2+\sigma^2}}{\sqrt{\frac{\sigma^2\tau^2}{n\tau^2+\sigma^2}}} > \frac{\theta_0 - \dots}{\dots} \;\middle|\; \vec{X}\right)$$

$$P\left(Z > \frac{\theta_0 - \dots}{\dots}\right) \geq .5$$

(11)

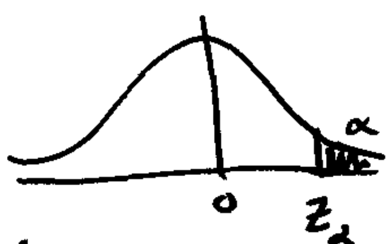
$$\Rightarrow \quad \frac{\theta_0 - \frac{n\tau^2\bar{x}+\sigma^2\mu}{n\tau^2+\sigma^2}}{\sqrt{\frac{\sigma^2\tau^2}{n\tau^2+\sigma^2}}} \le 0$$

$$\Rightarrow \quad \bar{x} \ge \theta_0 + \frac{\sigma^2(\theta_0-\mu)}{n\tau^2}$$

Compare this to the LRT, where we would have rejected H_0 if $\bar{x} \ge c$

Under H_0: $$\frac{\bar{x}-\theta_0}{\sigma/\sqrt{n}} \sim N(0,1)$$

(12)



Reject H_0 if

$$\frac{\bar{x}-\theta_0}{\sigma/\sqrt{n}} \ge z_\alpha \qquad \bar{x} \ge \theta_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$$

8.28 Let $f(x|\theta)$ be the logistic location pdf

$$f(x|\theta) = \frac{e^{(x-\theta)}}{(1+e^{(x-\theta)})^2}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

(a) Show that this family has an MLR.
(b) Based on one observation, X, find the most powerful size α test of $H_0: \theta = 0$ versus $H_1: \theta = 1$. For $\alpha = .2$, find the size of the Type II Error.
(c) Show that the test in part (b) is UMP size α for testing $H_0: \theta \le 0$ versus $H_1: \theta > 0$. What can be said about UMP tests in general for the logistic location family?

8.33 Let $X_1, \ldots, X_n$ be a random sample from the uniform$(\theta, \theta+1)$ distribution. To test $H_0: \theta = 0$ versus $H_1: \theta > 0$, use the test

$$\text{reject } H_0 \text{ if } Y_n \ge 1 \text{ or } Y_1 \ge k,$$

where k is a constant, $Y_1 = \min\{X_1, \ldots, X_n\}, Y_n = \max\{X_1, \ldots, X_n\}$.

(a) Determine k so that the test will have size α.
(b) Find an expression for the power function of the test in part (a).
(c) Prove that the test is UMP size α.
(d) Find values of n and k so that the UMP .10 level test will have power at least .8 if $\theta > 1$.