Contraction semigroups and Hille-Yosida theorem

Hille-Yosida: if $A : D(A) \subset H \to H$ is max monotone then $\forall u_0 \in D(A)$, $\exists! \, u \in C^1([0,\infty), H) \cap C([0,\infty), D(A))$ solution to

$$\begin{cases} \dfrac{du}{dt} + Au = 0 \\ u(0) = u_0 \end{cases} \qquad (*)$$

For $t \geq 0$ define the operator $S_A(t) : D(A) \to D(A)$

$$S_A(t) u_0 = u(t)$$

Notice that $S_A(t)$ is a linear and continuous operator:

$$S_A(t)[u_0 + \tilde{u}_0] = S_A(t) u_0 + S_A(t) \tilde{u}_0$$
$$S_A(t)[c u_0] = c \, S_A(t) u_0$$

$$\| S_A(t) u_0 \| = \| u(t) \| \leq \| u_0 \|$$

since

$$\left(\frac{du}{dt}, u \right) + (Au, u) = 0$$

implies

$$\frac{d}{dt} \| u(t) \|^2 \leq 0 \quad \text{thus}$$

$t \to \| u(t) \|$ is decreasing in $[0, \infty)$.

In addition, $S_A(0)u_0 = u_0$, $\forall u_0 \in D(A)$

and

$$S_A(t)S_A(s)u_0 = S_A(t+s)u_0 = S_A(s)S_A(t)u_0, \quad \forall t,s \geq 0$$

Since $D(A)$ is dense in H, $\left(\overline{D(A)} = H\right)$ the operator $S_A(t)$ may be extended as a <u>linear</u> and <u>continuous</u> operator from H to H,

$S_A(t) : H \to H$ as follows:

Let $u_0 \in H$, $\exists \{u_k\}_{k \geq 1} \subset D(A) : u_k \xrightarrow[k]{} u_0$

Then $\| S_A(t)u_k - S_A(t)u_m \| \leq \|u_k - u_m\| \xrightarrow[k,m]{} 0$

thus $\{S_A(t)u_k\}_{k \geq 1}$ is Cauchy in H thus

$$S_A(t)u_k \xrightarrow{def} S_A(t)u_0 \in H$$

Notice that the limit is well-defined since for any other sequence $\{v_k\}_{k \geq 1}$ such that $v_k \to u_0$ we have

$$\|S_A(t)u_K - S_A(t)v_K\| \leq \|u_K - v_K\| \xrightarrow[K]{} 0$$

such that $\lim_{K\to\infty} S_A(t)u_K = \lim_{K\to\infty} S_A(t)v_K = S_A(t)u_0$

and $S_A(t)u_0$ is independent of the choice of the sequence $\{u_K\}$.

$$\|S_A(t)u_0\| = \lim_{K\to\infty} \|S_A(t)u_K\| \leq \lim_{K\to\infty} \|u_K\| = \|u_0\|.$$

Therefore, the operator $S_A(t): H \to H$ satisfies the following properties:

a) $\forall t \geq 0$, $S_A(t)$ is linear and continuous,

$$\|S_A(t)\|_{\mathcal{L}(H)} \leq 1$$

b) $\begin{cases} S_A(t_1+t_2) = S_A(t_1) \circ S_A(t_2), & \forall t_1, t_2 \geq 0. \\ S_A(0) = I \end{cases}$

c) $\lim_{t\to 0^+} \|S_A(t)u_0 - u_0\| = 0, \quad \forall u_0 \in H$.

We say that the family of operators $\{S(t)\}_{t\geq 0}$ is a continuous <u>semigroup of contractions</u> (contraction semigroup).

<u>Solution to the nonhomogeneous problem</u>

$$\begin{cases} \dfrac{du}{dt} + Au = f \\ u(0) = u_0 \end{cases}$$

<u>Theorem</u> Assume that $f \in C^1([0,T], H)$ and $u_0 \in D(A)$. Then there is a unique solution to the nonhomogeneous problem,

$u \in C^1([0,T], H) \cap C([0,T], D(A))$ and it is given by

$$u(t) = \underbrace{S_A(t)u_0}_{u_h} + \underbrace{\int_0^t S_A(t-s)f(s)\,ds}_{u_p}$$

Notice:

$$\begin{cases} \dfrac{du_h}{dt} + Au_h = 0 \\ u_h(0) = u_0 \end{cases} \qquad \begin{cases} \dfrac{du_p}{dt} + Au_p = f \\ u_p(0) = 0 \end{cases}$$

since $\dfrac{du_p}{dt} = S_A(0)f(t) - A\int_0^t S_A(t-s)f(s)\,ds = f(t) - Au_p$

Ref: Evans, Ch. 7, Section 7.4

Semigroup theory Let X Banach space.

A semigroup is a family of operators $\{S(t)\}_{t\geq 0}$
$S(t): X \to X$ linear and continuous such that

1) $S(0)u = u$, $u \in X$

2) $S(t+s)u = S(t)S(s)u = S(s)S(t)u$, $\forall t, s \geq 0$
$\forall u \in X$

3) The mapping $t \to S(t)u$ is continuous from $[0,\infty)$ into X.

if in addition, $\|S(t)\|_{\mathcal{L}(X)} \leq 1$, $\forall t \geq 0$
(thus $\|S(t)u\| \leq \|u\|$)

then $\{S(t)\}_{t\geq 0}$ is called a <u>contraction</u> semigroup (semigroup of contractions).

Definition Let $\{S(t)\}_{t\geq 0}$ contraction semigroup.

Define $D(A) = \{u \in X : \lim_{t\to 0^+} \frac{S(t)u - u}{t} \text{ exists in } X\}$

if $u \in D(A)$, $Au \overset{\text{def}}{=} \lim_{t\to 0^+} \frac{S(t)u-u}{t}$

Then $A : D(A) \longrightarrow X$ is called the generator of the semigroup $\{S(t)\}_{t\geq 0}$

Theorem (Differential properties of semigroups)

if $u_0 \in D(A)$ then

i) $S(t)u_0 \in D(A)$ for each $t \geq 0$

ii) $AS(t)u_0 = S(t)Au_0$ for each $t \geq 0$

iii) The mapping $t \to S(t)u_0$ is differentiable and $\frac{d}{dt} S(t)u_0 = AS(t)u_0$, for $t > 0$.

[therefore $u(t) \overset{\text{def}}{=} S(t)u_0$ solves $\begin{cases} \frac{du}{dt} = Au \\ u(0) = u_0 \end{cases}$]

Resolvent set and resolvent operator

Let $A : D(A) \subset X \to X$ linear and closed operator.

Definition The resolvent set $\rho(A)$ is defined as

$$\rho(A) = \{\lambda \in \mathbb{R} : \lambda I - A : D(A) \to X \text{ is bijection}\}.$$

if $\lambda \in \rho(A)$, the resolvent operator $R_\lambda : X \to X$ is defined by

$$R_\lambda u = (\lambda I - A)^{-1} u$$

Hille-Yosida theorem Let A denote a linear and closed operator with dense domain $D(A) \subset X$, $A : D(A) \subset X \to X$. Then A is the generator of a contraction semigroup $\{S(t)\}_{t \geq 0}$ if and only if

$$(0, \infty) \subset \rho(A) \text{ and } \|R_\lambda\| \leq \frac{1}{\lambda} \text{ for } \lambda > 0$$

Remark Notice that the notation in Brézis corresponds to $A \to -A$.

Here, $(0,\infty) \subset \rho(A)$ if and only if

$\forall \lambda > 0$, $(\lambda I - A)u = f$ has unique solution, $\forall f \in X$.

and notice $(\lambda I - A)u = f$ is equivalent to

$$\left(I - \tfrac{1}{\lambda} A\right)u = \tfrac{1}{\lambda} f \qquad (*)$$

if $-A$ is maximal monotone, then $(*)$ has unique solution since $-\frac{1}{\lambda}A$ is max mon. for $\lambda > 0$

Also,

$$R_\lambda u = \tfrac{1}{\lambda}\left(I - \tfrac{1}{\lambda} A\right)$$

and

$$\|R_\lambda\| \le \tfrac{1}{\lambda} \iff \left\|\left(I - \tfrac{1}{\lambda}A\right)^{-1}\right\| \le 1, \ \forall \lambda > 0$$