Optimization approach to spectral analysis

Model problem: eigenvalues / eigenfunctions of the Laplacian

Strong: $\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u_{|\partial\Omega} = 0 \end{cases}$ | Weak: find $u \in H_0^1(\Omega)$, $u \neq 0$

$$\int_\Omega \nabla u \nabla v = \lambda \int_\Omega uv, \quad \forall v \in H_0^1(\Omega)$$

Denote $$\lambda = \inf_{\substack{v \in H_0^1 \\ v \neq 0}} \frac{\int_\Omega |\nabla v|^2}{\int_\Omega v^2} = \inf_{\substack{v \in H_0^1 \\ \|v\|_{L^2} = 1}} \int_\Omega |\nabla v|^2$$

The following properties hold (existence theorem)

i) $\lambda > 0$

ii) There is $u \in H_0^1$, $\|u\|_{L^2} = 1$: $\lambda = \int_\Omega |\nabla u|^2$

iii) in addition

$$\int_\Omega \nabla u \nabla v = \lambda \int_\Omega uv, \quad \forall v \in H_0^1(\Omega)$$

such that (λ, u) is a weak solution to the eigenvalue problem.

Proof: By Poincaré inequality,

$$\int_\Omega v^2 \leq C(\Omega) \int_\Omega |\nabla v|^2 \,, \quad \forall v \in H_0^1$$

therefore,
$$\frac{\int_\Omega |\nabla v|^2}{\int_\Omega v^2} \geq \frac{1}{C(\Omega)} > 0, \quad \forall v \in H_0^1$$

and it follows that $\lambda \geq \frac{1}{C(\Omega)} > 0$.

ii) Let $\{u_k\}_{k\geq 1}$ such that $\|u_k\|_{L^2} = 1$ and

$$\int_\Omega |\nabla u_k|^2 \longrightarrow \lambda = \inf_{\substack{v \in H_0^1 \\ \|v\|_{L^2}=1}} \int_\Omega |\nabla v|^2$$

Then $\{u_k\}$ is bounded in H^1-norm $\left(\|u_k\|_1^2 \to 1+\lambda \right)$ and after passing to a subsequence we may assume weak convergence in H_0^1

$$u_k \longrightarrow u \quad \text{in } H_0^1 \underset{\text{compact}}{\hookrightarrow} L^2$$

and strong convergence in L^2

$$u_k \longrightarrow u \quad \text{in } L^2$$

such that $1 = \|u_k\|_{L^2} \longrightarrow \|u\|_{L^2}$, thus $\boxed{\|u\|_{L^2} = 1}$

in addition, $u_k \rightharpoonup u$ in H_0^1 implies

$\|u\|_1 \le \liminf_k \|u_k\|_1 \quad \left(|u|_1 \le \liminf_k |u_k|_1 \right)$

such that $\int_\Omega |\nabla u|^2 \le \liminf_k \int_\Omega |\nabla u_k|^2 = \lambda$

Thus $u \in H_0^1$ is such that

$$\begin{cases} \|u\|_{L^2} = 1 \\ \int_\Omega |\nabla u|^2 \le \lambda = \inf\limits_{\substack{v \in H_0^1 \\ \|v\|_{L^2} = 1}} \int_\Omega |\nabla v|^2 \end{cases}$$

and we must have $\boxed{\lambda = \int_\Omega |\nabla u|^2}$

iii) Let $v \in H_0^1(\Omega)$ arbitrary.

Define $g : \mathbb{R} \to \mathbb{R}$, $\quad g(\varepsilon) = \int_\Omega |\nabla(u + \varepsilon v)|^2 - \lambda \int_\Omega (u + \varepsilon v)^2$

Then $g(0) = 0$ and $g(\varepsilon) \ge 0$, $\forall \varepsilon \ne 0$

such that $\varepsilon = 0$ is a global min point.

Notice that $g'(\varepsilon) = 2 \int_\Omega \nabla u \nabla v + 2\varepsilon \int_\Omega |\nabla v|^2 - 2\lambda \int_\Omega uv - 2\lambda\varepsilon \int_\Omega v^2$

thus $g'(0) = 0 \Rightarrow \boxed{\int_\Omega \nabla u \nabla v = \lambda \int_\Omega uv}$ with $v \in H_0^1$ arbitrary.

Denote $\lambda_1 = \inf\limits_{\substack{v\in H_0^1 \\ \|v\|_{L^2}=1}} \int_\Omega |\nabla v|^2$

$$\boxed{E_1 = \{u \in H_0^1 : \int_\Omega \nabla u \nabla v = \lambda_1 \int_\Omega uv, \quad \forall v \in H_0^1(\Omega)\}}$$

Remark: E_1 is a closed subspace of $H_0^1(\Omega)$

Property E_1 is finite dimensional, $0 < \dim(E_1) < \infty$

Proof We already know that $\dim(E_1) > 0$.
By contradiction, assume $\dim(E_1) = \infty$.
then there is a sequence of eigenvectors $\{u_k\}_{k\geq 1}$ such that $\{u_k\}$ is L^2-orthonormal.

$$\int_\Omega u_k^2 = 1, \quad \int_\Omega u_k u_m = 0, \quad k \neq m.$$

Then $\int_\Omega |\nabla u_k|^2 = \lambda_1$ such that $\{u_k\}$ is bounded in H_0^1-norm. Then, there must be a subsequence that converges in L^2: $u_k \to u$ in L^2.
However, $\|u_k - u_m\|_{L^2}^2 = 2$, so $\{u_k\}$ has no convergent subsequence. Contradiction.
Thus, $\dim(E_1) < \infty$.

Next we show the existence of a <u>sequence</u> of eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \cdots$$

such that $\lim_{k \to \infty} \lambda_k = \infty$

and that the sequence of eigenfunctions $\{u_k\}_{k \ge 1}$ forms a basis to $L^2(\Omega)$.

<u>Remark</u> Since $\lambda_1 \neq \lambda_2$ implies $\int_\Omega u_1 u_2 = \int_\Omega \nabla u_1 \nabla u_2 = 0$

we look for additional eigenvectors in the subspace of H_0^1 that is L^2-orthogonal on E_1

$$\boxed{X_1 = H_0^1 \cap E_1^\perp = \left\{ v \in H_0^1 : \int_\Omega v u_1 = 0, \ \forall u_1 \in E_1 \right\}}$$

Notice that X_1 is a <u>closed subspace</u> of H_0^1 and thus $(X_1, \| \ \|_1)$ is <u>Hilbert space.</u>

The previous steps are now performed in X_1 instead of H_0^1.

Denote $\lambda_2 = \inf\limits_{\substack{v \in X_1 \\ \|v\|_{L^2}=1}} \int_\Omega |\nabla v|^2$

with same approach as before, there is $u_2 \in X_1$ such that $\|u_2\|_{L^2} = 1$ and

$$\lambda_2 = \int_\Omega |\nabla u_2|^2 ,$$

$$\int_\Omega \nabla u_2 \nabla v = \lambda_2 \int_\Omega u_2 v , \quad \forall v \in X_1 = H_0^1 \cap E_1^\perp$$

Notice that $\lambda_2 \geq \lambda_1$ since $X_1 \subset H_0^1$.

in addition, if $u_1 \in E_1$ then

$$\int_\Omega \nabla u_1 \nabla u_2 = \lambda_1 \int_\Omega u_1 u_2 = 0$$

thus $\int_\Omega \nabla u_2 \nabla v = \lambda_2 \int_\Omega u_2 v, \quad \forall v \in H_0^1$.

such that (λ_2, u_2) is a weak eigenpair.

we must have $\lambda_2 > \lambda_1$ since $u_2 \in H_0^1 \cap E_1^\perp$ (otherwise $u_2 \in E_1$).

We denote $\boxed{E_2 = \{u \in H_0^1 : \int_\Omega \nabla u \nabla v = \lambda_2 \int_\Omega u v, \ \forall v \in H_0^1\}}$

Proceeding inductively, assume that we have the eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_K$ and the associated eigenspaces

$$E_j = \{ u \in H_0^1(\Omega) : \int_\Omega \nabla u \nabla v = \lambda_j \int_\Omega u v, \ \forall v \in H_0^1 \}$$

with

$$\lambda_j = \inf_{\substack{v \in X_{j-1} \\ \|v\|_{L^2} = 1}} \int_\Omega |\nabla v|^2 , \qquad j = 1 : K$$

where

$$X_0 = H_0^1(\Omega)$$

$$X_{j-1} = H_0^1 \cap \{ \bigcup_{i=1}^{j-1} E_i \}^\perp , \qquad j \geq 2$$

Define

$$X_K = H_0^1 \cap \{ \bigcup_{i=1}^{K} E_i \}^\perp =$$

$$= \{ v \in H_0^1 : \int_\Omega v u_i = 0, \ \forall u_i \in E_i, \ i = 1 : K \}$$

and

$$\lambda_{K+1} = \inf_{\substack{v \in X_K \\ \|v\|_{L^2} = 1}} \int_\Omega |\nabla v|^2 \geq \lambda_K$$

Since $X_K \subset H_0^1$ is closed subspace there is $u_{K+1} \in X_K$, $\|u_{K+1}\|_{L^2} = 1$: $\int_\Omega |\nabla u_{K+1}|^2 = \lambda_{K+1}$

$$\int_\Omega \nabla u_{k+1} \nabla v = \lambda_{k+1} \int_\Omega u_{k+1} v, \qquad \forall v \in X_k .$$

in addition, for $v \in E_i$, $i = 1:k$, say $v = u_i$

$$\int_\Omega \nabla u_i \nabla u_{k+1} = \lambda_i \int_\Omega u_i u_{k+1} = 0$$

thus $\int_\Omega \nabla u_{k+1} \nabla v = \lambda_{k+1} \int_\Omega u_{k+1} v, \quad \forall v \in H_0^1(\Omega)$.

such that (λ_{k+1}, u_{k+1}) is weak eigenpair.

Claim $\boxed{\lim_{k\to\infty} \lambda_k = \infty}$

Proof if not, then the sequence $\{u_k\}$ satisfies

$$\int_\Omega |\nabla u_k|^2 = \lambda_k \int_\Omega u_k^2 = \lambda_k .$$

Assuming that $\{\lambda_k\}_{k\geq 1}$ is bounded by M

then $\{u_k\}_{k\geq 1}$ is bounded in H_0^1-norm

$$\int_\Omega |\nabla u_k|^2 \leq M$$

thus it must have a subsequence that converges in L^2. However, $\|u_k - u_m\|_{L^2}^2 = 2$, $k \neq m$

Contradiction.

Next we show that there is a basis to $L^2(\Omega)$ that consists of the eigenfunctions $\{u_k\}_{k\geq 1}$.

$\forall f \in L^2(\Omega)$, $\|f - P_N f\|_{L^2} \xrightarrow[N]{} 0$ where

$P_N f$ denotes the projection of f on span$\{E_1, \dots, E_N\}$,

$$(f - P_N f, u_k)_{L^2} = 0, \quad \forall u_k \in E_k, \quad k = 1:N$$

$$\|f - P_N f\|_{L^2} \leq \|f - u\|_{L^2}, \quad \forall u = \sum_{k=1}^{N} \alpha_k u_k, \quad u_k \in E_k.$$

(Assume first that $f \in H_0^1(\Omega)$)

Notice that $f - P_N f \in X_N$ and therefore

$$\boxed{\|\nabla(f - P_N f)\|_{L^2}^2 \geq \lambda_{N+1} \|f - P_N f\|_{L^2}^2} \quad (*)$$

In addition, $(f - P_N f, u_k)_{L^2} = 0$, $\forall u_k \in E_k$

implies $\int_\Omega \nabla(f - P_N f) \nabla u_k = 0, \quad \forall u_k \in E_k$

and thus $\underline{\int_\Omega \nabla(f - P_N f) \nabla P_N f = 0}$.

such that

$$(**) \quad \boxed{\|\nabla(f - P_N f)\|_{L^2}^2 = \|\nabla f\|_{L^2}^2 - \|\nabla P_N f\|_{L^2}^2 \leq \|\nabla f\|_{L^2}^2}$$

From $(*)$ and $(**)$ we have

$$\lambda_{N+1} \|f - P_N f\|_{L^2}^2 \le \|\nabla f\|_{L^2}^2$$

and since $\lambda_N \xrightarrow[N]{} \infty$ this implies

$$\|f - P_N f\|_{L^2} \to 0 \quad (\underline{\text{for } f \in H_0^1(\Omega)})$$

Next consider $\underline{f \in L^2(\Omega)}$, let $\varepsilon > 0$.

There is $\tilde{f} \in H_0^1(\Omega)$ [in fact $C_c^\infty(\Omega)$]

such that $\|f - \tilde{f}\| < \frac{\varepsilon}{3}$

for $\tilde{f}$ we know that there is N_ε such that $\|\tilde{f} - P_N \tilde{f}\|_{L^2} < \frac{\varepsilon}{3}$, $\forall N \ge N_\varepsilon$.

Thus,
$$\begin{aligned}
\|f - P_N f\|_{L^2} &= \|f - \tilde{f} + \tilde{f} - P_N \tilde{f} + P_N \tilde{f} - P_N f\|_{L^2} \\
&\le \|f - \tilde{f}\|_{L^2} + \|\tilde{f} - P_N \tilde{f}\|_{L^2} + \|P_N \tilde{f} - P_N f\|_{L^2} \\
&\le \|f - \tilde{f}\|_{L^2} + 2\|\tilde{f} - P_N \tilde{f}\|_{L^2} < \varepsilon, \quad \forall N \ge N_\varepsilon.
\end{aligned}$$

Such that $\|f - P_N f\|_{L^2} \xrightarrow[N]{} 0$

The Max-min characterization of eigenvalues

For arbitrary $v_1, \dots, v_{k-1} \in H_0^1(\Omega)$ define

$$m\{v_1, \dots, v_{k-1}\} = \inf_{u \in H_0^1} \left\{ \int_\Omega |\nabla u|^2 : \|u\|_{L^2} = 1, \int_\Omega u v_i = 0 \right.$$
$$\left. i = 1 : k-1 \right\}$$

Then
$$\boxed{\lambda_k = \sup_{(v_1, \dots, v_{k-1})} m\{v_1, \dots, v_{k-1}\}}.$$

<u>Proof</u> Let $v_i = u_i$, $i = 1 : k-1$

then $m\{u_1, \dots, u_{k-1}\} = \lambda_k$ and therefore

(*) $$\lambda_k \le \sup_{(v_1, \dots, v_{k-1})} m(v_1, \dots, v_{k-1})$$

Next we show that $\lambda_k \ge \sup_{(v_1 \dots v_{k-1})} m(v_1, \dots, v_{k-1})$

Let $v_1, \dots, v_{k-1} \in H_0^1(\Omega)$ arbitrary.

Show that there is $u \in H_0^1(\Omega)$, $\|u\|_{L^2} = 1$

such that $\int_\Omega |\nabla u|^2 \le \lambda_k$

and $\int_\Omega u v_i = 0$, $i = 1 : k-1$.

Let $u = \sum_{i=1}^{K} c_i u_i$

Require $\|u\|_{L^2} = 1$: $\sum_{i=1}^{K} c_i^2 = 1$

$\int_\Omega u v_j = 0$, $j = 1:K-1$: $\sum_{i=1}^{K} c_i (u_i, v_j) = 0$, $j = 1:K-1$

Then $\int_\Omega |\nabla u|^2 = \sum_{i=1}^{K} \lambda_i c_i^2 \le \lambda_K \sum_{i=1}^{K} c_i^2 = \lambda_K$

and $m(v_1, \dots, v_{K-1}) \le \int_\Omega |\nabla u|^2 \le \lambda_K$

Since $(v_1, \dots, v_{K-1})$ are arbitrary, it follows that

$$\lambda_K \ge \sup_{(v_1, \dots, v_{K-1})} m(v_1, \dots, v_{K-1})$$

and with $(*)$ we then conclude that equality must hold.

<u>Consequence</u> (monotonicity of eigenvalues with respect to domain)

if $\hat{\Omega} \subset \Omega$ are open and bounded domains, then <u>$\lambda_K(\hat{\Omega}) \ge \lambda_K(\Omega)$, $\forall K \ge 1$</u>