An expository review of the quaternionization scheme for two player quantum games with maximal entanglement developed by Steve Landsburg
(http://rcer.econ.rochester.edu/RCERPAPERS/rcer_524.pdf)

Bleiler Group
(www.web.pdx.edu/~faisal/seminar)

Note: This review may contain typographical errors, and we would appreciate it if you notify us of any that you find at faisal@pdx.edu

Let G be a two player quantum game in which a player indicates the choice of a strategy from among two strategies of "defect" (D) or "cooperate" (C) by leaving a quantum penny in the state heads or flipping it over to the state tails, respectively. Hence we may say that his strategies are "flip" (F) and "no flip" (N).

Suppose the initial pennies (qubits) given to both players are maximally entangled. That is, the state of the initial qubits is $|HH\rangle + |TT\rangle$, which in the standard computational basis is $|00\rangle + |11\rangle = (1,0,0,1)^T$.

Let the strategies of the two players be $S_1 = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}$, $S_2 = \begin{pmatrix} P & Q \\ -\overline{Q} & \overline{P} \end{pmatrix}$ respectively. Then the "combined" strategy matrix acting on the combined state vector of the two initial qubits is

$$S_1 \otimes S_2 = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \otimes \begin{pmatrix} P & Q \\ -\overline{Q} & \overline{P} \end{pmatrix} = \begin{pmatrix} AP & AQ & BP & BQ \\ -A\overline{Q} & A\overline{P} & -B\overline{Q} & B\overline{P} \\ -\overline{B}P & -\overline{B}Q & \overline{A}P & \overline{A}Q \\ \overline{B}\overline{Q} & -\overline{B}\overline{P} & -\overline{A}\overline{Q} & \overline{A}\overline{P} \end{pmatrix} \begin{matrix} HH \\ HT \\ TH \\ TT \end{matrix}$$

The action of this matrix on the entangled state vector is

$$\begin{pmatrix} AP & AQ & BP & BQ \\ -A\overline{Q} & A\overline{P} & -B\overline{Q} & B\overline{P} \\ -\overline{B}P & -\overline{B}Q & \overline{A}P & \overline{A}Q \\ \overline{B}\overline{Q} & -\overline{B}\overline{P} & -\overline{A}\overline{Q} & \overline{A}\overline{P} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{matrix} HH \\ HT \\ TH \\ TT \end{matrix} = \begin{pmatrix} AP + BQ \\ -A\overline{Q} + B\overline{P} \\ -\overline{B}P + \overline{A}Q \\ \overline{B}\overline{Q} + \overline{A}\overline{P} \end{pmatrix} \begin{matrix} HH \\ HT \\ TH \\ TT \end{matrix} \qquad (1)$$

Note however, that the result needed to compute the payoffs has to be given in the F/N basis rather than the H/T basis. To compute the change of basis matrix, we note that the strategy (N) corresponds to the identity matrix $N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, while the strategy (F) corresponds to the matrix $F = \begin{pmatrix} 0 & \eta \\ -\overline{\eta} & 0 \end{pmatrix}$, where $\eta = e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}}(i+1)$ represents the phase.

The change of basis matrix for moving from the H/T basis to F/N basis can be computed as follows. First we represent the F/N basis in terms of the H/T basis by computing the action of the F/N matrices on the entangled pennies in the H/T basis. The combined matrices are:

$$N \otimes N = NN = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F \otimes N = FN = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & \eta \\ -\bar{\eta} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \eta & 0 & 0 \\ -\bar{\eta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta \\ 0 & 0 & -\bar{\eta} & 0 \end{pmatrix}$$

$$N \otimes F = NF = \begin{pmatrix} 0 & \eta \\ -\bar{\eta} & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & \eta \\ -\bar{\eta} & 0 & 0 & 0 \\ 0 & -\bar{\eta} & 0 & 0 \end{pmatrix}$$

$$F \otimes F = FF = \begin{pmatrix} 0 & \eta \\ -\bar{\eta} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \eta \\ -\bar{\eta} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \eta^2 \\ 0 & 0 & -\eta\bar{\eta} & 0 \\ 0 & -\bar{\eta}\eta & 0 & 0 \\ \bar{\eta}^2 & 0 & 0 & 0 \end{pmatrix}$$

and the result of their action on the entangled pennies in the H/T basis is:

$$N \otimes N = NN = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{matrix} HH \\ HT \\ TH \\ TT \end{matrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{matrix} HH \\ HT \\ TH \\ TT \end{matrix} = |HH\rangle + |TT\rangle$$

$$F \otimes N = FN = \begin{pmatrix} 0 & \eta & 0 & 0 \\ -\bar{\eta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta \\ 0 & 0 & -\bar{\eta} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{matrix} HH \\ HT \\ TH \\ TT \end{matrix} = \begin{pmatrix} 0 \\ -\bar{\eta} \\ \eta \\ 0 \end{pmatrix} \begin{matrix} HH \\ HT \\ TH \\ TT \end{matrix} = -\bar{\eta}|HT\rangle + \eta|TH\rangle$$

$$N \otimes F = NF = \begin{pmatrix} 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & \eta \\ -\bar{\eta} & 0 & 0 & 0 \\ 0 & -\bar{\eta} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{matrix} HH \\ HT \\ TH \\ TT \end{matrix} = \begin{pmatrix} 0 \\ \eta \\ -\bar{\eta} \\ 0 \end{pmatrix} \begin{matrix} HH \\ HT \\ TH \\ TT \end{matrix} = \eta \left| HT \right\rangle - \bar{\eta} \left| TH \right\rangle$$

$$F \otimes F = FF = \begin{pmatrix} 0 & 0 & 0 & \eta^2 \\ 0 & 0 & -\eta\bar{\eta} & 0 \\ 0 & -\bar{\eta}\eta & 0 & 0 \\ \bar{\eta}^2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{matrix} HH \\ HT \\ TH \\ TT \end{matrix} = \begin{pmatrix} \eta^2 \\ 0 \\ 0 \\ \bar{\eta}^2 \end{pmatrix} \begin{matrix} HH \\ HT \\ TH \\ TT \end{matrix} = \eta^2 \left| HH \right\rangle + \bar{\eta}^2 \left| TT \right\rangle = i \left| HH \right\rangle - i \left| TT \right\rangle$$

In short:

$$NN = \left| HH \right\rangle + \left| TT \right\rangle$$
$$FN = -\bar{\eta} \left| HT \right\rangle + \eta \left| TH \right\rangle$$
$$NF = \eta \left| HT \right\rangle - \bar{\eta} \left| TH \right\rangle$$
$$FF = i \left| HH \right\rangle - i \left| TT \right\rangle$$

That is

$$\begin{matrix} NN & FN & NF & FF \end{matrix}$$
$$\begin{pmatrix} 1 & 0 & 0 & i \\ 0 & -\bar{\eta} & \eta & 0 \\ 0 & \eta & -\bar{\eta} & 0 \\ 1 & 0 & 0 & -i \end{pmatrix} \begin{matrix} HH \\ HT \\ TH \\ TT \end{matrix}$$

So the basis change matrix going from the F/N basis to H/T basis is

$$X = \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & -\bar{\eta} & \eta & 0 \\ 0 & \eta & -\bar{\eta} & 0 \\ 1 & 0 & 0 & -i \end{pmatrix}, \text{ normalizing, this gives } X = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & -\bar{\eta} & \eta & 0 \\ 0 & \eta & -\bar{\eta} & 0 \\ 1 & 0 & 0 & -i \end{pmatrix}$$

Hence, the change of basis matrix from H/T to F/N basis is

$$X^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -\eta & \bar{\eta} & 0 \\ 0 & \bar{\eta} & -\eta & 0 \\ -i & 0 & 0 & i \end{pmatrix}$$

We can now express the outcome vectors from the action of the combined strategy matrix of the two players in terms of F/N basis vectors as follows.

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -\eta & \bar{\eta} & 0 \\ 0 & \bar{\eta} & -\eta & 0 \\ -i & 0 & 0 & i \end{pmatrix} \begin{pmatrix} AP+BQ \\ -A\bar{Q}+B\bar{P} \\ -\bar{B}P+\bar{A}Q \\ \bar{B}\bar{Q}+\bar{A}\bar{P} \end{pmatrix} \begin{matrix} HH \\ HT \\ TH \\ TT \end{matrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} (AP+BQ)+(\bar{B}\bar{Q}+\bar{A}\bar{P}) \\ -\eta(-A\bar{Q}+B\bar{P})+\bar{\eta}(-\bar{B}P+\bar{A}Q) \\ \bar{\eta}(-A\bar{Q}+B\bar{P})-\eta(-\bar{B}P+\bar{A}Q) \\ -i(AP+BQ)+i(\bar{B}\bar{Q}+\bar{A}\bar{P}) \end{pmatrix} \begin{matrix} NN \\ NF \\ FN \\ FF \end{matrix}$$

Rewriting as a linear combination we get:

$$\begin{aligned} &\frac{1}{\sqrt{2}}\left[(AP+BQ)+(\bar{B}\bar{Q}+\bar{A}\bar{P})\right]NN + \frac{1}{\sqrt{2}}\left[-\eta(-A\bar{Q}+B\bar{P})+\bar{\eta}(-\bar{B}P+\bar{A}Q)\right]NF \\ &+\frac{1}{\sqrt{2}}\left[\bar{\eta}(-A\bar{Q}+B\bar{P})-\eta(-\bar{B}P+\bar{A}Q)\right]FN + \frac{1}{\sqrt{2}}\left[-i(AP+BQ)+i(\bar{B}\bar{Q}+\bar{A}\bar{P})\right]FF \end{aligned} \qquad (1)$$

Now Let $A=(A_0+A_1 i),\ B=(B_0+B_1 i),\ P=(P_0+P_1 i),\ Q=(Q_0+Q_1 i)$. Then

$$\begin{aligned} &(AP+BQ)=(A_0+A_1 i)(P_0+P_1 i)+(B_0+B_1 i)(Q_0+Q_1 i) \\ &=(A_0P_0+A_1P_0 i+A_0P_1 i-A_1P_1)+(B_0Q_0+B_1Q_0 i+B_0Q_1 i-B_1Q_1) \\ &=(A_0P_0+B_0Q_0-A_1P_1-B_1Q_1)+(A_1P_0+A_0P_1+B_1Q_0+B_0Q_1)i \end{aligned}$$

and

$$\begin{aligned} &(\bar{B}\bar{Q}+\bar{A}\bar{P})=(\overline{BQ+AP}) \\ &=(A_0P_0+B_0Q_0-A_1P_1-B_1Q_1)-(A_1P_0+A_0P_1+B_1Q_0+B_0Q_1)i \end{aligned}$$

Hence, the coefficients of *NN* and *FF* if the expression in (1) are

$$(AP+BQ)+(\bar{B}\bar{Q}+\bar{A}\bar{P})=2\operatorname{Re}\{AP+BQ\}=(A_0P_0+B_0Q_0-A_1P_1-B_1Q_1) \qquad \text{(X)}$$

$$\begin{aligned} &-i(AP+BQ)+i(\bar{B}\bar{Q}+\bar{A}\bar{P})=-i\left[(AP+BQ)-(\bar{B}\bar{Q}+\bar{A}\bar{P})\right] \\ &=2i\operatorname{Im}\{(AP+BQ)\}i=-2(A_1P_0+A_0P_1+B_1Q_0+B_0Q_1) \qquad \text{(Y)} \end{aligned}$$

Also,

$$(-A\bar{Q}+B\bar{P}) = (-A_0 - A_1 i)(Q_0 - Q_1 i) + (B_0 + B_1 i)(P_0 - P_1 i)$$
$$= (-A_0Q_0 - A_1Q_0 i + A_0Q_1 i - A_1Q_1) + (B_0P_0 + B_1P_0 i - B_0P_1 i + B_1P_1)$$
$$= (B_0P_0 + B_1P_1 - A_0Q_0 - A_1Q_1) + (B_1P_0 + A_0Q_1 - B_0P_1 - A_1Q_0)i$$

$$(-\bar{B}P + \bar{A}Q) = -(\overline{-A\bar{Q}+B\bar{P}})$$
$$= -\left[(B_0P_0 + B_1P_1 - A_0Q_0 - A_1Q_1) - (B_1P_0 + A_0Q_1 - B_0P_1 - A_1Q_0)i\right]$$
$$= -(B_0P_0 + B_1P_1 - A_0Q_0 - A_1Q_1) + (B_1P_0 + A_0Q_1 - B_0P_1 - A_1Q_0)i$$

Therefore, the coefficients of *NF* and *FN* in the expression in (1) are

$$-\eta(-A\bar{Q}+B\bar{P}) + \bar{\eta}(-\bar{B}P+\bar{A}Q) = -\eta(-A\bar{Q}+B\bar{P}) - \bar{\eta}(\overline{-A\bar{Q}+B\bar{P}})$$
$$= -\left[\eta(-A\bar{Q}+B\bar{P}) + (\overline{\eta(-A\bar{Q}+B\bar{P})})\right] = -2\,\mathrm{Re}\{\eta(-A\bar{Q}+B\bar{P})\}$$
$$= (B_0P_0 + B_1P_1 - A_0Q_0 - A_1Q_1) - (B_1P_0 + A_0Q_1 - B_0P_1 - A_1Q_0) \qquad \text{(W)}$$
$$= \mathrm{Re}\{(-A\bar{Q}+B\bar{P})\} - \mathrm{Im}\{(-A\bar{Q}+B\bar{P})\}$$

$$\bar{\eta}(-A\bar{Q}+B\bar{P}) - \eta(-\bar{B}P+\bar{A}Q) = \bar{\eta}(-A\bar{Q}+B\bar{P}) - \eta(\overline{-A\bar{Q}+B\bar{P}})$$
$$= -i\left[\eta(-A\bar{Q}+B\bar{P}) - (\overline{\eta(-A\bar{Q}+B\bar{P})})\right] = -2\,\mathrm{Im}\{\eta(-A\bar{Q}+B\bar{P})\} \qquad \text{(Z)}$$
$$(B_0P_0 + B_1P_1 - A_0Q_0 - A_1Q_1) + (B_1P_0 + A_0Q_1 - B_0P_1 - A_1Q_0)$$
$$= \mathrm{Re}\{(-A\bar{Q}+B\bar{P})\} + \mathrm{Im}\{(-A\bar{Q}+B\bar{P})\}$$

Now identify the unitary matrices $\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$ and $\begin{pmatrix} P & -\bar{Q} \\ Q & \bar{P} \end{pmatrix}$ with unit quaternions as $p = A + B\eta j,\ q = P - \eta jQ$, then

$$pq = (A + B\eta j)(P - \eta jQ) = AP + B\eta jP - A\eta jQ - B\eta j\eta jQ$$

$$= AP + \eta B\overline{P}j - \eta A\overline{Q}j - \eta B\overline{\eta}\, j^2 Q$$

$$= AP + \eta B\overline{P}j - \eta A\overline{Q}j + \eta B\overline{\eta} Q$$

$$= AP + \eta B\overline{P}j - \eta A\overline{Q}j + BQ$$

$$= \left(AP + BQ\right) + \eta(B\overline{P}j - A\overline{Q}j)$$

$$= \left(AP + BQ\right) + \eta(B\overline{P} - A\overline{Q})\, j$$

$$= \left(AP + BQ\right) + (B\overline{P} - A\overline{Q})\eta\, j$$

$$= \left(AP + BQ\right) + (-A\overline{Q} + B\overline{P})\eta\, j$$

$$= \left[(A_0 + A_1 i)(P_0 + P_1 i) + (B_0 + B_1 i)(Q_0 + Q_1 i)\right] + \left[(-A_0 - A_1 i)(Q_0 - Q_1 i) + (B_0 + B_1 i)(P_0 - P_1 i)\right]\eta\, j$$

$$= (A_0P_0 + A_1P_0 i + A_0P_1 i - A_1P_1) + (B_0Q_0 + B_1Q_0 i + B_0Q_1 i - B_1Q_1) + (-A_0Q_0 - A_1Q_0 i + A_0Q_1 i - A_1Q_1)\eta\, j$$
$$+(B_0P_0 + B_1P_0 i - B_0P_1 i + B_1P_1)\eta\, j$$

$$= A_0P_0 + B_0Q_0 - A_1P_1 - B_1Q_1 + A_1P_0 i + A_0P_1 i + B_1Q_0 i + B_0Q_1 i - A_0Q_0\eta\, j + B_0P_0\eta\, j + B_1P_1\eta\, j - A_1Q_1\eta\, j$$
$$-A_1Q_0\eta ij + A_0Q_1\eta ij + B_1P_0\eta ij - B_0P\eta_1 ij$$

$$= (A_0P_0 + B_0Q_0 - A_1P_1 - B_1Q_1) + (A_1P_0 + A_0P_1 + B_1Q_0 + B_0Q_1)i + (-A_0Q_0 + B_0P_0 + B_1P_1 - A_1Q_1)\eta\, j$$
$$+ (-A_1Q_0 + A_0Q_1 + B_1P_0 - B_0P_1)\eta k$$

$$= (A_0P_0 + B_0Q_0 - A_1P_1 - B_1Q_1) + (A_1P_0 + A_0P_1 + B_1Q_0 + B_0Q_1)i + (-A_0Q_0 + B_0P_0 + B_1P_1 - A_1Q_1)(1+i)\, j$$
$$+ (-A_1Q_0 + A_0Q_1 + B_1P_0 - B_0P_1)(1+i)k$$

$$= (A_0P_0 + B_0Q_0 - A_1P_1 - B_1Q_1) + (A_1P_0 + A_0P_1 + B_1Q_0 + B_0Q_1)i + (-A_0Q_0 + B_0P_0 + B_1P_1 - A_1Q_1)\, j$$
$$(-A_0Q_0 + B_0P_0 + B_1P_1 - A_1Q_1)ij + (-A_1Q_0 + A_0Q_1 + B_1P_0 - B_0P_1)k + (-A_1Q_0 + A_0Q_1 + B_1P_0 - B_0P_1)ik$$

$$= (A_0P_0 + B_0Q_0 - A_1P_1 - B_1Q_1) + (A_1P_0 + A_0P_1 + B_1Q_0 + B_0Q_1)i + (-A_0Q_0 + B_0P_0 + B_1P_1 - A_1Q_1)j$$
$$(-A_0Q_0 + B_0P_0 + B_1P_1 - A_1Q_1)k \;\; + (-A_1Q_0 + A_0Q_1 + B_1P_0 - B_0P_1)k - (-A_1Q_0 + A_0Q_1 + B_1P_0 - B_0P_1)j$$

$$= (A_0P_0 + B_0Q_0 - A_1P_1 - B_1Q_1) + (A_1P_0 + A_0P_1 + B_1Q_0 + B_0Q_1)i$$
$$+(-A_0Q_0 + B_0P_0 + B_1P_1 - A_1Q_1)j - (-A_1Q_0 + A_0Q_1 + B_1P_0 - B_0P_1)j$$
$$(-A_0Q_0 + B_0P_0 + B_1P_1 - A_1Q_1)k \;\; + (-A_1Q_0 + A_0Q_1 + B_1P_0 - B_0P_1)k$$

$$= (A_0P_0 + B_0Q_0 - A_1P_1 - B_1Q_1) + (A_1P_0 + A_0P_1 + B_1Q_0 + B_0Q_1)i$$
$$+\left[(-A_0Q_0 + B_0P_0 + B_1P_1 - A_1Q_1) - (-A_1Q_0 + A_0Q_1 + B_1P_0 - B_0P_1)\right]j \qquad \text{(M)}$$
$$+\left[(-A_0Q_0 + B_0P_0 + B_1P_1 - A_1Q_1) \;\; + (-A_1Q_0 + A_0Q_1 + B_1P_0 - B_0P_1)\right]k$$

Note that in expression (M), the real part of the quaternion *pq* and the coefficients of the *i,j,k* are, respectively:

$\pi_1(pq) = (A_0P_0 + B_0Q_0 - A_1P_1 - B_1Q_1)$ Proportional to coefficient of expression (X) *NN*

$\pi_2(pq) = (A_1P_0 + A_0P_1 + B_1Q_0 + B_0Q_1)$ Proportional to coefficient of expression (Y) *FF*

$$\pi_3(pq) = (-A_0Q_0 + B_0P_0 + B_1P_1 - A_1Q_1)\eta$$
$$= (-A_0Q_0 + B_0P_0 + B_1P_1 - A_1Q_1) - (-A_1Q_0 + A_0Q_1 + B_1P_0 - B_0P_1)$$
$$= \mathrm{Re}\{(-A\bar{Q} + B\bar{P})\} - \mathrm{Im}\{(-A\bar{Q} + B\bar{P})\}$$

Proportional to coefficient of (W) *NF*

$$\pi_4(pq) = (-A_1Q_0 + A_0Q_1 + B_1P_0 - B_0P_1)\eta$$
$$= (-A_0Q_0 + B_0P_0 + B_1P_1 - A_1Q_1) \;\; + (-A_1Q_0 + A_0Q_1 + B_1P_0 - B_0P_1)$$
$$= \mathrm{Re}\{(-A\bar{Q} + B\bar{P})\} + \mathrm{Im}\{(-A\bar{Q} + B\bar{P})\}$$

Proportional to coefficient of (Z) *FN*

Hence, the lengths of the coefficients of (1) are the same as the lengths of the numbers above. So the pay off can be calculated as

$$\Pr ob(NN) = \pi_1(pq)^2$$

$$\Pr ob(NF) = \pi_4(pq)^2$$

$$\Pr ob(FN) = \pi_1(pq)^2$$

$$\Pr ob(FF) = \pi_2(pq)^2$$