Spectral theory of elliptic operators

Let $\Omega \subset \mathbb{R}^n$ an open and bounded domain of class C^1. Consider the elliptic operator

$$\mathcal{L}u = -\nabla \cdot (K \nabla u) + qu = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \sum_{j=1}^{n} K_{ij} \frac{\partial u}{\partial x_j} + qu$$

where the coefficients are $K_{ij} \in C^1(\bar{\Omega})$, $K_{ij}(x) = K_{ji}(x)$, and $q \in C(\bar{\Omega})$. Assume that there is $\bar{\alpha} > 0$ constant such that

$$\xi^T K \xi \geq \bar{\alpha} |\xi|^2, \quad \forall \xi \in \mathbb{R}^n$$

Consider the boundary value problem (BVP)

$$(*) \quad \begin{cases} \mathcal{L}u = \lambda u & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

where λ is a constant. We are interested in the existence of nontrivial solutions to $(*)$.

Definition A scalar (number) λ is called eigenvalue to the operator $\mathcal{L}$ if there is a nontrivial function $u \in C^2(\Omega)$ that solves the BVP $(*)$. In this case, the function u is called an eigenfunction associated with the eigenvalue λ and (λ, u) is an eigenpair.

The weak (variational) formulation to the BVP (*) is
Find $u \in H_0^1(\Omega)$ such that

$$(**) \qquad a(u,v) = \lambda (u,v)_0 \,, \quad \forall v \in H_0^1(\Omega)$$

where $a(u,v) = \int_\Omega \nabla v \cdot K \nabla u + \int_\Omega q u v$ is the bilinear form associated with the operator $\mathcal{L}$ and $(u,v)_0 = \int_\Omega uv$ denotes the $L^2(\Omega)$-inner product.

The ability to express a certain class of functions as a series (Fourier) of eigenfunctions was a key element to the Fourier series approach for solving Poisson, heat, or wave equations. Here we are interested to provide the <u>theoretical framework</u> to the existence of the eigenvalues and eigenfunctions of elliptic operators and their mathematical properties.

Spectral theorems for elliptic operators

Theorem (eigenvalues and eigenfunctions of the Laplacian)
Let $\Omega \subset \mathbb{R}^n$ an open and bounded domain. There is a Hilbert basis to $L^2(\Omega)$, $\{e_k\}_{k\geq 1}$ and a sequence of real numbers $\{\lambda_k\}_{k\geq 1}$ such that $\lambda_k > 0$, $k \geq 1$ and $\lim_{k\to\infty} \lambda_k = +\infty$ that satisfy

$$\begin{cases} -\Delta e_k = \lambda_k e_k & \text{in } \Omega \\ e_k = 0 & \text{on } \partial\Omega \end{cases}$$

in addition, $e_k \in H_0^1(\Omega) \cap C^\infty(\Omega)$ and $\{e_k\}_{k\geq 1}$ forms also a Hilbert basis to $H_0^1(\Omega)$.

Theorem (eigenvalues and eigenfunctions of an elliptic operator)

if $K_{ij} \in L^\infty(\Omega)$ satisfy the elliptic property

$$\exists \bar{\alpha} > 0 : \quad \xi^T K \xi \geq \bar{\alpha} |\xi|^2, \quad \forall \xi \in \mathbb{R}^n$$

and $q \in L^\infty(\Omega)$ then there is a Hilbert basis $\{e_k\}_{k\geq 1}$ to $L^2(\Omega)$ and a sequence of real numbers $\{\lambda_k\}_{k\geq 1}$ such that $\lim_{k\to\infty} \lambda_k = +\infty$ that satisfy

$$\begin{cases} e_k \in H_0^1(\Omega) \\ \int_\Omega \nabla v \cdot K \nabla e_k + \int_\Omega q e_k v = \lambda_k \int_\Omega e_k v, \quad \forall v \in H_0^1(\Omega) \end{cases}$$

Self-adjoint compact operators on Hilbert spaces

Definitions and elementary properties

Definition Let H denote a Hilbert space w.r.t. inner product $(\cdot,\cdot)$. A sequence $\{e_i\}_{i\geq 1}$ of elements of H is called a Hilbert basis (henceforth, basis) to H if it satisfies the following properties:

(i) The sequence is orthonormal

$$\|e_i\|=1, \quad (e_i, e_j) = 0, \quad \forall i\neq j .$$

ii) The vector space E generated by $\{e_i\}_{i\geq 1}$ is dense in H

$$\forall u\in H, \ \forall \varepsilon>0 : \exists \bar{u}\in E : \|u-\bar{u}\|<\varepsilon$$

Recall that the vector space E generated by $\{e_i\}_{i\geq 1}$ consists of finite linear combinations of the elements of $\{e_i\}_{i\geq 1}$, that is a generic $\bar{u}\in E$ is expressed as

$$\bar{u} = \sum_{j=1}^{m} c_{i_j} e_{i_j} = c_{i_1} e_{i_1} + c_{i_2} e_{i_2} + \cdots + c_{i_m} e_{i_m}$$

<u>Theorem</u> if $\{e_i\}_{i\geq 1}$ is a basis to H then

(i) Any element $u \in H$ may be expressed as

$$u = \sum_{i=1}^{\infty} (u, e_i) e_i = \lim_{k\to\infty} \sum_{i=1}^{k} (u, e_i) e_i$$

(ii) Bessel-Parseval's identity holds:

$$\|u\|^2 = \sum_{i=1}^{\infty} (u, e_i)^2$$

<u>Proof</u>: Let $u \in H$ arbitrary. Consider the projection of u on the k-dimensional space generated by the basis vectors $\{e_1, \dots, e_k\}$

$$P_k u = \sum_{i=1}^{k} (u, e_i) e_i$$

we show that $u = \lim_{k\to\infty} P_k u$.

Let $\varepsilon > 0$ arbitrary fixed. Since $\{e_i\}_{i\geq 1}$ is basis to H, there is $\bar{u} \in E$ such that

$$\|u - \bar{u}\| < \frac{\varepsilon}{2}$$

Since $\bar{u} \in E$, it is expressed as a finite linear combination

$$\bar{u} = \sum_{j=1}^{m} (\bar{u}, e_{i_j}) e_{i_j}$$

Therefore, there is $N = \max\{i_1, \dots, i_m\}$ such that $\bar{u}$ belongs to $\mathrm{Span}\{e_1, \dots, e_N\}$ and thus

$$\bar{u} = P_N \bar{u}$$

Recall that the projection operator is contractive

$$\|P_K u - P_K \bar{u}\| \leq \|u - \bar{u}\|, \quad \forall K > 0$$

For any $K \geq N$ we have

$$\|u - P_K u\| = \|u - \bar{u} + \bar{u} - P_K u\| = \|u - \bar{u} + P_K \bar{u} - P_K u\| \leq$$
$$\leq \|u - \bar{u}\| + \|P_K \bar{u} - P_K u\| \leq 2\|u - \bar{u}\| < \varepsilon$$

therefore, $u = \lim_{K \to \infty} P_K u = \lim_{K \to \infty} \sum_{i=1}^{K} (u, e_i) e_i$

Notice that

$$\|P_K u\|^2 = \sum_{i=1}^{K} (u, e_i)^2$$

such that $\|P_K u\| \xrightarrow[K]{} \|u\|$ implies

$$\|u\|^2 = \sum_{i=1}^{\infty} (u, e_i)^2$$

Self-adjoint and compact operators. Spectral properties

Definition Let H Hilbert space and $T: H \to H$ a linear and continuous operator. We say that the operator $T^*: H \to H$ is the adjoint of T if

$$(Tu, v) = (u, T^* v), \quad \forall u, v \in H.$$

T is called self-adjoint if $T = T^*$, that is

$$(Tu, v) = (u, Tv), \quad \forall u, v \in H.$$

Definition Let $T: H \to H$ linear and continuous. The resolvent of T is defined as

$$\rho(T) = \{\lambda \in \mathbb{R} : T - \lambda I \text{ is bijection from } H \text{ to } H\}$$

The spectrum $\sigma(T)$ of T is defined as the complement set to $\rho(T)$:

$$\sigma(T) = \mathbb{R} \setminus \rho(T).$$

A scalar $\lambda \in \mathbb{R}$ is an eigenvalue of T if

$N(T - \lambda I) \neq \{0\}$: $\exists u \neq 0$: $Tu = \lambda u$

then $N(T - \lambda I)$ is the eigenspace associated to λ.

The following is a fundamental property to the study of the eigenvalues of an elliptic operator

Property if the bilinear form $a(u,v) = \int_\Omega \nabla v \cdot K \nabla u + \int_\Omega g u v$ satisfies the hypotheses of the Lax-Milgram theorem on $H_0^1(\Omega)$, then the operator $T : L^2(\Omega) \to L^2(\Omega)$ defined as

$$T(f) = u \quad : \quad a(u,v) = (f,v)_{L^2}, \quad \forall v \in H_0^1$$

is a <u>compact and self-adjoint operator</u>

Proof By Lax-Milgram, $T : L^2 \to H_0^1 \hookrightarrow L^2$ is a well-defined linear operator. We have

$$\alpha \|u\|_1^2 \le a(u,u) = (f,u)_{L^2} \le \|f\|_{L^2} \|u\|_1$$

such that $T : L^2 \to H_0^1$ is continuous

$$\|Tf\|_1 = \|u\|_1 \le \frac{1}{\alpha} \|f\|_{L^2}$$

Since $I : H_0^1 \to L^2$ is a compact injection it follows that $T : L^2 \to L^2$ is a compact operator.

To show that T is self-adjoint we use the fact that $a(u,v)$ is symmetric:

$$a(u,v) = a(v,u), \quad \forall\, u, v \in H.$$

Let f, g arbitrary functions in $L^2(\Omega)$ and denote $u_f = T(f)$, $u_g = T(g)$. We have

$$a(u_f, v) = (f, v)_{L^2}, \quad \forall\, v \in H_0^1 \qquad (i)$$

$$a(u_g, v) = (g, v)_{L^2}, \quad \forall\, v \in H_0^1 \qquad (ii)$$

in particular, if we let $v = u_g$ in (i) and $v = u_f$ in (ii) we have

$$(f, u_g)_{L^2} = a(u_f, u_g) = a(u_g, u_f) = (g, u_f)_{L^2}$$

thus $(f, Tg)_{L^2} = (g, Tf)_{L^2}$, so $T = T^*$.

Notice that the study of the eigenvalues of the $\mathcal{L}$ operator is closely related with the study of the eigenvalues of the T operator. in particular,

<u>Property</u> if $a(\cdot,\cdot)$ is elliptic (Lax-Milgram) on $H_0^1(\Omega)$ then (λ, u) is an eigenpair for the weak problem

$$(**) \qquad a(u,v) = \lambda (u,v)_{L^2}, \quad \forall v \in H_0^1(\Omega)$$

<u>if and only if</u> $(\frac{1}{\lambda}, u)$ is eigenpair to T

$$(***) \qquad Tu = \frac{1}{\lambda} u$$

<u>Proof</u>: Notice that if $a(u,v) = \lambda(u,v)_{L^2}$ then $\lambda > 0$ (since $u \not\equiv 0$).

Also, if $Tu = 0$ then $u = 0$ so $N(T) = \{0\}$

Then $a(u,v) = \lambda(u,v)_{L^2}$ if and only if

$a(\frac{1}{\lambda}u, v) = (u,v)_{L^2}$ that is $Tu = \frac{1}{\lambda}u$.

<u>Hilbert-Schmidt theorem</u> Let H denote a separable Hilbert space and $T: H \to H$ a self-adjoint and compact operator.
Then there is a Hilbert basis to H consisting of the eigenvectors of T.

<u>Consequence:</u> For the Laplacian operator $L = -\Delta$ Hilbert-Schmidt theorem shows that there is a sequence $\{e_k\}_{k \geq 1}$ of functions in $L^2(\Omega) \cap H_0^1(\Omega)$ and scalars $\{\lambda_k\}_{k \geq 1}$, $\lambda_k > 0$ such that $\{e_k\}_{k \geq 1}$ is a Hilbert basis to $L^2(\Omega)$ and

$$\int_\Omega \nabla e_k \nabla v = \lambda_k \int_\Omega e_k v, \quad \forall v \in H_0^1(\Omega).$$

that is, e_k is the weak solution to

$$\begin{cases} -\Delta e_k = \lambda_k e_k & \text{in } \Omega \\ e_k|_{\partial\Omega} = 0 \end{cases}$$

From elliptic regularity we have $\underline{e_k \in C^\infty(\Omega)}$.

<u>Remark</u> if $\lambda_i \neq \lambda_j$ then the associated eigenfunctions e_i and e_j are orthogonal in both $L^2(\Omega)$ and $H_0^1(\Omega)$

$$a(e_i, e_j) = \lambda_i (e_i, e_j)_{L^2} = \lambda_j (e_i, e_j)_{L^2} \Rightarrow \begin{cases} (e_i, e_j)_{L^2} = 0 \\ a(e_i, e_j) = 0 \end{cases}$$

in particular, for $L = -\Delta$ it follows that the sequence $\{e_i\}_{i \geq 1}$ is dense in $H_0^1(\Omega)$.

Let $v \in H_0^1(\Omega)$ arbitrary such that

$$\int_\Omega \nabla e_i \nabla v = 0, \quad \forall i \geq 1$$

since

$$\int_\Omega \nabla e_i \nabla v = \lambda_i \int_\Omega e_i v$$

it follows that $\int_\Omega e_i v = 0, \quad \forall i \geq 1.$

since $\{e_i\}_{i \geq 1}$ is dense in $L^2(\Omega)$ it follows that $\underline{v = 0}$