Stat 563
4-3-24

(1)

Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be unbiased estimators of θ.

Defn: The relative efficiency of $\hat{\theta}_1$ to $\hat{\theta}_2$ is

$$\frac{V(\hat{\theta}_2)}{V(\hat{\theta}_1)}$$

Defn: The efficiency of $\hat{\theta}_1$ is $\frac{CRLB}{V(\hat{\theta}_1)}$

Defn: $\hat{\theta}_1$ is efficient if the efficiency is 1.

(2)

Defn: $\hat{\theta}$ is asymptotically unbiased

if $\lim_{n\to\infty} Bias(\hat{\theta}) = 0$

Defn: If $\hat{\theta}_1$ and $\hat{\theta}_2$ are asymptotically unbiased estimators of θ, then the asymptotic relative efficiency of $\hat{\theta}_1$ to $\hat{\theta}_2$

is $\lim_{n\to\infty} \frac{V(\hat{\theta}_2)}{V(\hat{\theta}_1)}$.

③

<u>Defn</u>: If $\hat{\theta}_1$ is asymptotically unbiased, then its asymptotic efficiency is

$$\lim_{n\to\infty} \frac{CRLB}{V(\hat{\theta}_1)}.$$

<u>Example</u>: $X_1, \ldots, X_n \sim iid\ Exp(\lambda)$

Compare $\hat{\lambda}_{MLE}$ to an unbiased version of it.

④

$$L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda \Sigma x_i}$$

$$\ell(\lambda) = \ln L(\lambda) = n \ln \lambda - \lambda \Sigma x_i$$

$$\ell'(\lambda) = \frac{n}{\lambda} - \Sigma x_i \overset{set}{=} 0$$

$$\hat{\lambda}_{MLE} = \frac{n}{\Sigma X_i} = \frac{1}{\bar{X}}$$

$$E[\hat{\lambda}_{MLE}] = E\left[\frac{n}{\Sigma X_i}\right] = E\left[\frac{n}{Y}\right] = n\,E[Y^{-1}]$$

$$Y = \Sigma X_i \sim Gamma(\alpha = n, \beta = \tfrac{1}{\lambda})$$

(5)

$$E[Y^{-1}] = \beta^{-1}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha)} = \lambda\frac{(n-2)!}{(n-1)!} = \frac{\lambda}{n-1}$$

$$E[\hat{\lambda}_{MLE}] = n\,E[Y^{-1}] = \frac{n}{n-1}\lambda$$

$$\text{Bias}(\hat{\lambda}_{MLE}) = \frac{n}{n-1}\lambda - \lambda = \frac{\lambda}{n-1}$$

$\therefore$ $\hat{\lambda}_{MLE}$ is biased, but asymptotically unbiased.

Define $\hat{\lambda}_u = \frac{n-1}{n}\hat{\lambda}_{MLE} = \frac{n-1}{n}\frac{n}{\Sigma x_i} = \frac{n-1}{\Sigma x_i}$

(6)

$$V(\hat{\lambda}_{MLE}) = V\left(\frac{n}{\Sigma x_i}\right) = V\left(\frac{n}{Y}\right)$$

$$= n^2 V[Y^{-1}]$$

$$= n^2\Big[\underbrace{E[Y^{-2}]} - (\underbrace{E[Y^{-1}]}_{\frac{\lambda}{n-1}})^2\Big]$$

$$\beta^{-2}\frac{\Gamma(\alpha-2)}{\Gamma(\alpha)}$$

$$\lambda^2\frac{(n-3)!}{(n-1)!}$$

$$\frac{\lambda^2}{(n-1)(n-2)}$$

7

$$V(\hat{\lambda}_{MLE}) = n^2\left[\frac{\lambda^2}{(n-1)(n-2)} - \frac{\lambda^2}{(n-1)^2}\right]$$

$$= \frac{n^2\lambda^2}{(n-1)^2(n-2)}$$

$$V(\hat{\lambda}_u) = V\left(\frac{n-1}{n}\hat{\lambda}_{MLE}\right) = \left(\frac{n-1}{n}\right)^2 \frac{n^2\lambda^2}{(n-1)^2(n-2)}$$

$$= \frac{\lambda^2}{n-2}$$

the asymptotic relative efficiency of $\hat{\lambda}_u$ to $\hat{\lambda}_{MLE}$

8

is $$\lim_{n\to\infty} \frac{\dfrac{n^2\lambda^2}{(n-1)^2\cancel{(n-2)}}}{\dfrac{\lambda^2}{\cancel{n-2}}} = \lim_{n\to\infty} \frac{n^2}{(n-1)^2} = 1$$

Find the Cramér-Rao Lower Bound (CRLB):

$$f(x) = \lambda e^{-\lambda x}$$

$$\ln f(x) = \ln\lambda - \lambda x$$

$$\frac{\partial \ln f}{\partial \lambda} = \frac{1}{\lambda} - x$$

$$\frac{\partial^2 \ln f}{\partial \lambda^2} = -\frac{1}{\lambda^2}$$

(9)

$$I(\lambda) = E\left[\frac{\partial \ln f}{\partial \lambda}\right]^2 = -E\left[\frac{\partial^2 \ln f}{\partial \lambda^2}\right]$$

Fisher Information

$$E\left[\frac{1}{\lambda} - X\right]^2 = V(X) = \frac{1}{\lambda^2}$$

$$-E\left[-\frac{1}{\lambda^2}\right] = \frac{1}{\lambda^2}$$

$$CRLB = \frac{1}{n I(\lambda)} = \frac{1}{n \cdot \frac{1}{\lambda^2}} = \frac{\lambda^2}{n}$$

(10)

The efficiency of $\hat{\lambda}_u$ is $\dfrac{CRLB}{V(\hat{\lambda}_u)} = \dfrac{\frac{\lambda^2}{n}}{\frac{\lambda^2}{n-2}} = \dfrac{n-2}{n}$

The asymptotic efficiency of $\hat{\lambda}_u = 1$

The asymptotic efficiency of $\hat{\lambda}_{MLE}$ is

$$\lim_{n\to\infty} \frac{CRLB}{V(\hat{\lambda}_{MLE})} = \frac{\frac{\lambda^2}{n}}{\frac{n^2\lambda^2}{(n-1)^2(n-2)}} = \lim_{n\to\infty} \frac{(n-1)^2(n-2)}{n^3}$$

$$= 1$$

(11)

$$MSE(\hat{\theta}) = B^2(\hat{\theta}) + V(\hat{\theta})$$

$$MSE(\hat{\lambda}_u) = V(\hat{\lambda}_u) = \frac{\lambda^2}{n-2}$$

$$MSE(\hat{\lambda}_{MLE}) = \left(\frac{\lambda}{n-1}\right)^2 + \frac{n^2\lambda^2}{(n-1)^2(n-2)}$$

$$= \lambda^2 \frac{(n-2) + n^2}{(n-1)^2(n-2)}$$

$$= \lambda^2 \frac{n^2+n-2}{(n-1)^2(n-2)} = \frac{\lambda^2(n+2)}{(n-1)(n-2)}$$

$$\therefore MSE(\hat{\lambda}_{MLE}) > MSE(\hat{\lambda}_u)$$

Stat 563 HW #1

7.12 Let $X_1, \ldots, X_n$ be a random sample from a population with pmf

$$P_\theta(X = x) = \theta^x(1-\theta)^{1-x}, \quad x = 0 \text{ or } 1, \quad 0 \le \theta \le \frac{1}{2}.$$

(a) Find the method of moments estimator and MLE of θ.
(b) Find the mean squared errors of each of the estimators.
(c) Which estimator is preferred? Justify your choice.

7.48 Suppose that $X_i, i = 1, \ldots, n$, are iid Bernoulli(p).
(a) Show that the variance of the MLE of p attains the Cramér–Rao Lower Bound.
(b) For $n \ge 4$, show that the product $X_1X_2X_3X_4$ is an unbiased estimator of p^4, and use this fact to find the best unbiased estimator of p^4.

7.59 Let $X_1, \ldots, X_n$ be iid $\mathrm{n}(\mu, \sigma^2)$. Find the best unbiased estimator of σ^p, where p is a known positive constant, not necessarily an integer.