Hypothesis Testing

Stat 563
4-10-24

1

There will be 2 competing hypotheses, H_0 & H_1, that make contradictory claims about a parameter.

Simple example: $H_0: \theta = 2$
$H_1: \theta = 3$

H_0 is the <u>null hypothesis</u>

H_1 is the <u>alternative hypothesis</u>

There will be a test statistic Λ and a decision rule based on Λ

2

That is, $\mathbb{R}$ will be partitioned onto 2 sets R_1 and R_2, so that

if $\Lambda \in R_1$, H_0 is selected &

if $\Lambda \in R_2$, H_1 is selected.

		Decision: H_0	Decision: H_1
Actually true	H_0	✓	Type I
	H_1	Type II	✓

③

Let $\alpha = P(\text{rejecting } H_0 \mid H_0 \text{ was actually true})$

$= P(\text{Type I error})$

Select $H_1 \Rightarrow$ "Reject H_0"

Select $H_0 \Rightarrow$ "Fail to reject H_0"

Likelihood Ratio Test (LRT)

$H_0: \Theta \in \Omega_0$ where $\Omega_0 \subset \Omega$ (parameter space)

$H_1: \Theta \notin \Omega_0$

④

Let $$\Lambda = \frac{\sup_{H_0} L(\Theta)}{\sup_{\Omega} L(\Theta)}$$

Decision rule: Reject H_0 when $\Lambda \le c$,

where c is selected so that $P_{H_0}(\Lambda \le c) = \alpha$.

<u>Example</u>: $X_1, \ldots, X_n \sim$ iid $\text{Exp}(\lambda)$ $\quad f(x) = \lambda e^{-\lambda x} \quad x > 0$

$H_0: \lambda = \lambda_0$ $\qquad \Omega = \{\lambda \mid \lambda > 0\}$

$H_1: \lambda \neq \lambda_0$ $\qquad \Omega_0 = \{\lambda_0\}$

(5)

$$L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda \Sigma x_i}$$

Denominator of Λ : $\hat{\lambda}_{MLE} = \frac{1}{\bar{x}}$

$$L(\hat{\lambda}_{MLE}) = \frac{1}{\bar{x}^n} e^{-\frac{1}{\bar{x}} \Sigma x_i} = \frac{e^{-n}}{\bar{x}^n}$$

Numerator of Λ : $L(\lambda_0) = \lambda_0^n e^{-\lambda_0 \Sigma x_i}$

$$\Lambda = \frac{\lambda_0^n e^{-\lambda_0 \Sigma x_i}}{e^{-n}/\bar{x}^n} = e^n \lambda_0^n \bar{x}^n e^{-n\lambda_0 \bar{x}}$$

$$= e^n (\lambda_0 \bar{x})^n e^{-n\lambda_0 \bar{x}}$$

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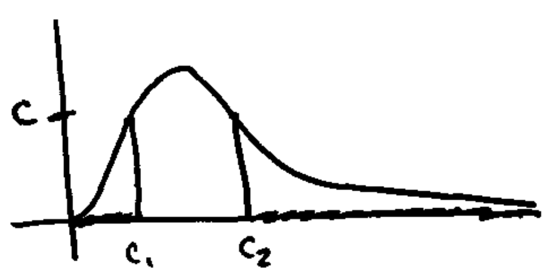
Decision rule: reject H_0 when $\Lambda \le c$.

What does $g(t) = t^n e^{-nt}$ look like?



So $\Lambda \le c$ is equivalent to $\lambda_0 \bar{x} \ge c_2$
or $\lambda_0 \bar{x} \le c_1$

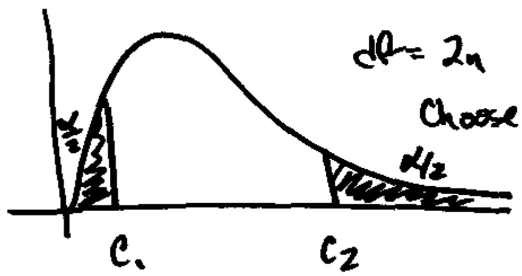
OR $\Sigma x_i \ge c_2'$
$\Sigma x_i \le c_1'$

Under H_0, $X_1, \ldots, X_n \sim \text{Exp}(\lambda_0)$

So $\sum X_i \sim \text{Gamma}(\alpha = n, \beta = \frac{1}{\lambda_0})$

Thus $2\lambda_0 \sum X_i \sim \text{Gamma}(\alpha = n, \beta = 2)$

$$\sim \chi^2_{2n}$$



Choose c_1, c_2 so that the total area $= \alpha$

Usually, we split the area into 2 equal parts

(8)

Final decision rule:

Reject H_0 if $2\lambda_0 \sum X_i \geq C_2$ or

$2\lambda_0 \sum X_i \leq C_1$, where

C_2 cuts off the upper $\alpha/2$ area

and C_1 " " " lower $\alpha/2$ area in

the χ^2 distr. with $2n$ df.

<u>Wording of your conclusion:</u>

Reject H_0: found sufficient evidence favoring H_1

Fail to reject H_0: failed to find sufficient evidence favoring H_1.

<u>Example</u>: $X_1, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$ ← known

$H_0: \mu \le \mu_0$

$H_1: \mu > \mu_0$

$$L(\mu) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}$$

$$= \sigma^{-n}(2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2}$$

Know $\hat{\mu}_{MLE} = \bar{x}$

$$L(\hat{\mu}_{MLE}) = \sigma^{-n}(2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum(x_i-\bar{x})^2}$$

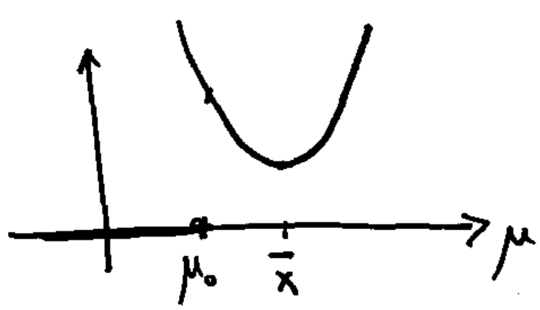
$$= \sigma^{-n}(2\pi)^{-\frac{n}{2}} e^{-\frac{(n-1)s^2}{2\sigma^2}}$$

For the numerator, we need to maximize $L(\mu)$, subject to $\mu \le \mu_0$

$$\ell(\mu) = \ln L(\mu) = -n\ln\sigma - \frac{n}{2}\ln(2\pi) - \frac{1}{2\sigma^2}\underbrace{\sum(x_i-\mu)^2}$$

$$\sum x_i^2 - 2\mu\sum x_i + n\mu^2$$

$$n\left(\frac{1}{n}\sum x_i^2 - 2\mu\bar{x} + \mu^2\right)$$



Subject to H_0, $\ell(\mu)$ is maximized at $\mu = \mu_0$

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$$\text{Now } \Lambda = \frac{\cancel{\sigma^{-n}}\,\cancel{(2\pi)^{-\frac{n}{2}}}\; e^{-\frac{1}{2\sigma^2}\sum (x_i-\mu_0)^2}}{\cancel{\sigma^{-n}}\,\cancel{(2\pi)^{-\frac{n}{2}}}\; e^{-\frac{1}{2\sigma^2}(n-1)s^2}}$$

$$= e^{-\frac{1}{2\sigma^2}\left(\cancel{\sum x_i^2} - 2\mu_0 \sum x_i + n\mu_0^2 - \left(\cancel{\sum x_i^2} - 2\bar{x}\sum x_i + n\bar{x}^2\right)\right)}$$

$$= e^{-\frac{n}{2\sigma^2}\left(-2\mu_0\bar{x} + \mu_0^2 + 2\bar{x}^2 - \bar{x}^2\right)}$$

$$= e^{-\frac{n}{2\sigma^2}(\bar{x}-\mu_0)^2}$$

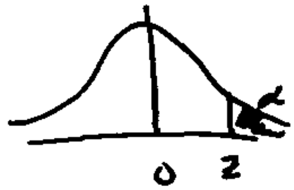
Rule: Reject H_0 when $\Lambda \le c$,

i.e. when $\underbrace{\frac{(\bar{x}-\mu_0)^2}{\sigma^2/n}}_{\sim\,\chi^2_1} \ge c'$

(12)

This is equivalent to rejecting H_0 when

$$\underline{\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \ge \sqrt{c_1}} \quad \text{or} \quad \underbrace{\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \le \sqrt{c_2}}_{\text{Not feasible}}$$



Stat 563 HW #2

8.5 A random sample, $X_1, \dots, X_n$, is drawn from a Pareto population with pdf

$$f(x|\theta, \nu) = \frac{\theta \nu^\theta}{x^{\theta+1}} I_{[\nu,\infty)}(x), \quad \theta > 0, \quad \nu > 0.$$

(a) Find the MLEs of θ and ν.

(b) Show that the LRT of

$$H_0: \theta = 1,\ \nu \text{ unknown}, \qquad \text{versus} \qquad H_1: \theta \neq 1,\ \nu \text{ unknown},$$

has critical region of the form $\{\mathbf{x}: T(\mathbf{x}) \le c_1 \text{ or } T(\mathbf{x}) \ge c_2\}$, where $0 < c_1 < c_2$ and

$$T = \log\left[\frac{\prod_{i=1}^{n} X_i}{(\min_i X_i)^n}\right].$$

(c) Show that, under H_0, $2T$ has a chi squared distribution, and find the number of degrees of freedom. (*Hint:* Obtain the joint distribution of the $n-1$ nontrivial terms $X_i/(\min_i X_i)$ conditional on $\min_i X_i$. Put these $n-1$ terms together, and notice that the distribution of T given $\min_i X_i$ does not depend on $\min_i X_i$, so it is the unconditional distribution of T.)

8.17 Suppose that $X_1, \dots, X_n$ are iid with a beta$(\mu, 1)$ pdf and $Y_1, \dots, Y_m$ are iid with a beta$(\theta, 1)$ pdf. Also assume that the Xs are independent of the Ys.

(a) Find an LRT of $H_0: \theta = \mu$ versus $H_1: \theta \neq \mu$.

(b) Show that the test in part (a) can be based on the statistic

$$T = \frac{\Sigma \log X_i}{\Sigma \log X_i + \Sigma \log Y_i}.$$

(c) Find the distribution of T when H_0 is true, and then show how to get a test of size $\alpha = .10$.

8.19 The random variable X has pdf $f(x) = e^{-x}, x > 0$. One observation is obtained on the random variable $Y = X^\theta$, and a test of $H_0: \theta = 1$ versus $H_1: \theta = 2$ needs to be constructed. Find the UMP level $\alpha = .10$ test and compute the Type II Error probability.