The Hille-Yosida theorem

→ provides the theoretical framework to the linear initial-value problems set in Hilbert spaces.
For functions evolving in time & space, $u(x,t)$, at any fixed time can view $u(\cdot,t) \in H$ where H denotes an appropriate Sobolev space ($H_0^1, H^1, \ldots$)
Thus the theory will provide the framework to the existence, uniqueness, and regularity of the solutions to PDEs (IBVP).

Let H Hilbert space w.r.t. $(\cdot,\cdot)$.

Definition Let $A : D(A) \subset H \to H$ a linear unbounded operator. A is called monotone if
$(Av, v) \geq 0, \quad \forall v \in D(A)$

A is called maximal monotone if in addition

$$R(I+A) = H$$

that is $\forall f \in H$, there is $u \in D(A)$: $u + Au = f$

Remarks : Notice that we do not require A to be a continuous (bounded) operator.

For example, the Laplacian operator $A = -\Delta$:

$$-\Delta : H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$$

Here $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$ is viewed as a subspace of $L^2(\Omega)$. The Hilbert space is $H = L^2(\Omega)$

$-\Delta$ is unbounded operator w.r.t. L^2-norm since there is no constant $C \geq 0$ such that

$$\int_\Omega |\Delta u|^2 \, dx \leq C \int_\Omega u^2 \, dx, \quad \forall u \in H^2(\Omega) \cap H_0^1(\Omega)$$

Example in $L^2(0,\pi)$ $\qquad u_k = (\sin kx) \cdot \sqrt{\frac{2}{\pi}}$

$$u_k'' = -k^2 (\sin kx) \sqrt{\frac{2}{\pi}}$$

$\|u_k\|_{L^2}^2 = 1 \qquad \|u_k''\|_{L^2}^2 = k^4$, thus $\|u_k''\|_{L^2} \xrightarrow[k]{} \infty$.

$-\Delta$ is monotone operator on $H^2(\Omega) \cap H_0^1(\Omega)$

$$\int_\Omega -(\Delta u) u = \int_\Omega |\nabla u|^2 \geq 0.$$

$-\Delta$ is maximal monotone operator since

$\forall f \in L^2(\Omega), \exists u \in H^2(\Omega) \cap H_0^1(\Omega) : \quad -\Delta u + u = f$

Hille-Yosida theorem Let $A : D(A) \subset H \to H$ denote a maximal monotone operator in a Hilbert space H. Then $\forall u_0 \in D(A)$ there is a unique function

$u \in C^1([0,\infty); H) \cap C([0,\infty), D(A))$

such that
$$\begin{cases} \dfrac{du}{dt} + Au = 0, & t \geq 0 \\ u(0) = u_0 \end{cases}$$

in addition,
$$\begin{cases} \|u(t)\| \leq \|u_0\|, & \forall t \geq 0 \\ \left\|\dfrac{du}{dt}(t)\right\| = \|Au(t)\| \leq \|Au_0\|, & \forall t \geq 0 \end{cases}$$

if A is in addition a self-adjoint operator, then $\forall u_0 \in H$ there is a unique

$u \in C([0,\infty), H) \cap C^1((0,\infty); H) \cap C((0,\infty); D(A))$

such that
$$\begin{cases} \dfrac{du}{dt} + Au = 0, & t > 0 \\ u(0) = u_0 \end{cases}$$

in this case, the solution satisfies for any $t > 0$: $\|u(t)\| \leq \|u_0\|$ and $\left\|\frac{du}{dt}(t)\right\| = \|Au(t)\| \leq \frac{1}{t}\|u_0\|$

Regularity Define $D(A^k) = \{v \in D(A^{k-1}) : Av \in D(A^{k-1})\}$ for $k \geq 2$.

Then if $u_0 \in D(A^k)$, $u \in C^{k-j}([0,\infty); D(A^j))$, $j = 0, 1, \ldots, k$.

<u>Remark</u> : The major merit of the Hille-Yosida theorem is that it reduces the study of the evolution problem (time-dependent) to the study of the stationary (time-independent) problem $u + \lambda A u = f \qquad (\lambda > 0)$

to show that the operator A is max monotone.

Direct application: heat equation:

$$(*) \quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = 0 & \text{in } \Omega \times (0, \infty) \\ u = 0 & \text{on } \partial\Omega \times [0, \infty) \\ u(x,0) = u_0 & \text{in } \Omega . \end{cases}$$

<u>$\forall u_0 \in L^2(\Omega)$</u> there is a unique function $u(x,t)$ that solves $(*)$ and

$$u \in C([0,\infty) ; L^2(\Omega)) \cap C((0,\infty) ; H^2(\Omega) \cap H_0^1(\Omega))$$
$$u \in C^1((0,\infty) ; L^2(\Omega)) .$$

(more on this later ...).

Properties of maximal monotone operators

Property 1 if A is maximal monotone then

$I + A : D(A) \subset H \to H$ is bijection,

$\forall f \in H : \quad \exists!\, u \in D(A) : \quad u + Au = f$.

Proof Only need to show uniqueness (since A is max monotone).

Let $w = u_1 - u_2$ then $w + Aw = 0$

$\Rightarrow \ (w, w) + (Aw, w) = 0 \ \Rightarrow \ \|w\| = 0$.

Property 2 $T : H \to H$ defined as $T = (I+A)^{-1}$ is linear and continuous operator

$Tf = u \quad : \quad u + Au = f \qquad u = (I+A)^{-1} f$

Proof: $(u, u) + (Au, u) = (f, u) \ \Rightarrow$

$\Rightarrow \ \|u\|^2 \le \|f\| \|u\| \ \Rightarrow \ \|u\| \le \|f\|$

Thus $\|Tf\| \le \|f\|$ such that $\|T\| \le 1$

therefore $\|(I+A)^{-1}\|_{\mathcal{L}(H)} \le 1$ $\qquad \sup\limits_{\|x\| \le 1} \|Tx\|$

<u>Property 3</u> if A is maximal monotone then $D(A)$ is a <u>dense subset</u> of H.

<u>Proof</u> Let $f \in H$ such that $(f, v) = 0$, $\forall v \in D(A)$. Show that then we must have $\underline{f = 0}$.

Since A is max. mon., $\exists u_0 \in D(A) : u_0 + Au_0 = f$

Then $(f, u_0) = (u_0, u_0) + \underbrace{(Au_0, u_0)}_{\geq 0} = 0 \Rightarrow \|u_0\| = 0$

$\Rightarrow u_0 = 0 \Rightarrow f = 0$.

<u>Property 4</u> if A is maximal monotone then $\forall \lambda > 0$, λA is maximal monotone.

(if $(I + A)u = f$ has unique solution then $(I + \lambda A)u = f$ has unique solution).

<u>Proof</u> Clearly $\forall \lambda > 0$, λA is monotone. To prove that λA is max monotone we show that if $\lambda_0 A$ is max monotone for some $\lambda_0 > 0$ then λA is maximal monotone for any $\lambda > \frac{\lambda_0}{2}$

Use Banach fixed point theorem.

$u + \lambda A u = f$ is equivalent to $\left(\cdot \frac{\lambda_0}{\lambda} \right)$

$$u + \lambda_0 A u = \frac{1}{\lambda}\left[\lambda_0 f + (\lambda - \lambda_0) u \right]$$

Then $$u = \frac{1}{\lambda}(I + \lambda_0 A)^{-1}\left[\lambda_0 f + (\lambda - \lambda_0) u \right]$$

a fixed point problem.

Let $$T(v) = \frac{1}{\lambda}(I + \lambda_0 A)^{-1}\left[\lambda_0 f + (\lambda - \lambda_0) v \right]$$

$$\| T(v_1) - T(v_2) \| \leq \left| \frac{\lambda - \lambda_0}{\lambda} \right| \| (I + \lambda_0 A)^{-1} (v_1 - v_2) \| \leq$$

$$\leq \left| 1 - \frac{\lambda_0}{\lambda} \right| \| v_1 - v_2 \|$$

T is contraction if $\left| 1 - \frac{\lambda_0}{\lambda} \right| < 1$ which holds for any $\lambda > \frac{\lambda_0}{2}$

<u>Consequence:</u> Let A maximal monotone. Then $\forall \lambda > 0$, $I + \lambda A : D(A) \to H$ is bijection and $(I + \lambda A)^{-1}$ is a bounded (continuous) operator with $\| (I + \lambda A)^{-1} \|_{\mathcal{L}(H)} \leq 1$

<u>Definition</u> Given X, Y Banach spaces and $T: X \to Y$, the <u>graph of T</u> is defined as

$$G(T) = \{(x, Tx)\} \subset X \times Y$$

<u>Closed Graph Theorem</u> Let $T: X \to Y$, $\begin{pmatrix} X, Y \\ \text{Banach} \end{pmatrix}$ <u>linear operator</u>. T is <u>continuous</u> if and only if $G(T)$ is a closed set in $X \times Y$

$$\left(\left. \begin{array}{l} x_n \to x \\ Tx_n \to y \end{array} \right\} \text{ implies } \underline{Tx = y} \right).$$

<u>Property</u> Let A maximal monotone, $A: D(A) \subset H \to H$. Then A is a <u>closed operator</u> i.e. $G(A)$ is closed subset of $H \times H$.

<u>Proof</u> Let $\{u_n\} \subset D(A)$ such that

$$u_n \to u, \quad Au_n \to f$$

Then $u_n + Au_n \to u + f$ and since A is maximum

$$u_n = (I + A)^{-1}(u_n + Au_n) \to (I + A)^{-1}(u + f)$$

$$\Rightarrow \quad u = (I + A)^{-1}(u + f) \Rightarrow \left\{ \begin{array}{l} \underline{u \in D(A)} \quad \underline{\text{and}} \\ u + Au = u + f \text{ so } \underline{Au = f} \end{array} \right.$$

On $D(A)$ define the graph norm

$$\|u\|_{D(A)} = \|u\| + \|Au\|$$

Remark $(D(A), \|\cdot\|)$ is not complete in general unless $D(A) = H$ and in which case A must be continuous.
However,

Theorem $(D(A), \| \|_{D(A)})$ is Banach space and thus Hilbert space for the inner product

$$(u,v)_{D(A)} = (u,v) + (Au, Av)$$

Proof Let $\{u_n\}$ Cauchy in $D(A)$ w.r.t. $\| \|_{D(A)}$
Then $\{u_n\}$, $\{Au_n\}$ are Cauchy in H, thus $(u_n \to u\ ,\ Au_n \to f)$ and since A is closed operator we must have $Au = f$ such that $u_n \to u$ in $\| \|_{D(A)}$: $\|u_n - u\|_{D(A)} = \|u_n - u\| + \|Au_n - Au\| \to 0$

$$\boxed{\text{Fundamental example / consequence}}$$

$$-\Delta : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$$

if $\begin{cases} v_n \to v & \text{in } L^2(\Omega) \\ -\Delta v_n \to f & \text{in } L^2(\Omega) \end{cases}$ then $\begin{cases} v \in H^2(\Omega) \cap H_0^1(\Omega) \\ \text{and} \\ -\Delta v = f \end{cases}$

in addition, $H_0^1(\Omega) \cap H^2(\Omega)$ is Hilbert w.r.t.

$\|u\|^2_{D(A)} = \|u\|^2_{L^2} + \|\Delta u\|^2_{L^2}$ thus this is equivalent to the H^2-norm $\|u\|_2^2 = \int_\Omega \sum_{|\alpha| \le 2} |D^\alpha u|^2$

<u>Definition</u> Let A maximal monotone operator.
For each $\lambda > 0$ define

i) The <u>resolvent of A</u> as the operator

$$J_\lambda : H \to D(A), \quad J_\lambda = (I + \lambda A)^{-1}$$

ii) <u>Yosida regularization</u> of A as the operator

$$A_\lambda : H \to H, \quad A_\lambda = \frac{1}{\lambda}(I - J_\lambda)$$

<u>Remark</u> $\|J_\lambda\|_{\mathcal{L}(H)} \leq 1$ since λA is max. monotone

<u>Proposition</u> Let A maximal monotone and $\lambda > 0$.
Then

a) $A_\lambda v = A(J_\lambda v), \ \forall v \in H$

b) $A_\lambda v = J_\lambda (Av), \ \forall v \in D(A)$

(this implies that A and J_λ commute on $D(A)$)

c) $\|A_\lambda v\| \leq \|Av\|, \ \forall v \in D(A)$

d) $\lim_{\lambda \to 0} A_\lambda v = Av, \quad \forall v \in D(A)$

e) $\lim_{\lambda \to 0} J_\lambda v = v, \quad \forall v \in H$

f) $(A_\lambda v, v) \geq 0, \quad \forall v \in H$ (so A_λ is monotone)

g) $\|A_\lambda v\| \leq \frac{1}{\lambda}\|v\|, \ \forall v \in H$ (so $\|A_\lambda\| \leq \frac{1}{\lambda}$)

<u>Remark</u> $\{A_\lambda\}_{\lambda > 0}$ is a family of <u>bounded</u> (continuous) operators that approaches A as $\lambda \to 0$, pointwise.
Clearly, $\|A_\lambda\|_{\mathcal{L}(H)} \to \infty$ as $\lambda \to 0$

Proof of the Hille-Yosida theorem (sketch)

Uniqueness: Let u_1, u_2 solutions, denote $u = u_1 - u_2$

Then $\left(\frac{du}{dt}, u\right) = -(Au, u) \le 0 \Rightarrow \frac{1}{2}\frac{d}{dt}\|u(t)\|^2 \le 0$

thus $t \to \|u(t)\|^2$ is decreasing,

$$\|u(t)\| \le \|u(0)\| = 0 \Rightarrow u(t) = 0, \quad \forall t \ge 0.$$

Existence Replace the unbounded operator A by the bounded Yosida regularization operator A_λ,

$$\begin{cases} \frac{du_\lambda}{dt} + A_\lambda u_\lambda = 0 \quad \text{on } [0,\infty) \\ u_\lambda(0) = u_0 \in D(A) \end{cases}$$

and show that $u_\lambda(t) \xrightarrow[\lambda \to 0]{} u(t)$ uniformly on $[0,\infty)$

$$\frac{du_\lambda}{dt}(t) \xrightarrow[\lambda \to 0]{} \frac{du}{dt}(t) \text{ uniformly on } [0,\infty).$$

Then show that $J_\lambda u_\lambda(t) \to u(t)$

thus $A_\lambda u_\lambda(t) = A(J_\lambda u_\lambda(t)) \to -\frac{du}{dt}$ $\Big\} \Rightarrow$

since A is closed operator

$$\Rightarrow \boxed{\frac{du}{dt} + Au = 0, \quad u \in D(A)}$$