Homework #2 - due February 18, in class

Hand in solutions to the following problems:

P1 (35 points) Consider the eigenvalue problem:

$$\begin{aligned} u''''(x) &= \lambda u(x), \quad 0 < x < 1 && (1) \\ u(0) &= u'(0) = 0 && (2) \\ u(1) &= u'(1) = 0 && (3) \end{aligned}$$

(5p) Show that all eigenvalues are strictly positive.

(30 points): Show that there is a basis to $L^2(0,1)$ that consists of the eigenfunctions $\{u_k\}_{k\geq 1}$ to the problem above and the eigenvalues are such that $\lambda_k \to \infty$. In your proof complete the following steps.

(5p) Let $f \in L^2(0,1)$. In the Hilbert space $H_0^2(0,1)$, give a variational formulation to the problem

$$\begin{aligned} u''''(x) &= f(x), \quad 0 < x < 1 && (4) \\ u(0) &= u'(0) = 0 && (5) \\ u(1) &= u'(1) = 0 && (6) \end{aligned}$$

(10p) Show that there is a unique weak solution $u \in H_0^2(0,1)$.

(10p) Show that the operator $T : L^2(0,1) \to L^2(0,1)$ defined as $T(f) = u$ is linear, continuous, compact, and self-adjoint.

(5p) Use the Hilbert-Schmidt theorem to complete the proof. Explain why the set of eigenvalues is unbounded, $\lambda_k \to \infty$.

Optional bonus points: (5p) Find an equation that is satisfied by the eigenvalues and provide the expression of the eigenfunctions.

P2 (15 points) Let $\Omega = B(0,1)$ denote the open unit ball in $\mathbb{R}^2$. Consider the eigenvalue problem

$$\begin{aligned} -\Delta u &= \lambda u, \text{ in } \Omega && (7) \\ u &= 0, \text{ on } \partial\Omega && (8) \end{aligned}$$

Let $\lambda_1 > 0$ denote the principal (smallest) eigenvalue. Use an analytical approach (no calculator/numerical approximation) to find two constants $c_1 > 0$ and $c_2 > 0$ such that $c_1 \leq \lambda_1 \leq c_2$ (that is, provide a nontrivial lower bound and an upper bound to λ_1).