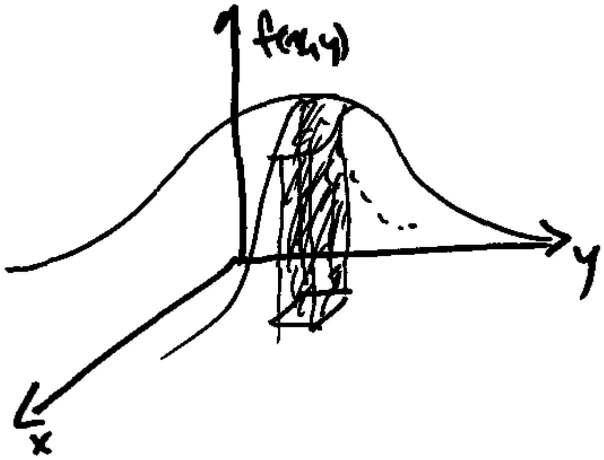
Stat 451
2-15-18

For 2 continuous random variables, there is a joint density function $f(x,y)$.

(1)

Probabilities are volumes underneath this surface



(2)

Example: $f(x,y) = \begin{cases} 8xy & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

Check to see if the total volume is 1.

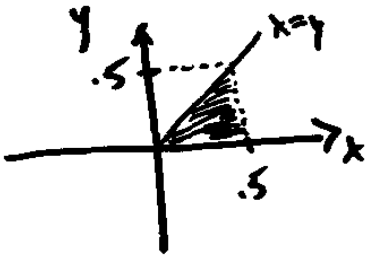
$$\underbrace{\int_0^1 \int_0^x 8xy \, dy \, dx}_{} \quad \text{OR} \quad \int_0^1 \int_y^1 8xy \, dx \, dy$$

$$\int_0^1 8x \frac{y^2}{2} \Big|_{y=0}^{y=x} dx = \int_0^1 8x\left(\frac{x^2}{2} - 0\right) dx$$

$$= \int_0^1 4x^3\,dx = x^4\Big|_0^1 = 1-0 = 1 \checkmark$$

Find the probability that X and Y are both less than .5.

$$P(X<.5 \cap Y<.5) = \int_0^{.5}\int_0^{x} 8xy\,dy\,dx$$



$$= \int_0^{.5} 8x\,\frac{y^2}{2}\Big|_{y=0}^{y=x}\,dx$$

$$= \int_0^{.5} 8x\left(\frac{x^2}{2}-0\right)dx$$

$$= \int_0^{.5} 4x^3\,dx = x^4\Big|_0^{.5} = .5^4-0 = \frac{1}{16}$$

If $f(x,y)$ is the joint density, then the marginal density of X is $g(x) = \displaystyle\int_{\substack{\text{all } y \\ (\text{in terms of } x)}} f(x,y)\,dy$

and the marginal density of Y is $h(y) = \displaystyle\int_{\text{all } x\ (\text{in terms of } y)} f(x,y)\,dx$

⑤

Back to our example:

$$g(x) = \int_{\text{all } y} f(x,y)\,dy = \int_0^x 8xy\,dy$$

$$= 8x \left.\frac{y^2}{2}\right|_{y=0}^{y=x} = 8x\left(\frac{x^2}{2} - 0\right)$$

$$= 4x^3$$

$$g(x) = \begin{cases} 4x^3 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

⑥

$$h(y) = \int_y^1 8xy\,dx = 8y \left.\frac{x^2}{2}\right|_{x=y}^{x=1}$$

$$= 8y\left(\frac{1}{2} - \frac{y^2}{2}\right) = 4y - 4y^3$$

$$h(y) = \begin{cases} 4y - 4y^3 & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Check: $\int_0^1 4y - 4y^3\,dy = 2y^2 - y^4 \Big|_0^1 = 1 - 0 = 1$ ✓

<u>Defn</u>: The continuous random variables X & Y are independent if $f(x,y) = g(x)\,h(y)$ $\forall\, x,y$

In our example, $f(x,y) = 8xy \quad 0 \le y \le x \le 1$

$g(x) = 4x^3 \quad 0 \le x \le 1$

$h(y) = 4y - 4y^3 \quad 0 \le y \le 1$

$$8xy \stackrel{?}{=} 4x^3(4y - 4y^3) \qquad \underline{\text{No}}$$

So X & Y are <u>not</u> independent.

(8)

<u>Defn</u>: $$E[XY] = \iint\limits_{\text{all }(x,y)} xy\, f(x,y)\, dy\, dx$$

In our example,

$$E[XY] = \int_0^1 \int_0^x xy\; 8xy\; dy\, dx$$

$$= \int_0^1 8x^2 \left.\frac{y^3}{3}\right|_{y=0}^{y=x} dx$$

$$= \int_0^1 8x^2\left(\frac{x^3}{3} - 0\right) dx = \int_0^1 \frac{8}{3} x^5\, dx$$

$$= \frac{8}{3}\frac{x^6}{6}\Big|_0^1 = \frac{8}{3}\left(\frac{1}{6}-0\right) = \frac{4}{9}$$ (9)

Defn: The covariance of X and Y is

$$\sigma_{XY} = Cov(X,Y)$$
$$= E[XY] - E[X]\,E[Y]$$

In our example,

$$E[X] = \int_{\text{all } x} x\, g(x)\,dx = \int_0^1 x\, 4x^3\,dx$$

$$= \int_0^1 4x^4\,dx = \frac{4x^5}{5}\Big|_0^1 = \frac{4}{5}$$ (10)

$$E[Y] = \int_{\text{all } y} y\, h(y)\,dy = \int_0^1 y(4y-4y^3)\,dy$$

$$= \int_0^1 4y^2 - 4y^4\,dy = \frac{4y^3}{3} - \frac{4y^5}{5}\Big|_0^1$$

$$= \frac{4}{3} - \frac{4}{5} = \frac{8}{15}$$

$$\sigma_{XY} = E[XY] - E[X]E[Y] = \frac{4}{9} - \frac{4}{5}\cdot\frac{8}{15}$$
$$= 4/225$$

Defn: The correlation between X & Y

is $\rho_{XY} = Corr(X,Y) = \dfrac{\sigma_{XY}}{\sigma_X \sigma_Y}$

For our example,

$$E[X^2] = \int_0^1 x^2 \, 4x^3 \, dx = \int_0^1 4x^5 \, dx$$

$$= 4\frac{x^6}{6}\Big|_0^1 = 2/3$$

$$\sigma_X^2 = E[X^2] - (E[X])^2 = 2/3 - \left(\tfrac{4}{5}\right)^2 = \frac{2}{75}$$

$$E[Y^2] = \int_0^1 y^2(4y - 4y^3)\, dy = \int_0^1 4y^3 - 4y^5 \, dy$$

$$= y^4 - \frac{4y^6}{6}\Big|_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\sigma_Y^2 = E[Y^2] - (E[Y])^2 = \frac{1}{3} - \left(\frac{8}{15}\right)^2$$

$$= \frac{11}{225}$$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{4/225}{\sqrt{\frac{2}{75}}\sqrt{\frac{11}{225}}} = .492$$

"Pearson Product-Moment Correlation"

(13)

Given $f(x,y)$, what steps do you take to get ρ_{XY}?

① Find the marginal densities $g(x)$, $h(y)$

② Use the marginals to find $E[X], E[X^2], E[Y], E[Y^2]$

③ Use the joint density to find $E[XY]$

④ Compute
$$\sigma_X^2 = E[X^2] - (E[X])^2,$$
$$\sigma_Y^2 = E[Y^2] - (E[Y])^2,$$
$$\sigma_{XY} = E[XY] - E[X]E[Y]$$

(14)

⑤ Compute
$$\rho_{XY} = \frac{\sigma_{XY}}{\sqrt{\sigma_X^2}\sqrt{\sigma_Y^2}}$$

HW #5 due 2/22

p. 165 #3

p. 180 #17

p. 191 #8

3. A local diner offers entrees in three prices, \$8.00, \$10.00, and \$12.00. Diner customers are known to tip either \$1.50, \$2.00, or \$2.50 per meal. Let X denote the price of the meal ordered, and Y denote the tip left, by a random customer. The joint PMF of X and Y is

		y		
	$p(x, y)$	\$1.50	\$2.00	\$2.50
	\$8.00	0.3	0.12	0
x	\$10.00	0.15	0.135	0.025
	\$12.00	0.03	0.15	0.09

(a) Find $P(X \leq 10, Y \leq 2)$ and $P(X \leq 10, Y = 2)$.

(b) Compute the marginal PMFs of X and Y.

(c) Given that a customer has left a tip of \$2.00, find the probability that the customer ordered a meal of \$10.00 or less.

17. A type of steel has microscopic defects that are classified on a continuous scale from 0 to 1, with 0 the least severe and 1 the most severe. This is called the defect index. Let X and Y be the static force at failure and the defect index, respectively, for a particular type of structural member made from this steel. For a member selected at random, X and Y are jointly distributed random variables with joint PDF

$$f(x, y) = \begin{cases} 24x & \text{if } 0 \leq y \leq 1 - 2x \quad \text{and} \quad 0 \leq x \leq .5 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Sketch the support of this PDF, that is, the region of (x, y) values where $f(x, y) > 0$.

(b) Are X and Y independent? Justify your answer in terms the support of the PDF sketched above.

(c) Find each of the following: $f_X(x)$, $f_Y(y)$, $E(X)$, and $E(Y)$.

8. Suppose (X, Y) have the joint PDF

$$f(x, y) = \begin{cases} 24xy & 0 \le x \le 1, 0 \le y \le 1, x + y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $\mathrm{Cov}(X, Y)$. (*Hint.* Use the marginal PDF of X, which was derived in Example 4.3-9, and note that by the symmetry of the joint PDF in x, y, it follows that the marginal PDF of Y is the same as that of X.)