Homework #3 - due March 8, in class

Problem 1 (50 points) Consider the wave equation

$$\begin{aligned} u_{tt} - \Delta u &= 0, \quad x \in \Omega, \quad t > 0 &(1)\\ u &= 0, \quad \text{on } \partial\Omega \times (0,\infty) &(2)\\ u(x,0) &= u_0(x) \quad x \in \Omega &(3)\\ u_t(x,0) &= v_0(x) \quad x \in \Omega &(4) \end{aligned}$$

Introduce a new variable $v = u_t$ to obtain an equivalent formulation

$$\begin{aligned} u_t - v &= 0 &(5)\\ v_t - \Delta u &= 0 &(6)\\ u(x,0) &= u_0(x) &(7)\\ v(x,0) &= v_0(x) &(8)\\ u|_{\partial\Omega} &= 0 &(9) \end{aligned}$$

For $t \geq 0$ we view

$$U = \begin{bmatrix} u \\ v \end{bmatrix}$$

as a function of t with values in an appropriate Hilbert space. Consider the Hilbert space[1]

$$H = H_0^1(\Omega) \times L^2(\Omega)$$

with the inner product

$$(U_1, U_2)_H = \int_\Omega \nabla u_1 \nabla u_2 \, dx + \int_\Omega u_1 u_2 \, dx + \int_\Omega v_1 v_2 \, dx$$

where

$$U_1 = \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \quad U_2 = \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}$$

Define the linear operator

$$A : D(A) \subset H \to H, \quad AU = \begin{bmatrix} -v \\ -\Delta u \end{bmatrix}$$

where $D(A) = \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega)$. Given $U_0 \in D(A)$, we are interested to study the existence of the solution of the IVP

$$\begin{aligned} \frac{dU}{dt} + AU &= 0 &(10)\\ U(0) &= U_0 &(11) \end{aligned}$$

[1] notice that the boundary conditions are included in the definition of H

(10 points) Show that $A + I : D(A) \to H$ is monotone[2]

(20 points) Show that $A + I : D(A) \to H$ is maximal monotone

Given $U_0 \in D(A)$, apply the Hille-Yosida theorem to conclude that there is an unique solution to

$$
\begin{aligned}
\frac{d\tilde{U}}{dt} + A\tilde{U} + \tilde{U} &= 0 \qquad (12)\\
\tilde{U}(0) &= U_0 \qquad (13)
\end{aligned}
$$

(10 points) Show that $U(t) = e^t \tilde{U}(t)$ solves (10-11).

(10 points) The Hille-Yosida theorem states that the solution $U(t)$ has regularity

$$U \in C([0,\infty); D(A)) \cap C^1([0,\infty); H)$$

Using this result, what is the regularity of the solution $u(t)$?

(fill in the dots $u \in C([0,\infty); \ldots) \cap \ldots([0,\infty); \ldots) \cap \ldots([0,\infty); \ldots)$).

Bonus Problem (20 points): Telegraph Equation[3]

(5 points) Show that there is at most one smooth solution to the initial boundary value problem

$$
\begin{aligned}
u_{tt} + du_t - u_{xx} &= 0, \quad (x,t) \in (0,1) \times (0,T) \qquad (14)\\
u &= 0, \quad (x,t) \in \{0,1\} \times (0,T) \qquad (15)\\
u = g,\ u_t &= h, \quad (x,t) \in (0,1) \times \{0\} \qquad (16)
\end{aligned}
$$

where d is a constant.

(15 points) Write the problem above in the form

$$
\begin{aligned}
\frac{dU}{dt} + AU &= 0 \qquad (17)\\
U(0) &= U_0 \qquad (18)
\end{aligned}
$$

and use the Hille-Yosida theorem to give a result of existence of the solution.

[2] $(A + I)U = AU + U$

[3] See also Evans, section 7.5, problem 9