Compact sets, compact operators, weak convergence

Review of concepts (Ref: Real Analysis, H.L. Royden)

Definition A metric space X is said to be compact if any open covering $\{V_i\}_{i\in I}$ of X has a finite subcovering, that is, given any collection of open sets $\{V_i, i \in I\}$ such that $\bigcup_{i\in I} V_i \supset X$ there is a finite collection of sets $\{V_1, \dots, V_N\}$ such that $\bigcup_{i=1}^{N} V_i \supset X$

Definition A metric space X is sequentially compact if any sequence $\{x_k\}_{k\geq 1} \subset X$ has a convergent subsequence $\{x_{k_p}\}$

Theorem (Borel-Lebesgue) Let X be a metric space. The following are equivalent

i) X is compact

ii) X is sequentially compact

Proposition A closed subset of a compact space is compact. A compact subset of a metric space is closed and bounded.

Theorem (Heine-Borel) A set $\Omega \subset \mathbb{R}^n$ is compact if and only if is closed and bounded.

Remark This is not true in an infinite dimensional space. Example,

$$\ell^2 = \{ \{x_k\} : \sum x_k^2 < \infty \}$$

Consider the sequence of elements of ℓ^2 (sequence of sequences) defined as

$$X_{k;i} = \begin{cases} 0, & i \neq k \\ 1, & i = k \end{cases}$$

then $\|\{X\}_i\|_{\ell^2} = 1, \quad \|\{X\}_i - \{X\}_j\|_{\ell^2} = \sqrt{2}$

(the unit sphere is closed, bounded, but not compact).

Similarly, for $L^2(0, 2\pi)$, $f_k(x) = \frac{1}{\sqrt{\pi}} \sin kx$ is such that $\|f_k - f_m\|_{L^2} = \sqrt{2}, \quad \|f_k\|_{L^2} = 1$.

Theorem (Riesz) Let X normed vector space. Then the closed unit ball $\bar{B}(0,1)$ is compact if and only if $\dim(X) < \infty$.

(i.e., the closed unit ball is never compact in an infinite dimensional space).

Proof For simplicity, we give the proof in the particular case when $X = H$ is a Hilbert space.

Assume that H is infinite dimensional. Consider a sequence of finite dimensional subspaces $\{H_n\}$ such that $H_n \subsetneqq H_{n+1}$, $\dim(H_n) = n$, $n \geq 1$

From the projection theorem, since $H_n \subsetneqq H_{n+1}$ let $v_{n+1} \in H_{n+1} \setminus H_n$ and $u_{n+1} \stackrel{def}{=} \dfrac{v_{n+1} - P_{H_n} v_{n+1}}{\| v_{n+1} - P_{H_n} v_{n+1} \|}$

then $(u_{n+1}, h_n) = 0, \ \forall h_n \in H_n$

and $\begin{cases} \|u_{n+1}\| = 1, \quad \|u_{n+1} - h_n\|^2 = \|u_{n+1}\|^2 + \|h_n\|^2 \geq 1 \\ u_{n+1} \in H_{n+1} \setminus H_n \end{cases}$

Therefore, the sequence $\{u_n\}$ is such that

$\|u_m - u_n\| \geq 1, \quad \forall m > n > 1$

and has no convergent subsequence.

Compact operators on Banach spaces

Definition Let X, Y Banach spaces and $T: X \to Y$ a linear and continuous operator. T is a compact operator if $T(\bar{B}_X(0,1))$ is precompact (relatively compact) subset of Y, i.e., $\overline{T(\bar{B}_X(0,1))}$ is compact set in Y.

Remark if $T: X \to Y$ is a compact operator and $\{x_n\}$ is a bounded sequence in X then $\{T(x_n)\}$ has a convergent subsequence in Y.

$$\|x_n\| \leq M \Rightarrow \|\tfrac{1}{M} x_n\| \leq 1 \Rightarrow \{\tfrac{1}{M} T(x_n)\} \subset T(\bar{B}_X(0,1))$$

and therefore, it has a convergent subsequence

$$\frac{1}{M} T(x_{n_k}) \xrightarrow[Y]{} \xi \quad , \quad T(x_{n_k}) \xrightarrow[Y]{} M\xi$$

<u>Definition</u> Let X, Y Banach spaces such that $X \subset Y$. We say that X is <u>compactly embedded</u> in Y and denote $X \subset\subset Y$ or $X \xhookrightarrow[\text{compact}]{} Y$ if the injection operator $\mathcal{I} : X \to Y$, $\mathcal{I}(x) = x$ is compact:

i) $\exists\, C \;:\; \|x\|_Y \leq C \|x\|_X , \; \forall x \in X$ (continuity)

ii) $\forall \{x_n\} \subset X$ bounded sequence in the X-norm there is a convergent subsequence $\{x_{n_k}\}$ in Y

$\|x_n\|_X \leq M$ implies there is $\{x_{n_k}\} \subseteq \{x_n\}$ and $x \in Y$ such that $\|x_{n_k} - x\|_Y \to 0$.

<u>Sobolev compact embeddings</u>

<u>Definition</u> if $1 \leq p < n$, the Sobolev conjugate of p is defined as $\boxed{p^* = \dfrac{np}{n-p}}$

<u>Remark</u>: $p^* > p$ and $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}$.

Sobolev - Rellich - Kondrachov theorem

Let $\Omega \subset \mathbb{R}^n$ bounded domain of class C^1.

if $1 \le p < n$ then $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$, $1 \le q < p^*$

if $p = n$ then $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$, $1 \le q < \infty$.

if $p > n$ then $W^{1,p}(\Omega) \subset\subset C(\bar{\Omega})$

are compact embeddings (injections).

$$\boxed{\begin{array}{l}\text{in particular, for any } 1 \le p \le \infty \\ W^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \text{ is compact injection}\end{array}}$$

Consequence: if $\{u_k\}_{k \ge 1} \subset W^{1,p}(\Omega)$ is a bounded sequence in the $\| \|_{1,p}$ norm, then there is a convergent subsequence in the

Weak convergence; reflexive spaces

Let X denote a Banach space.

Definition A sequence $\{x_n\} \subset X$ is said to converge weakly to $x \in X$ if $\forall f \in X^*$: $f(x_n) \to f(x)$ (for any linear and continuous functional $f: X \to \mathbb{R}$ we have $f(x_n) \to f(x)$).

We denote the weak convergence: $x_n \rightharpoonup x$

Remark in particular, if $X = H$ is a Hilbert space $(H, (\cdot,\cdot))$ then $u_n \rightharpoonup u$ if and only if

$$(v, u_n) \longrightarrow (v, u), \quad \forall v \in H.$$

Remark Strong convergence implies weak convergence but the reverse is not true

Example: $\ell^2 = \{ \{x_i\}_{i \geq 1}, \ \sum x_i^2 < \infty \}$

Let $x_{n,i} = \begin{cases} 0, & i \neq n \\ 1, & i = n \end{cases}$ (sequence of sequences)

Then $\|\{x_n\} - \{x_m\}\|_{\ell^2} = \sqrt{2}$ $\longrightarrow$ not Cauchy.

For any sequence $\{y_i\} \in \ell^2$

$(\{x_n\}, \{y\}) = y_n \xrightarrow[n \to \infty]{} 0$ since $\sum y_i^2 < \infty$ therefore $x_n \rightharpoonup 0$.

Example: in $L^2(0,2\pi)$, $f_n(x) = \frac{1}{\sqrt{\pi}} \sin nx$

is such that $\|f_n - f_m\|_{L^2} = \sqrt{2}$

and $f_n \rightharpoonup 0$ in $L^2(0,2\pi)$.

if $\varphi \in C_c^\infty(0,2\pi)$,

$$\int_0^{2\pi} \varphi(x) f_n(x)\, dx = \frac{1}{n\sqrt{\pi}} \int_0^{2\pi} \varphi'(x) \cos nx\, dx \xrightarrow[n\to\infty]{} 0$$

Remark: This example shows that the set (unit sphere) $S = \{x : \|x\| = 1\}$ is not closed w.r.t. the weak convergence, in an infinite dimensional space.

it can be shown that $\bar{S}^w = \{x \in X : \|x\| \leq 1\}$

(the closure of the unit sphere w.r.t. the weak convergence (topology) is the closed unit ball, $\bar{S}^w = \bar{B}(0,1)$.

<u>Reflexive spaces</u> let X Banach space.

$X^* =$ dual of $X = \{ f : X \to \mathbb{R},\ f$ is linear and continuous $\}$

$$\|f\|_{X^*} = \sup_{\|x\| \le 1} |f(x)|$$

$X^{**} =$ dual of $X^* = \{ F : X^* \to \mathbb{R},$ linear & continuous $\}$

$$\|F\|_{X^{**}} = \sup_{\substack{f \in X^* \\ \|f\| \le 1}} |F(f)|$$

The canonical injection $J : X \to X^{**}$ is defined as follows : to $x \in X$ we associate $J(x) \in X^{**}$ defined as

$$J(x)[f] = f(x), \quad \forall f \in X^*.$$

<u>Definition</u> X is reflexiv space if $J(X) = X^{**}$ (i.e. if J is <u>surjective</u>).

<u>Remark</u> <u>All Hilbert spaces are reflexive</u>

All L^p spaces with $1 < p < \infty$ are reflexive

L^1 and L^∞ <u>are not</u> reflexive.

$$(L^\infty)^* \supsetneq L^1$$

<u>Definition</u> A subset $S \subset X$ is <u>weakly sequentially compact</u> if any sequence $\{x_n\} \subset S$ has a weakly convergent subsequence $x_{n_k} \rightharpoonup x \in S$.

<u>Theorem (Kakutani)</u> A Banach space X is reflexiv if and only if the closed unit ball $\bar{B}(0,1) = \{x \in X : \|x\| \leq 1\}$ is <u>weakly compact</u>.

Consequently, in a reflexive space any <u>bounded sequence</u> has a <u>weakly convergent subsequence</u>

<u>Properties</u>

1. Strong convergence implies weak convergence.

$$u_n \to u \quad \text{then} \quad u_n \rightharpoonup u$$

2. Weak convergence <u>and</u> convergence of the norm implies strong convergence.

$$\boxed{\text{if } u_n \rightharpoonup u \text{ and } \|u_n\| \to \|u\| \text{ then } u_n \to u}$$

3. $$\left.\begin{array}{l} f_n \to f \text{ in } X^* \\ u_n \rightharpoonup u \text{ in } X \end{array}\right\} \Rightarrow f_n(u_n) \to f(u).$$

4. $$\text{if } u_n \rightharpoonup u \text{ then } \left\{\begin{array}{l} \{u_n\} \text{ is bounded and} \\ \|u\| \leq \liminf \|u_n\| \end{array}\right.$$

Remark Let X, Y reflexive spaces and $T: X \to Y$ a compact operator.

if $\{x_n\} \subset X$ is a weakly convergent sequence in X, $x_n \rightharpoonup x$ Then

$\{T(x_n)\} \subset Y$ has a convergent subsequence (strongly) in Y, $T(x_{n_k}) \to T(x) \in Y$.

in particular, fundamental property:

if $u_n \rightharpoonup u$ in $H^1(\Omega)$ then there is a subsequence

$$u_{n_k} \to u \text{ in } L^2(\Omega).$$

(weak convergence in H^1 implies the existence of a subsequence that is strongly convergent in L^2).

Applications to Poincaré inequalities

Theorem Let $\Omega \subset \mathbb{R}^n$ bounded domain with boundary $\partial\Omega$ partitioned as $\partial\Omega = \Gamma \cup \Gamma'$ and $|\Gamma| > 0$. Let $H = \{u \in H^1(\Omega) : u_{/\Gamma} = 0\}$

Then there is a constant $C(\Omega)$ such that

$$\int_\Omega u^2 \leq C(\Omega) \int_\Omega |\nabla u|^2 \ , \quad \forall\, u \in H .$$

Proof By contradiction: Assume that for any constant c, there is $u \in H$: $\int_\Omega u^2 > c \int_\Omega |\nabla u|^2$

Let $C_k = k$: $\exists\, u_k \in H$: $\int_\Omega u_k^2 > k \int_\Omega |\nabla u_k|^2$

Redefining $\bar{u}_k = \dfrac{u_k}{\|u_k\|_1}$, we may assume that

$\|u_k\|_1 = 1$ such that $\{u_n\}$ is bounded in $H^1(\Omega)$

Therefore, $\{u_k\}$ has a weakly convergent subsequence, u_{n_k} redenote it $\{u_k\}$,

$u_k \longrightarrow u$ in $H^1(\Omega)$.

Since $H^1(\Omega) \hookrightarrow L^2(\Omega)$ compact, we may extract a subsequence <u>convergent in</u> L^2.

$u_k \longrightarrow u$ in $L^2(\Omega)$.

Notice, $\int_\Omega u_k^2 > k \int_\Omega |\nabla u_k|^2 \Rightarrow$

$$\Rightarrow \underbrace{\|u_k\|_1^2}_{=1} > (k+1) \int_\Omega |\nabla u_k|^2$$

$$\Rightarrow \int_\Omega |\nabla u_k|^2 < \frac{1}{k+1} \xrightarrow[k\to\infty]{} 0 \Rightarrow \|u_k\|_{L^2} \to \underset{\|u\|_{L^2}}{\overset{\|}{1}}$$

Then, $u_k \rightharpoonup u$ in H^1 implies

$$\int_\Omega u_k v + \int_\Omega \nabla u_k \nabla v \longrightarrow \int_\Omega uv + \int_\Omega \nabla u \nabla v, \quad \forall v \in H^1$$

Since $u_k \to u$ in L^2 we have $\int_\Omega u_k v \to \int_\Omega uv$ such that $\int_\Omega \nabla u_k \nabla v \longrightarrow \int_\Omega \nabla u \nabla v \Rightarrow$

$$\downarrow k$$
$$0$$

$\Rightarrow \int_\Omega \nabla u \nabla v = 0, \quad \forall v \in H^1 \Rightarrow \nabla u = 0$ a.e.

$$\left.\begin{array}{l} \Rightarrow u = \text{constant a.e.} \\ u_{|\Gamma} = 0, \quad |\Gamma| > 0 \end{array}\right\} \Rightarrow u = 0 \text{ a.e. contradiction with } \|u\|_{L^2} = 1.$$

Application to the mixed boundary conditions problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_1, \quad |\Gamma_1| > 0 \\ \dfrac{\partial u}{\partial \vec{\nu}} = 0 & \text{on } \Gamma_2 \end{cases}$$

Let $H = \{ v \in H^1(\Omega) : v_{/\Gamma_1} = 0 \}$

<u>Remark</u>: H is a closed subspace of $H^1(\Omega)$ due to the trace operator theorem.

<u>Variational formulation</u> Find $u \in H$ such that

$$\int_\Omega \nabla u \nabla v = \int_\Omega f v, \quad \forall v \in H.$$

→ Poincaré inequality implies

$a(u,v) = \int_\Omega \nabla u \nabla v$ is <u>elliptic on H</u>.

such that, $\forall f \in L^2(\Omega)$, $\exists!\, u \in H$ solution.

Poincaré inequality for $H^1(\Omega)$

Let $\Omega \subset \mathbb{R}^n$ a bounded domain with boundary of class C^1. Then there is a constant $C(\Omega)$ such that

$$(*) \quad \|u-\bar{u}\|^2_{L^2(\Omega)} \le C(\Omega)\|\nabla u\|^2_{L^2(\Omega)}, \quad \forall u \in H^1(\Omega)$$

where $\bar{u} = \frac{1}{|\Omega|}\int_\Omega u\,dx$ denotes the mean value of u over Ω

Remark Denote $H = \{u \in H^1(\Omega) : \int_\Omega u\,dx = 0\}$

Then H is a closed subspace of $H^1(\Omega)$ and thus a Hilbert space w.r.t. $\| \ \|_1$.

Notice that $(*)$ is equivalent to $(**)$

$$(**) \quad \|u\|^2_{L^2(\Omega)} \le C(\Omega)\|\nabla u\|^2_{L^2(\Omega)}, \quad \forall u \in H.$$

Clearly, $(*) \Rightarrow (**)$. The reverse $(**) \Rightarrow (*)$ is as follows

Let $u \in H^1(\Omega)$ and denote $u_0 = u - \bar{u}$.

Then $\bar{u}_0 = \int_\Omega u_0\,dx = 0$ such that $u_0 \in H$.

in addition, $\nabla u_0 = \nabla u$ since $\bar{u}$ is a constant.

Since $u_0 \in H$, $(**)$ implies

$\int_\Omega u_0^2 \le C(\Omega)\int_\Omega |\nabla u_0|^2$ which implies $(*)$ since $\nabla u_0 = \nabla u$.

Therefore, is enough to prove $(**)$.

By contradiction, assume that (**) is not true. Then, there is a sequence $\{u_k\} \subset H$ such that

$\int_\Omega u_k^2 > k \int_\Omega |\nabla u_k|^2$ and therefore,

$\|u_k\|_1^2 > (k+1) \int_\Omega |\nabla u_k|^2$

Denote $w_k = \frac{u_k}{\|u_k\|_1}$ then $\boxed{\|w_k\|_1 = 1}$. and

$\int_\Omega |\nabla w_k|^2 < \frac{1}{k+1}$ thus $\nabla w_k \longrightarrow 0$ in $[L^2(\Omega)]$

Since $\{w_k\}$ is bounded in H^1-norm and $H^1 \hookrightarrow L^2$ compact, there is a subsequence (for convenience redenote it w_k) such that

$w_k \longrightarrow w$ in $L^2(\Omega)$.

Since in addition $\nabla w_k \longrightarrow 0$ in $L^2(\Omega)$ it follows that $\{w_k\}$ is Cauchy in $H^1(\Omega)$ and thus (up to passing to a subsequence) convergent in H^1.

$w_k \to w$ in $H^1(\Omega) \Rightarrow \int_\Omega |\nabla w|^2 = 0 \Rightarrow w =$ constant a.e. in Ω

In one hand, since $\int_\Omega w = 0$ we must have $w = 0$ a.e., however, $\|w\|_1 = 1 \longrightarrow$ contradiction

Thus (**) holds.

Nonhomogeneous Neumann Boundary Conditions

Consider

$$(*)\quad \begin{cases} -\Delta u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \vec{\nu}} = g & \text{on } \partial\Omega \\ \int_\Omega u\,dx = 0 \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ bounded, of class C^1.

Variational formulation:

$H = \{ v \in H^1(\Omega) : \int_\Omega v\,dx = 0 \}$.

$$\boxed{\text{find } u \in H: \quad \int_\Omega \nabla u \nabla v = \int_\Omega f v + \int_{\partial\Omega} g v, \quad \forall v \in H}$$

$a(u,v) = \int_\Omega \nabla u \nabla v$ is elliptic on H since Poincaré inequality holds on H.

Therefore, $\forall f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$ there is a unique solution to the variational problem.

Solvability condition is

$$\int_\Omega f + \int_{\partial\Omega} g = 0$$

for $(*)$ to have solution