Linear evolution equations : Parabolic PDEs

Let $\Omega \subset \mathbb{R}^n$ open, bounded domain. Let $T > 0$ fixed and define $\Omega_T = \Omega \times (0, T]$. Consider the initial-condition boundary value problem (ibvp)

$$(*) \quad \begin{cases} u_t + \mathcal{L}u = f & \text{in } \Omega_T & \text{PDE} \\ u = 0 & \text{on } \partial\Omega \times [0,T] & \text{Boundary condition} \\ u = g & \text{on } \Omega \times \{t=0\} & \text{initial condition} \end{cases}$$

where $u : \Omega_T \to \mathbb{R}$ and

$$\mathcal{L}u = -\nabla \cdot [K(x,t)\nabla u] + b(x,t)\cdot \nabla u + q(x,t)u$$

Definition The operator $\boxed{\frac{\partial}{\partial t} + \mathcal{L}}$ is parabolic if there is a constant $\alpha > 0$ such that

$$\xi^T K(x,t)\xi \geq \alpha |\xi|^2, \quad \begin{matrix} \forall (x,t) \in \Omega_T \\ \forall \xi \in \mathbb{R}^n \end{matrix}$$

[for each fixed time t, $\mathcal{L}$ is elliptic in x, and uniformly in x, t].
(as usual, K is a symmetric matrix)

To define a weak solution to the IBVP (*), we view $u(x,t)$ at each fixed time t as a function of x, $u(\cdot,t):\Omega\to\mathbb{R}$ such that $u:[0,T]\longrightarrow H_0^1(\Omega)$, $u(t)(x)=u(x,t)$.

Define the operator

$$B(u,v;t)=\int_\Omega \nabla v\cdot(K\nabla u)\,dx+\int_\Omega(b\cdot\nabla u)\,v\,dx+$$
$$+\int_\Omega q\,u\,v,\quad \forall u,v\in H_0^1(\Omega)$$
$$0\le t\le T$$

Assume that the coefficients are such that $K_{ij}\in L^\infty(\Omega_T)$, $b_j\in L^\infty(\Omega_T)$, $q\in L^\infty(\Omega_T)$, $f\in L^2(\Omega_T)$, $g\in L^2(\Omega)$.

Define the subspace of $H^1(\Omega_T)$ as

$$H(\Omega_T)=\{\,w\in H^1(\Omega_T)\;:\;w(\cdot,t)\in H_0^1(\Omega),\ \text{a.e. } t\in[0,T]\,\}.$$

The H^1 norm on $H(\Omega_T)$ is

$$\|w\|_{1,1}^2=\int_{\Omega\times(0,T)}\left[w^2+|\nabla w|^2+\left(\frac{\partial w}{\partial t}\right)^2\right]dx\,dt$$

Weak formulation to the iBVP:

Find $u \in H(\Omega_T)$ such that

$$\begin{cases} \int_\Omega u_t v \, dx + B(u,v;t) = \int_\Omega f v \, dx, & \forall v \in H_0^1(\Omega) \\ & \text{a.e. } t \in [0,T] \\ \text{and} & \\ u(x,0) = g(x) & \end{cases}$$

Property (uniqueness of the weak solution)

There is at most one weak solution.

Proof Let $u_1, u_2 \in H(\Omega_T)$ weak solutions and define $u = u_1 - u_2$. Then, at $v = u(\cdot, t)$

$$\int_\Omega u_t u \, dx + B(u,u;t) = 0, \quad \text{a.e. } t.$$

or $$\frac{1}{2}\frac{d}{dt}\left[\int_\Omega u^2 \, dx\right] + B(u,u;t) = 0$$

if $B(u,u;t)$ is elliptic $(B(u,u;t) > 0)$

then $\frac{d}{dt}\int_\Omega u^2 \le 0$ such that $\int_\Omega u^2(x,t)\,dx \le$

$$\le \int_\Omega u^2(x,0)\,dx = 0$$

thus $u = 0$, a.e. Ω_T.

For the general case, there is a constant $\delta > 0$ such that

$$B(v, v; t) + \delta \|v\|_{L^2}^2 \geq \tilde{\alpha} \|v\|_{H_0^1}^2 \quad \text{for some}$$

$\tilde{\alpha} > 0$. Then $u = u_1 - u_2$ satisfies

$$\begin{cases} \frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 \leq \delta \|u\|_{L^2}^2 \\ \|u(0)\|_{L^2} = 0 \end{cases}$$

Denoting $\|u(t)\|_{L^2}^2 = y(t)$, we have

$$\begin{cases} \frac{dy}{dt} \leq \delta y \\ y(0) = 0 \\ y(t) \geq 0 \end{cases}$$ then by Gronwall's inequality it follows that $y(t) = 0$ a.e.

Since $\frac{dy}{dt} \leq \delta y \Rightarrow e^{-\delta t} \frac{dy}{dt} \leq \delta e^{-\delta t} y$

$$\Rightarrow \frac{d}{dt}\left[e^{-\delta t} y\right] \leq 0 \Rightarrow e^{-\delta t} y(t) \leq y(0) = 0$$

such that $0 \leq e^{-\delta t} y(t) \leq 0$. Thus $y(t) = 0$.

<u>Remark</u> : $u(x,t)$ solves IBVP

$$\begin{cases} u_t + \mathcal{L}u = f \\ u_{|\partial\Omega} = 0 \\ u(x,0) = g(x) \end{cases}$$

if and only if $\tilde{u}(x,t) = e^{-\gamma t} u(x,t)$ solves IBVP

$$\begin{cases} \tilde{u}_t + \mathcal{L}\tilde{u} + \gamma \tilde{u} = e^{-\gamma t} f \\ \tilde{u}_{|\partial\Omega} = 0 \\ \tilde{u}(x,0) = g(x) \end{cases}$$

Therefore, to study the existence and uniqueness of the solution we may assume that the operator B satisfies

$$B(v,v;t) \geq \tilde{\alpha} \|v\|^2_{H_0^1}, \quad \forall v \in H_0^1(\Omega), \text{ a.e. } t.$$

Existence of the weak solution (Ritz-Galerkin)

Specialize in the model problem

$$
\begin{aligned}
u_t - \Delta u &= f && \text{in } \Omega_T \\
u &= 0 && \text{on } \partial\Omega \times [0,T] \\
u &= g && \text{on } \Omega \times \{t=0\}
\end{aligned}
$$

where $f \in L^2(\Omega_T)$ and $g \in H_0^1(\Omega)$.

Let $\{e_1, \dots, e_k, \dots\}$ basis to $H_0^1(\Omega)$ such that

$$
\begin{cases}
-\Delta e_k = \lambda_k e_k & \text{in } \Omega \\
e_k|_{\partial\Omega} = 0
\end{cases}, \quad \|e_k\|_{L^2} = 1 .
$$

Let $E_k = \operatorname{span}\{e_1, \dots, e_k\}$ and consider the finite dimensional problem:

Find $\boxed{u_k(x,t) = \sum_{j=1}^{k} q_j(t) e_j(x)}$ such that

$$
\begin{cases}
\displaystyle\int_\Omega \frac{\partial u_k}{\partial t} v + \int_\Omega \nabla u_k \nabla v = \int_\Omega f v, \quad \forall v \in E_k \quad \text{a.e. } t \\
\displaystyle u_k(x,0) = \tilde{g}_k(x) \overset{\text{def}}{=} P_{E_k} g = \sum_{j=1}^{k} \left(g, \frac{1}{\sqrt{\lambda_j}} e_j\right)_1 \frac{1}{\sqrt{\lambda_j}} e_j
\end{cases}
$$

where $\tilde{g}_k = P_{E_k} g = \sum_{j=1}^{k} \left(g, \frac{1}{\sqrt{\lambda_j}} e_j\right)_1 \frac{1}{\sqrt{\lambda_j}} e_j$

Equivalent to: for each $i = 1:K$,

$$\sum_{j=1}^{K} a_j'(t) \int_\Omega e_j e_i + \sum_{j=1}^{K} a_j(t) \int_\Omega \nabla e_j \nabla e_i = \int_\Omega f e_i$$

or, since $\int_\Omega \nabla e_j \nabla e_i = \begin{cases} \lambda_i, & i = j \\ 0, & j \neq i \end{cases}$

$$\int_\Omega e_j e_i = \delta_{ij}$$

~~th~~

$$\boxed{a_i'(t) + \lambda_i a_i(t) = f_i(t), \quad i = 1:K}$$

where $f_i = \int_\Omega f e_i$

From the initial condition we have

$$u_K(x,0) = \tilde{g}_K(x) \Rightarrow \sum_{j=1}^{K} a_j(0) e_j(x) = \sum_{j=1}^{K} \frac{1}{\lambda_j}(g, e_j)_1 e_j$$

$$\boxed{a_i(0) = \frac{1}{\lambda_i}(g, e_i)_1, \quad i = 1:K} \quad [= (g, e_i)_0]$$

since $\int_\Omega \nabla e_i \nabla g = \lambda_i \int_\Omega e_i g$

Next we prove that $\{u_K\}_{K \geq 1}$ is uniformly bounded in $H(\Omega_T)$ and therefore $u_K \longrightarrow u$ (there is a weakly convergent subsequence)

Remark: u_K is $C^\infty(\Omega_T)$!

Theorem Let $g \in H_0^1(\Omega)$, $f \in L^2(\Omega_T)$. Then

$$\|u_k\|_{1,1}^2 \overset{def}{=} \int_0^T \int_\Omega |u_k|^2 + |\nabla u_k|^2 + \left|\frac{\partial u_k}{\partial t}\right|^2 dx\,dt \leq$$

$$\leq c\left[\|g\|^2_{H_0^1(\Omega)} + \|f\|^2_{L^2(\Omega_T)}\right]$$

for some constant $c > 0$, independent on k.

Proof We show that each of the terms in the $H_0^{1,1}$-norm of u_k is bounded by $c\left[\|g\|^2_{H_0^1(\Omega)} + \|f\|^2_{L^2(\Omega_T)}\right]$ for some constant $c > 0$.

Since $u_k = \sum_{j=1}^{k} q_j(t) e_j(x)$,

$$\nabla u_k = \sum_{j=1}^{k} q_j(t) \nabla e_j(x)$$

such that $\int_\Omega |\nabla u_k|^2 = \sum_{j=1}^{k} q_j^2(t) \int_\Omega |\nabla e_j|^2 = \sum_{j=1}^{k} \lambda_j q_j^2(t)$

and

$$\boxed{\int_0^T \int_\Omega |\nabla u_k|^2 \,dx\,dt = \sum_{j=1}^{k} \int_0^T \lambda_j q_j^2(t)\,dt} \quad (*)$$

Next, we provide an uniform bound for the right side term in the equality above.

Since $a_j'(t) + \lambda_j a_j(t) = f_j(t) \quad / \cdot a_j(t)$

we have

$$\int_0^T \lambda_j a_j^2(t) = \int_0^T f_j a_j - \int_0^T a_j a_j' =$$

$$= \int_0^T f_j a_j - \frac{1}{2} a_j^2 \Big|_0^T \le$$

$$\le \int_0^T f_j a_j + \frac{1}{2} a_j^2(0)$$

Notice that $f_j a_j \le \dfrac{\lambda_j a_j^2}{2} + \dfrac{1}{2\lambda_j} f_j^2$

such that

$$\int_0^T \lambda_j a_j^2 \le \frac{1}{2}\int_0^T \lambda_j a_j^2 + \frac{1}{2\lambda_j}\int_0^T f_j^2 + \frac{1}{2} a_j^2(0)$$

and therefore

$$\int_0^T \lambda_j a_j^2 \le \frac{1}{\lambda_j}\int_0^T f_j^2 + a_j^2(0) \le C(\Omega)\int_0^T f_j^2 + a_j^2(0)$$

where $C(\Omega) = \frac{1}{\lambda_1}$ denotes the Poincaré constant

Then $\displaystyle\sum_{j=1}^k \int_0^T \lambda_j a_j^2 \le C(\Omega)\sum_{j=1}^k \int_0^T f_j^2 + \sum_{j=1}^k a_j^2(0)$

Notice $\displaystyle\sum_{j=1}^k f_j^2 \le \sum_{j=1}^\infty f_j^2 = \|f(x,t)\|^2_{L^2(\Omega)}$

Thus $\displaystyle\int_0^T \sum_{j=1}^k f_j^2 \le \|f\|^2_{L^2(\Omega_T)} \qquad (\xi)$

$$q_j(0) = (g, e_j) \Rightarrow \sum_{j=1}^{K} q_j^2(0) \leq \|g\|^2_{L^2(\Omega)} \qquad (\xi\xi)$$

From (ξ) and $(\xi\xi)$ we have

$$\sum_{j=1}^{K} \int_0^T \lambda_j q_j^2 \leq C(\Omega)\|f\|^2_{L^2(\Omega_T)} + \|g\|^2_{L^2(\Omega)}$$

and with $(*)$ we get

$$(**) \quad \boxed{\|\nabla u_K\|^2_{L^2(\Omega_T)} \leq C\left[\|f\|^2_{L^2(\Omega_T)} + \|g\|^2_{H_0^1(\Omega)}\right]}.$$

At each fixed t, $u_K(\cdot, t)$ belongs to $H_0^1(\Omega)$, by Poincaré:

$$\int_\Omega u_K^2 \, dx \leq C(\Omega)\int_\Omega |\nabla u_K|^2 \, dx$$

then $\int_0^T \int_\Omega u_K^2 \, dx\,dt \leq C(\Omega)\int_0^T \int_\Omega |\nabla u_K|^2 \, dx\,dt$

such that $\|u_K\|^2_{L^2(\Omega_T)} \leq C(\Omega)\|\nabla u_K\|^2_{L^2(\Omega_T)}$

so $(**)$ implies $\boxed{\|u_K\|^2_{L^2(\Omega_T)} \leq C \cdot C(\Omega)\left[\|f\|^2_{L^2(\Omega_T)} + \|g\|^2_{H_0^1(\Omega)}\right]}$

Next we provide the bound on $\int_{\Omega_T} \left|\frac{\partial u_k}{\partial t}\right|^2 dx\,dt$

we have $\quad u_{k,t} = \sum_{j=1}^{k} a_j'(t) e_j = \sum_{j=1}^{k} (f_j - \lambda_j a_j) e_j$

$$\Rightarrow \int_\Omega u_{k,t}^2 = \sum_{j=1}^{k} (f_j - \lambda_j a_j)^2 \le 2\sum_{j=1}^{k} f_j^2 + 2\sum_{j=1}^{k} \lambda_j^2 a_j^2$$

Since $\quad a_j'(t) + \lambda_j a_j(t) = f_j(t) \quad / \cdot \lambda_j a_j(t)$

we have

$$(\lambda_j a_j)^2 = \lambda_j f_j a_j - \lambda_j a_j a_j'$$

such that

$$\int_0^T (\lambda_j a_j)^2 = \int_0^T (\lambda_j a_j) f_j - \frac{1}{2}\lambda_j a_j^2 \Big|_0^T$$

$$\le \frac{1}{2}\int_0^T (\lambda_j a_j)^2 + \frac{1}{2}\int_0^T f_j^2 + \frac{1}{2}\lambda_j a_j^2(0)$$

$$\Rightarrow \int_0^T (\lambda_j a_j)^2 \le \int_0^T f_j^2 + \lambda_j a_j^2(0).$$

$$\Rightarrow \sum_{j=1}^{k} \int_0^T (\lambda_j a_j)^2 \le \int_0^T \sum_{j=1}^{k} f_j^2 + \sum_{j=1}^{k} \lambda_j a_j^2(0).$$

$$\le \|f\|^2_{L^2(\Omega_T)} + \underbrace{\|g\|^2_{H_0^1(\Omega)}}_{= \sum_{j=1}^{\infty} \lambda_j a_j^2(0)}$$

Since $\quad \|g\|^2_{H_0^1(\Omega)} = \sum_{j=1}^{\infty} \lambda_j a_j^2(0)$

it follows that

$$\int_0^T \int_\Omega u_{k,t}^2 \le 2\int_0^T \sum_{j=1}^k f_j^2 + 2\sum_{j=1}^k \int_0^T \lambda_j^2 a_j^2 \le$$

$$\le \tilde{C}\left[\|f\|^2_{L^2(\Omega_T)} + \|g\|^2_{H_0^1(\Omega)} \right].$$

Existence of the weak solution

Since $\{u_k\}_{k\ge 1}$ is (uniformly) bounded in $H_0^{1,1}$, it has a weakly convergent subsequence, so we may assume

$$u_k \longrightarrow u \quad \text{in } H_0^{1,1} = H(\Omega_T).$$

We show that u is the weak solution to the original problem, that is

$$\begin{cases} \int_\Omega \frac{\partial u}{\partial t} v + \int_\Omega \nabla u \nabla v = \int_\Omega f v, & \forall v \in H_0^1(\Omega) \\ & \text{a.e. } t \\ u(x,0) = g(x). \end{cases}$$

$u_K \rightharpoonup u$ in $H(\Omega_T)$ implies weak convergence of the weak derivatives in $L^2(\Omega_T)$:

$$\frac{\partial u_K}{\partial t} \rightharpoonup \frac{\partial u}{\partial t}, \quad \frac{\partial u_K}{\partial x_i} \rightharpoonup \frac{\partial u}{\partial x_i}$$

Let $v \in H_0^1(\Omega)$ and $N > 0$ arbitrary fixed.
Denote $v_N = P_{E_N} v$ projection on span$\{e_1, \dots, e_N\}$

Then, for any $K > N$

$$\begin{cases} \int_\Omega \frac{\partial u_K}{\partial t} v_N + \int_\Omega \nabla u_K \nabla v_N = \int_\Omega f v_N \\ u_K(x,0) = P_{E_K} g \end{cases}$$

For $K \to \infty$,

$$\begin{cases} \int_\Omega \frac{\partial u}{\partial t} v_N + \int_\Omega \nabla u \nabla v_N = \int_\Omega f v_N \\ u(x,0) = \lim_{K \to \infty} P_{E_K} g = g \end{cases}$$

For $N \to \infty$, $v_N \to v$ in $H_0^1(\Omega)$ so

$$\boxed{\int_\Omega \frac{\partial u}{\partial t} v + \int_\Omega \nabla u \nabla v = \int_\Omega f v}$$