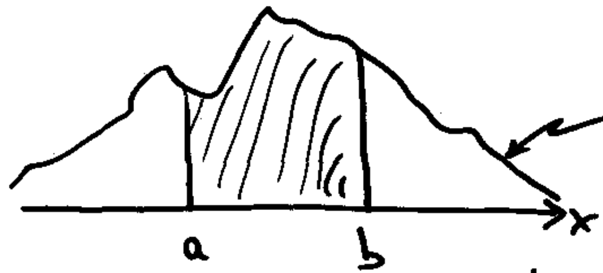
Stat 451
2-1-18

1

Continuous Random Variables



$f(x)$ = probability density function (pdf)

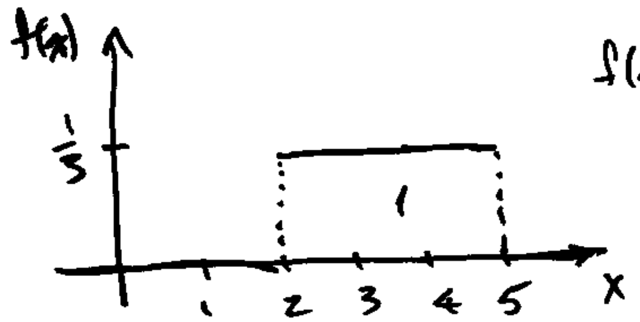
$$P(a < X < b) = \int_a^b f(x)\,dx$$

Properties of pdfs:

1. $f(x) \ge 0 \quad \forall x$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

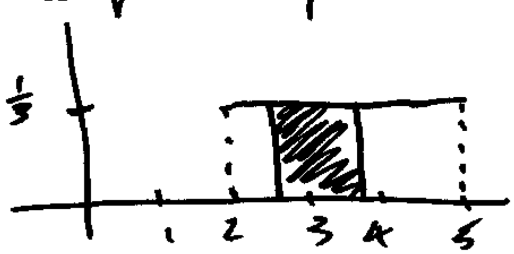
2

Example 1 X takes on any value between 2 & 5, with constant density throughout.



$$f(x) = \begin{cases} 1/3 & 2 < x < 5 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability that X lies between 2.5 and 3.7



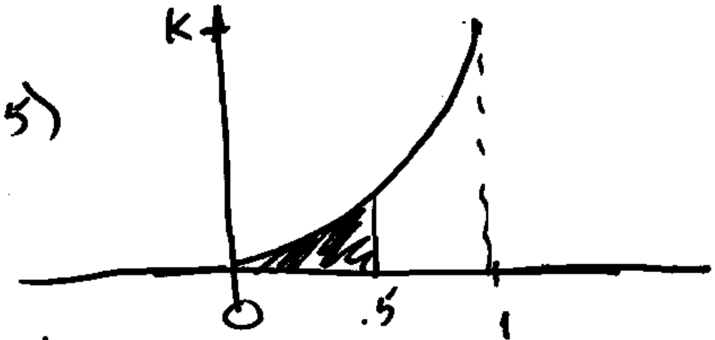
$$P(2.5 < X < 3.7) = (1.2)\left(\tfrac{1}{3}\right) = .4$$

OR $\int_{2.5}^{3.7} \frac{1}{3}\,dx$

③

Example 2 $f(x) = \begin{cases} kx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

① Find k
② Find $P(X < .5)$



① $\int_0^1 kx^2\,dx \overset{\text{set}}{=} 1$

$k\frac{x^3}{3}\Big|_0^1 = \frac{k}{3} - 0 \quad \therefore k = 3$

④

② $P(X<.5) = \int_0^{.5} 3x^2\,dx$

$= x^3\Big|_0^{.5} = .5^3 - 0^3 = .125$

The cumulative distribution function (cdf)

is still $F(x) = P(X \le x)$

$= \int_{-\infty}^{x} f(t)\,dt$

Graph this for Ex1 & Ex2.

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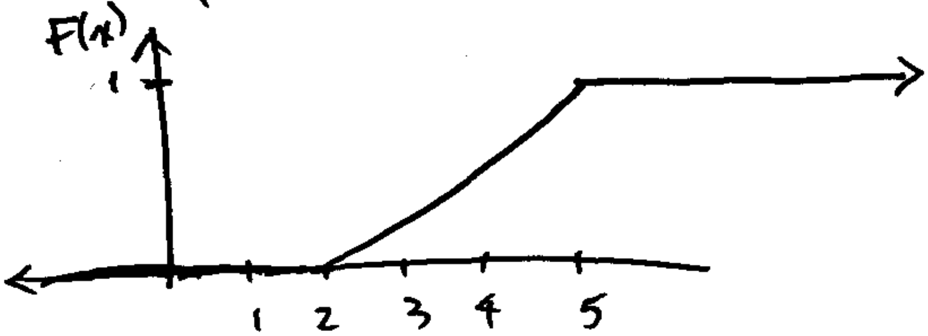
Ex 1: $f(x) = \begin{cases} \frac{1}{3} & 2 < x < 5 \\ 0 & \text{o.w.} \end{cases}$

$$F(x) = \begin{cases} 0 & x \le 2 \\ \frac{1}{3}(x-2) & 2 < x < 5 \\ 1 & x \ge 5 \end{cases} \qquad F(x) = \int_2^x \frac{1}{3}\,dt = \frac{1}{3}t\Big|_2^x$$



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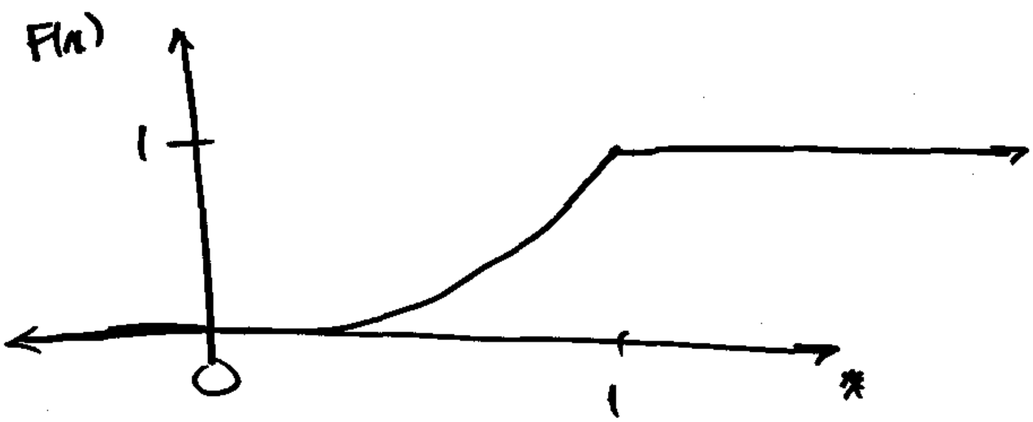
Ex 2: $f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$

$$F(x) = \begin{cases} 0 & x \le 0 \\ x^3 & 0 < x < 1 \\ 1 & x \ge 1 \end{cases} \qquad F(x) = \int_0^x 3t^2\,dt = t^3\Big|_0^x$$

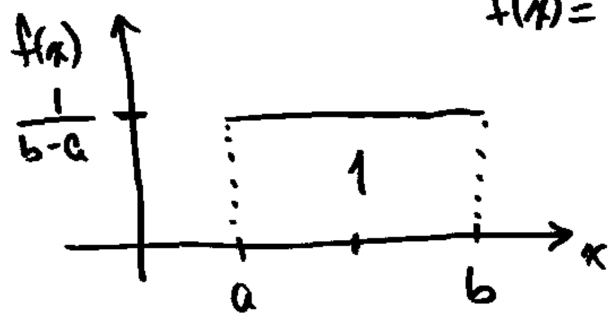


7

The Uniform Distribution

X takes on values in the interval (a,b) with constant density

$$f(x) = \begin{cases} \frac{1}{b-a} & a<x<b \\ 0 & \text{o.w.} \end{cases}$$



For a continuous distribution,

$$\mu = E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$

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$$\mu = \int_a^b x \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a}\,\frac{x^2}{2}\Bigg|_a^b$$

$$= \frac{1}{b-a}\cdot\frac{1}{2}\cdot\overbrace{(b^2-a^2)}^{(b+a)(b-a)} = \frac{b+a}{2}$$

For continuous random variables,

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

and $\sigma^2 = E[X^2] - \mu^2$

(9)

$$E[X^2] = \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{1}{b-a}\,\frac{x^3}{3}\Big|_a^b$$

$$= \frac{1}{b-a}\cdot\frac{1}{3}\underbrace{(b^3-a^3)}_{(b-a)(b^2+ab+a^2)} = \frac{1}{3}(b^2+ab+a^2)$$

$$\sigma^2 = \frac{1}{3}(b^2+ab+a^2) - \left(\frac{a+b}{2}\right)^2$$

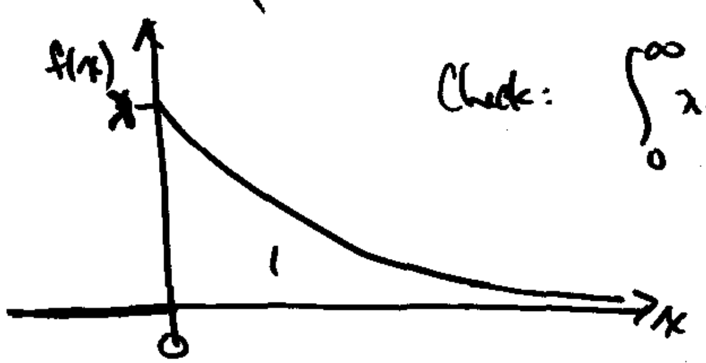
$$= \frac{4b^2+4ab+4a^2-(3a^2+6ab+3b^2)}{12}$$

$$= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}$$

(10)

<u>The exponential distribution</u>

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{o.w.} \end{cases} \qquad \lambda > 0$$



Check: $\int_0^\infty \lambda e^{-\lambda x}\,dx$

$$= -e^{-\lambda x}\Big|_0^\infty$$

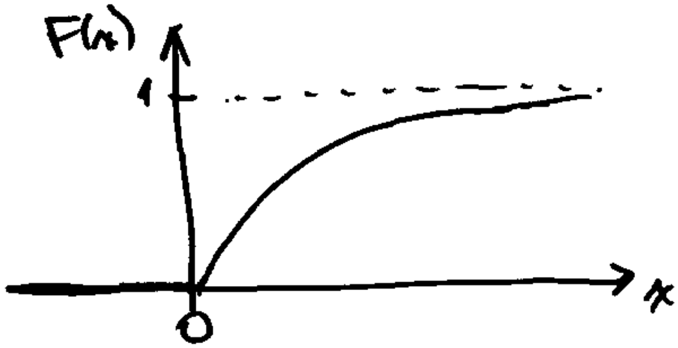
$$= (0-(-1))$$

$$= 1 \checkmark$$

Find $F(x)$, μ, σ^2 (11)

$$F(x) = \begin{cases} 0 & x \le 0 \\ 1-e^{-\lambda x} & x > 0 \end{cases}$$

$$F(x) = \int_0^x \lambda e^{-\lambda t}\,dt = -e^{-\lambda t}\Big|_0^x = -e^{-\lambda x} - (-1) = 1 - e^{-\lambda x}$$



(12)

$$\mu = E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} x \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_0^{\infty} x e^{-\lambda x}\,dx$$

let $u = x$ $du = dx$

$dv = e^{-\lambda x}dx$

$v = -\frac{1}{\lambda}e^{-\lambda x}$

$$= \lambda\left[uv - \int v\,du\right]$$

$$= \lambda\left(\left[-\frac{x}{\lambda}e^{-\lambda x}\right]_0^{\infty} - \int_0^{\infty} -\frac{1}{\lambda}e^{-\lambda x}dx\right)$$

$$= \lambda\left(\underset{\substack{\uparrow \\ \text{by L'Hôpital}}}{0} - 0 + \int_0^{\infty}\frac{1}{\lambda}e^{-\lambda x}dx\right)$$

$$\mu = -\frac{1}{\lambda}e^{-\lambda x}\Big|_0^\infty = 0 - \left(-\frac{1}{\lambda}\right) = \frac{1}{\lambda}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^\infty x^2 \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_0^\infty x^2 e^{-\lambda x}\,dx \qquad \text{Let } u = x^2 \quad du = 2x\,dx$$

$$dv = e^{-\lambda x}dx \qquad v = -\frac{1}{\lambda}e^{-\lambda x}$$

$$= \lambda\left[\underbrace{\cancel{-\frac{x^2}{\lambda}e^{-\lambda x}\Big|_0^\infty}}_{0 \text{ by L'Hôpital}} - \int_0^\infty -\frac{2x}{\lambda}e^{-\lambda x}\,dx\right]$$

$$= 2\frac{1}{\lambda}\underbrace{\int_0^\infty \lambda x e^{-\lambda x}\,dx}_{\mu} = 2\cdot\frac{1}{\lambda}\cdot\frac{1}{\lambda} = \frac{2}{\lambda^2}$$

$$\therefore \sigma^2 = E[X^2] - \mu^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Relationship between the Poisson & Exponential

Poisson ↑ discrete

Exponential ↑ continuous

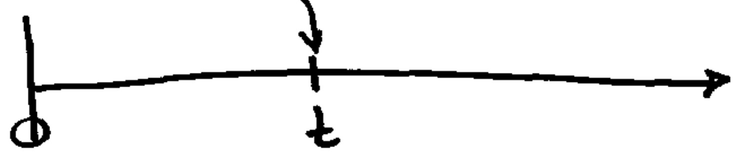
Assume $X \sim \text{Poisson}(\alpha)$

α: expected # occurrences per unit time

(15)

Let T be the time until the 1st occurrence

Find $P(T > t) = P(X = 0)$ ← # occurrences in $(0, t)$



$$P(X=0) = \frac{(\alpha t)^x e^{-\alpha t}}{x!}\Bigg|_{x=0} \quad \left(\begin{array}{l}\alpha \text{ was expected} \\ \# \text{ per unit time}\end{array}\right)$$

$$= e^{-\alpha t}$$

$P(T > t) = e^{-\alpha t}$ $\quad \therefore \; P(T \le t) = 1 - e^{-\alpha t}$,

which is the cdf for the exponential dist

(16)

If X has a Poisson distribution,

then the waiting time until the 1st occurrence has an exponential distribution

The Poisson was a limit of the binomial, so the exponential must be a limit of the geometric

HW #4 due 2/8

p.156 # 1, 2, 5, 7, 10

Stat 451 HW#4

1. The lifespan of a car battery averages six years. Suppose the battery lifespan follows an exponential distribution.

(a) Find the probability that a randomly selected car battery will last more than four years.

(b) Find the variance and the 95th percentile of the battery lifespan.

(c) Suppose a three-year-old battery is still going strong.
   (i) Find the probability the battery will last an additional five years.
   (ii) How much longer is this battery expected to last?

2. The number of wrongly dialed phone calls you receive can be modeled as a Poisson process with the rate of one per month.

(a) Find the probability that it will take between two and three weeks to get the first wrongly dialed phone call.

(b) Suppose that you have not received a wrongly dialed phone call for two weeks. Find the expected value and variance of the additional time until the next wrongly dialed phone call.

5. The yield strength (ksi) for A36 steel is normally distributed with $\mu = 43$ and $\sigma = 4.5$.

(a) What is the 25th percentile of the distribution of A36 steel strength?

(b) What strength value separates the strongest 10% from the others?

(c) What is the value of c such that the interval $(43 - c, 43 + c)$ includes 99% of all strength values?

(d) What is the probability that at most three of 15 independently selected A36 steels have strength less than 43?

7. The resistance for resistors of a certain type is a random variable X having the normal distribution with mean 9 ohms and standard deviation 0.4 ohms. A resistor is acceptable if its resistance is between 8.6 and 9.8 ohms.

(a) What is the probability that a randomly chosen resistor is acceptable?

(b) What is the probability that out of four randomly and independently selected resistors, two are acceptable?

10. A machine manufactures tires with a tread thickness that is normally distributed with mean 10 millimeters (mm) and standard deviation 2 mm. The tire has a 50,000-mile warranty. In order to last for 50,000 miles the tread thickness must be at least 7.9 mm. If the thickness of tread is measured to be less than 7.9 mm, then the tire is sold as an alternative brand with a warranty of less than 50,000 miles.

(a) Find the expected proportion of tires sold under the alternative brand.

(b) The demand for the alternative brand of tires is such that 30% of the total output should be sold under the alternative brand name. What should the critical thickness, originally 7.9 mm, be set at in order to meet the demand?