Stat 563

4-8-24

## The Rao-Blackwell Theorem

(1)

Let $X_1, \dots, X_n \sim \text{iid } f(x|\theta)$

Let $T$ be a sufficient statistic for $\theta$.

Let $W$ be an unbiased estimator of $\tau(\theta)$.

Let $\phi(T) = E[W|T]$

Then $\phi(T)$ is an unbiased estimator of $\tau(\theta)$

and $V[\phi(T)] \leq V[W] \;\; \forall \, \theta$.

(2)

<u>Proof</u>: $E[\phi(T)] = E[E[W|T]]$

$= E[W] = \tau(\theta)$

$$V[W] = \underbrace{E[V[W|T]]}_{\geq 0} + \underbrace{V[E[W|T]]}_{V[\phi(T)]}$$

$V[W] \geq V[\phi(T)]$

Also, $f(\vec{x}|T)$ is free of $\theta$, so distribution of $W|T$ is free of $\theta$, as is $\phi(T) = E(W|T)$.

③

*Example:* $X_1, X_2 \sim iid \ \frac{1}{\theta} e^{-x/\theta} \quad x > 0$

Let $W = X_1 \qquad E[W] = \theta \qquad V[W] = \theta^2$

Find a sufficient statistic for $\theta$.

$$L(\theta) = \frac{1}{\theta} e^{-\frac{x_1}{\theta}} \frac{1}{\theta} e^{-\frac{x_2}{\theta}}$$

$$= \frac{1}{\theta^2} e^{-\frac{1}{\theta}(x_1 + x_2)} \cdot 1$$

So $T = X_1 + X_2$ is sufficient for $\theta$.

④

Find $E[W|T]$

Start with the joint distribution of $W, T$.

| | | |
|---|---|---|
| $W = X_1$ | $X_1 = w$ | $X_1 > 0$ |
| $T = X_1 + X_2$ | $X_2 = T - w$ | $X_2 > 0$ |
| | | $w > 0$ |
| | | $t - w > 0$ |
| | | $t > w$ |

$$|J| = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

$$g(w,t) = f(x_1, x_2) \cdot 1 = \frac{1}{\theta^2} e^{-\frac{1}{\theta}(x_1 + x_2)}$$

$$= \frac{1}{\theta^2} e^{-t/\theta} \quad 0 < w < t$$

⑤

$$g(w|t) = \frac{g(w,t)}{h(t)} = \frac{\frac{1}{\theta^2} e^{-t/\theta}}{\frac{1}{\theta^2} t e^{-t/\theta}}$$

Because $T \sim \text{Gamma}(\alpha = 2, \beta = \theta)$

$$= \frac{1}{t}, \quad 0 < w < t$$

That is, $W|T \sim \text{Unif}(0, T)$

So $E[W|T] = \frac{0+T}{2} = \frac{T}{2} = \frac{X_1 + X_2}{2} = \bar{X}$

Note $E[\bar{X}] = \theta$ and $V[\bar{X}] = \frac{\theta^2}{2}$

⑥

A simpler approach might be to

① Find a sufficient statistic for $\theta$, called $T$

② Find a function of $T$ that is an unbiased estimator of $\theta$.

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<u>Defn</u>: $W$ is a minimum variance unbiased (MVUE) estimator of $\theta$ if $E[W] = \theta$ and its variance is less than or equal to the variance of any other unbiased estimator of $\theta$, $\forall \theta$.

⑦

<u>Theorem</u>: If $W$ is a MVUE of $\theta$, then $W$ is unique.

<u>Proof</u>: let $W'$ also be MVUE of $\theta$.

Let $W^* = \frac{1}{2}(W + W')$

$$E[W^*] = \frac{1}{2}[E(W) + E(W')] = \theta$$

$$V[W^*] = \frac{1}{4}[\underline{V[W] + V[W']} + 2\,Cov(W,W')]$$

$$= \frac{1}{4}[2V[W] + 2\,Cov(W,W')]$$

⑧

$$V[W^*] = \frac{1}{2}V[W] + \frac{1}{2}Cov(W,W')$$

Recall $\rho^2 = \dfrac{Cov^2(W,W')}{V[W]V[W']} \leq 1$

So $Cov^2(W,W') \leq (V[W])^2$

$$|Cov(W,W')| \leq V[W] \quad \bigstar$$

Now $V[W^*] \leq \frac{1}{2}V[W] + \frac{1}{2}V[W]$

$$V[W^*] \leq V[W]$$

(9)

If, for any $\theta$, $V[W^*] < V[W]$,

our assumption that $W$ was MVUE

would be violated.

$\therefore V[W^*] = V[W] \quad \forall \theta$

Equivalently, $\rho = \mathrm{Cor}(W, W') = 1$

Cauchy-Schwarz says $W' = aW + b$

By $\bigstar$, $\mathrm{Cov}(W, W') = V[W]$

(10)

$$\mathrm{Cov}(W,W') = \mathrm{Cov}(W, aW+b)$$

$$= a\,\mathrm{Cov}(W,W) + \mathrm{Cov}(W,b)$$

$$= a\,V[W] + 0$$

$$\therefore a = 1$$

So $W' = W + b$

$$E[W'] = E[W] + b$$

$$\theta = \theta + b \quad \therefore b = \theta$$

$$\therefore W = W'$$

(11)

## The Lehmann - Scheffé Theorem

Let $T$ be a complete sufficient statistic for $\theta$. If there exists $W$, a function of $T$, that is an unbiased estimator of $\tau(\theta)$, then $W$ is the unique MVUE of $\tau(\theta)$.

Proof: Assume $W$ exists.

Let $W'$ be another unbiased estimator of $\tau(\theta)$.

(12)

By the Rao-Blackwell Theorem,

$E[W' \mid T]$ will be an unbiased estimator of $\tau(\theta)$ and its variance will be $\leq V[W']$.

Also note that $E[W' \mid T]$ is a function of $T$.

Let $\varphi(T)$ and $\psi(T)$ be functions of $T$ that are both unbiased estimators of $\tau(\theta)$.

$$E[\varphi(T) - \psi(T)] = E[\varphi(T)] - E[\psi(T)]$$

$$= \tau(\theta) - \tau(\theta) = 0 \quad \forall \theta$$

By completeness, this implies

$$\varphi(T) - \psi(T) = 0$$

So $W$ and $E[W'|T]$ must be equal.

Then $V[W] = V[E[W'|T]] \leq V[W']$

So $W$ is MVUE and is unique.