Homework #1 - due January 26, in class

Hand in solutions to the following problems:

**P1 (30 points)** Let $\Omega \subset \mathcal{R}^n$ an open and bounded domain with smooth boundary $\partial\Omega$. Consider the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathcal{R}$ defined as

$$a(u,v) = \int_\Omega \nabla u \nabla v \, dx + \left(\int_\Omega u \, dx\right)\left(\int_\Omega v \, dx\right) \tag{1}$$

(i) [5p] Show that $a(\cdot,\cdot)$ is $H^1$-continuous

(ii) [15p] Show that $a(\cdot,\cdot)$ is $H^1$-elliptic

(iii) [2p] From (i) and (ii), conclude that for any $f \in L^2(\Omega)$ there is a unique solution $u \in H^1(\Omega)$ to the problem

$$a(u,v) = \int_\Omega f v \, dx, \quad \forall v \in H^1(\Omega) \tag{2}$$

(iv) [3p] Show that the solution $u$ to (2) satisfies

$$\int_\Omega u \, dx = \frac{1}{|\Omega|} \int_\Omega f \, dx \tag{3}$$

(v) [5p] Assuming that the solution $u$ is a smooth function, what PDE does it solve?

**P2 (20 points)** Let $\Omega \subset \mathcal{R}^n$ an open and bounded domain with smooth boundary $\partial\Omega$. Consider the boundary value problem

$$-\Delta u = f \quad \text{in}\, \Omega \tag{4}$$

$$-\frac{\partial u}{\partial \vec{\nu}} = \tau u \quad \text{on}\, \partial\Omega \tag{5}$$

where $\tau > 0$ is a positive constant and $f \in L^2(\Omega)$ .

(i) [5p] Give a variational formulation to the problem (4)-(5).

(ii) [15p] Show that the variational problem has a unique solution $u \in H^1(\Omega)$ (show that the hypotheses of the Lax-Milgram theorem are satisfied).