

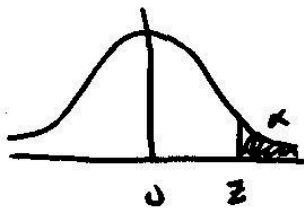
Additional comments about the example from last time:

$$H_0: \mu \leq \mu_0$$
$$H_1: \mu > \mu_0$$

Stat 523

4-15-24

①



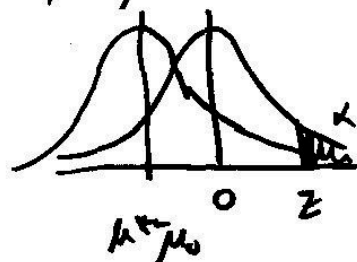
Our solution was to reject H_0 when

$$\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \geq z$$

What if the true mean was actually $\mu^* < \mu_0$?

$$\text{Then } E[\bar{x}] = \mu^*$$

$$\text{So } E[\bar{x} - \mu_0] = \mu^* - \mu_0 < 0$$



So the new tail area to the right of z is $< \alpha$. ②

$$\text{Now } \sup_{\Omega_0} P(\Lambda \leq c) = \alpha$$

Defn: The power function of a test is

$$\pi(\theta) = P(\text{Reject } H_0 \mid \theta) \text{ for } \theta \in \Omega_0^c$$

Power = 1 - Prob. (Type II error)

Note: this is a function of θ

Defn. Let C be a class of hypothesis tests of size α for ③

tests of size α for

$$H_0: \theta \in \Omega_0$$

$$H_1: \theta \notin \Omega_0$$

$$\alpha = \text{Prob}(\text{Type I})$$

= "size"

= "level of significance"

Then a test in C is (UMP)

uniformly most powerful if its power

function $\pi(\theta)$ is greater than or equal to

$\pi'(\theta)$, the power function for any other test in C , $\forall \theta \in \Omega_0^c$.

Neyman-Pearson Theorem

Suppose we are testing $H_0: \theta = \theta_0$

$$H_1: \theta = \theta_1$$

$$\text{(i.e. } \Omega = \{\theta_0, \theta_1\}, \Omega_0 = \{\theta_0\}\text{)}$$

And our rejection region is

$$R = \{\vec{x} : \Lambda \geq c\}$$

and c is chosen so that $P_{H_0}(\vec{x} \in R) = \alpha$.

Then this test is MP.

(5)

Proof: Let A be the rejection region
for any other test of size α

We need to show

$$P(\vec{x} \in R | \theta_1) \geq P(\vec{x} \in A | \theta_1),$$

$$\text{i.e. } \int_R f(\vec{x} | \theta_1) d\vec{x} \geq \int_A f(\vec{x} | \theta_1) d\vec{x}$$

$$= \int_R L(\theta_1) d\vec{x} \geq \int_A L(\theta_1) d\vec{x}$$

$$\text{Consider } \int_R L(\theta_1) d\vec{x} - \int_A L(\theta_1) d\vec{x} = \star \quad (6)$$

We need to show that $\star \geq 0$

$$\star = \int_{R \setminus A} L(\theta_1) d\vec{x} + \int_{R \cap A^c} L(\theta_1) d\vec{x} - \left[\int_{A \setminus R} L(\theta_1) d\vec{x} + \int_{A \cap R^c} L(\theta_1) d\vec{x} \right]$$

$$= \int_{R \setminus A} L(\theta_1) d\vec{x} - \int_{A \setminus R} L(\theta_1) d\vec{x}$$

Suppose $\vec{x} \in R$. Then $\Lambda \leq c$

$$\frac{L(\theta_0)}{L(\theta_1)} \leq c$$

$$L(\theta_1) \geq \frac{1}{c} L(\theta_0)$$

$$\text{So } \int_{R \cap A^c} L(\theta_1) d\vec{x} \geq \frac{1}{c} \int_{R \cap A^c} L(\theta_0) d\vec{x}$$

Suppose $\vec{x} \notin R$. Then $\Lambda > c$

$$\text{So } L(\theta_1) < \frac{1}{c} L(\theta_0)$$

$$\text{Now } \int_{A \cap R^c} L(\theta_1) d\vec{x} < \frac{1}{c} \int_{A \cap R^c} L(\theta_0) d\vec{x}$$

$$\text{So } \star \geq \frac{1}{c} \int_{R \cap A^c} L(\theta_0) d\vec{x} - \frac{1}{c} \int_{A \cap R^c} L(\theta_0) d\vec{x}$$

$$= \frac{1}{c} \left[\int_{R \cap A^c} L(\theta_0) d\vec{x} + \int_{R \cap A} L(\theta_0) d\vec{x} - \int_{R \cap A} L(\theta_0) d\vec{x} - \int_{A \cap R^c} L(\theta_0) d\vec{x} \right]$$

(7)

(8)

(9)

$$= \frac{1}{c} \left[\int_R L(\theta_0) d\vec{x} - \int_A L(\theta_0) d\vec{x} \right]$$

$$= \frac{1}{c} \left[\int_R f(\vec{x} | \theta_0) d\vec{x} - \int_A f(\vec{x} | \theta_0) d\vec{x} \right]$$

$$= \frac{1}{c} \left[P(\vec{x} \in R | \theta_0) - P(\vec{x} \in A | \theta_0) \right]$$

$$= \frac{1}{c} [\alpha - \alpha] = 0$$

$$\therefore \star \geq 0$$

(10)

Recap: For testing $H_0: \theta = \theta_0$, the L.R.T. is most powerful.
 $H_1: \theta = \theta_1$

Consider the simple null vs. composite alternative hypotheses.

$$H_0: \theta = \theta_0$$

$$H_1: \theta > \theta_0$$

Neyman-Pearson Theorem says

that LRT is most powerful

for $H_0: \theta = \theta_0$

$H_1: \theta = \theta_1$ for any $\theta_1 > \theta_0$

If it happens that the structure of the test does not involve the actual value of θ_1 , then the LRT would be UMP. (11)

Example: $X_1, \dots, X_n \sim \text{iid Poisson}(\theta)$
 $f(x|\theta) = \frac{e^{-\theta} \theta^x}{x!}$
 $x = 0, 1, 2, \dots$
 $H_0: \theta = .1$
 $H_1: \theta > .1$

Consider $H_0: \theta = .1$ where $\theta_1 > .1$ (12)
 $H_1: \theta = \theta_1$
 $L(\theta) = \frac{e^{-10\theta} \theta^{\sum x_i}}{\prod (x_i!)}$

$$\Lambda = \frac{L(\theta_0)}{L(\theta_1)} = \frac{e^{-10(.1)} (.1)^{\sum x_i}}{\prod (x_i!)} \frac{\prod (x_i!)}{e^{-10(\theta_1)} \theta_1^{\sum x_i}}$$

$$\Lambda = \frac{e^{-1} (.1)^{\sum x_i}}{e^{-10\theta_1} \theta_1^{\sum x_i}} \leq c$$

$$\frac{1}{(10\theta_1)^{\sum x_i}} \leq c'$$

$$\sum x_i \ln\left(\frac{1}{10\theta_i}\right) \leq c'' \quad (13)$$

$$\sum x_i \geq c'''$$

Under H_0 , $\sum_{i=1}^n x_i \sim \text{Poisson}(1)$

Choose c''' so that $P(\sum x_i \geq c''') = \alpha$

Note that this no longer involves θ_i , so we have a test that is most powerful $\forall \theta_i \notin \Omega_0$, so it is UMP.

Table of Poisson(1) distribution:

k	P(X=k)
0	1
1	.632
2	.264
3	.08
4	.019

Question: how do you construct a test with $\alpha = .05$?

$$\text{let } p = \frac{.05 - .019}{.08 - .019} = \frac{31}{61}$$

Decision rule: If $\sum x_i \geq 4$, Reject H_0

If $\sum x_i \leq 2$, Fail to reject H_0 .

If $\sum x_i = 3$, flip a coin whose "heads" prob is p .

Reject H_0 if you get "heads"

(15)

$$P_{H_0}(\text{Reject } H_0) = P_{H_0}(\sum x_i \geq 4) + pP(\sum x_i = 3)$$

$$= .019 + \frac{31}{61} (.08 - .019) = .05$$