Stat 563
4-1-24

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Defn: Let X have pdf or pmf $f(x|\theta)$.

The score function for X is $\frac{\partial}{\partial\theta}\ln f(x|\theta)$

Consider $\frac{\partial}{\partial\theta}\ln f(X|\theta)$ as a random variable that is a function of X.

Find its expectation and variance.

$$E\left[\frac{\partial}{\partial\theta}\ln f(X|\theta)\right] = \int_{-\infty}^{\infty}\frac{\partial}{\partial\theta}\ln f(x|\theta)\, f(x|\theta)\, dx$$

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$$= \int_{-\infty}^{\infty}\frac{1}{\cancel{f(x|\theta)}}\frac{\partial}{\partial\theta}f(x|\theta)\,\cancel{f(x|\theta)}\, dx$$

$$= \frac{\partial}{\partial\theta}\underbrace{\int_{-\infty}^{\infty} f(x|\theta)\, dx}_{1}$$

* subject to some regularity conditions

$$= 0$$

$$V\left[\frac{\partial}{\partial\theta}\ln f(X|\theta)\right] = E\left[\left[\frac{\partial}{\partial\theta}\ln f(X|\theta)\right]^2\right]$$

Defn: This is the Fisher Information $I(\theta)$

There is an alternate form that is more common. ③

From the previous derivation,

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial \theta} \ln f(x|\theta)\, f(x|\theta)\, dx = 0$$

Take $\frac{\partial}{\partial \theta}$ of both sides

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial \theta}\left[\frac{\partial}{\partial \theta} \ln f(x|\theta)\, f(x|\theta)\right] dx = 0$$

④

$$= \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial \theta} \ln f(x|\theta)\, \frac{\partial}{\partial \theta} f(x|\theta) + f(x|\theta)\, \frac{\partial^2}{\partial \theta^2} \ln f(x|\theta)\right] dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial \theta} \ln f(x|\theta)\right)\left(\frac{\frac{\partial}{\partial \theta} f(x|\theta)}{f(x|\theta)}\right) f(x|\theta)\, dx$$

$$+ \int_{-\infty}^{\infty} \frac{\partial^2}{\partial \theta^2} \ln f(x|\theta)\, f(x|\theta)\, dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial \theta} \ln f(x|\theta)\right)^2 f(x|\theta)\, dx + E\left[\frac{\partial^2}{\partial \theta^2} \ln f(X|\theta)\right]$$

⑤

So $0 = I(\theta) + E\left[\frac{\partial^2}{\partial\theta^2} \ln f(X|\theta)\right]$

$$\therefore \quad I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2} \ln f(X|\theta)\right]$$

Let $X_1, \ldots, X_n \sim$ iid $f(x|\theta)$

Let W be a function of $X_1, \ldots, X_n$ (free of θ)

Suppose $E[W] = k(\theta)$

⑥

$$k(\theta) = E[W] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} w \underbrace{\prod_{i=1}^{n} f(x_i|\theta)}_{L(\theta)} dx_1 \cdots dx_n$$

$$k'(\theta) = \frac{\partial}{\partial\theta} E[W] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} w L'(\theta) \, dx_1 \cdots dx_n$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} w \underbrace{\left(\frac{L'(\theta)}{L(\theta)}\right)}_{Z} L(\theta) \, dx_1 \cdots dx_n$$

(7)

$$= E[WZ]$$

So far: $k(\theta) = E[W]$

$k'(\theta) = E[WZ]$

$$Z = \frac{L'(\theta)}{L(\theta)} = \frac{\partial}{\partial\theta} \ln L(\theta)$$

$$= \frac{\partial}{\partial\theta} \ln \prod_{i=1}^{n} f(x_i|\theta)$$

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$$= \frac{\partial}{\partial\theta} \sum_{i=1}^{n} \ln f(x_i|\theta)$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial\theta} \ln f(x_i|\theta)$$

$$E[Z] = \sum_{i=1}^{n} E\left[\frac{\partial}{\partial\theta} \ln f(X_i|\theta)\right]$$

$$= 0$$

(Expected value of score function was 0)

This means that

$$Cov(W,Z) = E[WZ] - E[W]\underbrace{E[Z]}_{0}$$
$$= E[WZ]$$

Also, $V[Z] = V\left[\sum_{i=1}^{n} \frac{\partial}{\partial\theta}\ln f(X_i|\theta)\right]$

$$= \sum_{i=1}^{n} V\left[\frac{\partial}{\partial\theta}\ln f(X_i|\theta)\right] \text{ by independence}$$

$$= n\,I(\theta)$$

$$Corr(W,Z) = \frac{Cov(W,Z)}{\sqrt{V(W)V(Z)}}$$

$$\frac{Cov^2(W,Z)}{V[W]V[Z]} \leq 1 \quad \text{since } -1 \leq \rho \leq 1$$

$$\frac{[k'(\theta)]^2}{V[W]\,n\,I(\theta)} \leq 1$$

(11)

$$V[W] \geq \frac{[k'(\theta)]^2}{n I(\theta)} \qquad (k(\theta) = E[W])$$

This is called the Cramér-Rao Inequality

Suppose that W is an unbiased estimator of θ.

Then $k(\theta) = E[W] = \theta$ and $k'(\theta) = 1$

So $V[W] \geq \frac{1}{n I(\theta)}$ $\left.\right\}$ C.R.L.B.
This is the lower bound on the variance of an unbiased estimator.

(12)

Example: $X_1, \dots, X_n \sim$ iid $f(x|\theta) = \frac{1}{\theta} e^{-x/\theta}$ $\quad x > 0$

Know that $\hat{\theta}_{MLE} = \bar{X}$

$E[\hat{\theta}] = E[\bar{X}] = \mu = \theta$ So $\hat{\theta}$ is unbiased

$V[\hat{\theta}] = V[\bar{X}] = \frac{\sigma^2}{n} = \frac{\theta^2}{n}$

Find the C.R.L.B.

$\ln f(x|\theta) = -\ln\theta - \frac{x}{\theta}$

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$$\frac{\partial}{\partial\theta}\ln f(x|\theta) = -\frac{1}{\theta} + \frac{x}{\theta^2}$$

$$\frac{\partial^2}{\partial\theta^2}\ln f(x|\theta) = \frac{1}{\theta^2} - \frac{2x}{\theta^3}$$

$$I(\theta) = -E\left[\frac{1}{\theta^2} - \frac{2x}{\theta^3}\right] = -\left[\frac{1}{\theta^2} - \frac{2\theta}{\theta^3}\right]$$

$$= \frac{1}{\theta^2}$$

$$\text{C.R.L.B.} = \frac{1}{nI(\theta)} = \frac{1}{n\cdot\frac{1}{\theta^2}} = \frac{\theta^2}{n}$$

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Defn. The efficiency of an estimator is $\dfrac{CRLB}{V[\hat{\theta}]}$

If the efficiency $= 1$, then the estimator is efficient