<u>Remark</u> : $\Omega \subset \mathbb{R}^n$ open, bounded

$$\begin{cases} -\Delta e_k = \lambda_k e_k & \text{in } \Omega \\ e_k|_{\partial\Omega} = 0 \end{cases}$$

we have that $\lambda_k > 0$, $\lambda_k \xrightarrow[k]{} \infty$, and $\{e_k\}_{k\geq 1}$ is a basis to $L^2(\Omega)$. We notice that $\{\frac{1}{\sqrt{\lambda_k}} e_k\}_{k\geq 1}$ is an orthonormal basis to $H_0^1(\Omega)$ also.

Variational formulation

$$\int_\Omega \nabla e_k \nabla v = \lambda_k \int_\Omega e_k v, \quad \forall v \in H_0^1.$$

On H_0^1 we consider $(u,v)_a = \int_\Omega \nabla u \nabla v$ which induced norm is equivalent to the standard H^1- norm.

$$a(u,v) = \int_\Omega \nabla u \nabla v$$

Then $\quad \|e_k\|_a^2 = \int_\Omega |\nabla e_k|^2 = \lambda_k$

and $\quad (e_k, e_m)_a = \lambda_k \int_\Omega e_k e_m = \lambda_m \int_\Omega e_m e_k = 0$

if $\lambda_m \neq \lambda_k$.

Thus, $\{\frac{1}{\sqrt{\lambda_k}} e_k\}$ are $(\cdot,\cdot)_a$ <u>orthonormal</u>.

We show that $\left\{ \frac{1}{\sqrt{\lambda_k}} e_k \right\}_{k \geq 1}$ is dense in $H_0^1(\Omega)$.

Let $v \in H_0^1(\Omega)$: $a(e_k, v) = 0, \quad \forall k \geq 1$

Then $\int_\Omega e_k v = 0, \quad \forall k \geq 1$

Since $\{e_k\}$ is dense in $L^2(\Omega)$ then $v = 0$ (as element of L^2), thus $v = 0$, a.e.

The same procedure may be used to show that if the bilinear form

$$a(u, v) = \int_\Omega \nabla v \cdot K \nabla u + \int_\Omega q u v$$ is

elliptic, then the eigenfunctions $\left\{ \frac{1}{\sqrt{\lambda_k}} e_k \right\}_{k \geq 1}$ are a basis to $H_0^1(\Omega)$.

$(\cdot, \cdot)_a$ - orthonormal

<u>Theorem</u> (Minimization principle). Consider the elliptic operator $\mathcal{L}u = -\nabla\cdot(K\nabla u) + quv$

such that $a(u,v) = \int_\Omega \nabla v\cdot K\nabla u + quv$ is elliptic on $H_0^1(\Omega)$. Let $\lambda_1 > 0$ denote the <u>principal</u> (smallest, first) eigenvalue. Then

$$\lambda_1 = \inf_{\substack{v\in H_0^1(\Omega)\\ v\not\equiv 0}} \frac{a(v,v)}{(v,v)}$$

<u>Proof</u>: Let $v\in H_0^1(\Omega)$, $v\not\equiv 0$. We know that the eigenfunctions $\{e_k\}_{k\geq 1}$ form a basis to $L^2(\Omega)$ and $\left\{\frac{1}{\sqrt{\lambda_k}}e_k\right\}_{k\geq 1}$ form a basis to $H_0^1(\Omega)$, $(\cdot,\cdot)_a$ - orthonormal.

Then, $v = \sum_{k=1}^{\infty}(v,e_k)e_k$, $\|v\|_{L^2}^2 = (v,v) = \sum_{k=1}^{\infty}(v,e_k)^2$

and $v = \sum_{k=1}^{\infty}\frac{1}{\lambda_k}a(v,e_k)e_k$,

$$a(v,v) = \|v\|_a^2 = \sum_{k=1}^{\infty}\frac{1}{\lambda_k}a(v,e_k)^2 = \sum_{k=1}^{\infty}\lambda_k(v,e_k)^2$$

$$\geq \min_{k\geq 1}\{\lambda_k\}\sum_{k=1}^{\infty}(v,e_k)^2 = \lambda_1\sum_{k=1}^{\infty}(v,e_k)^2 =$$

$$= \lambda_1(v,v)$$

Thus, $\forall v \in H_0^1$, $a(v,v) \geq \lambda_1 (v,v)$

and $a(e_1, e_1) = \lambda_1 (e_1, e_1)$

inf is min since it is reached at e_1.

such that $\lambda_1 = \dfrac{a(e_1,e_1)}{(e_1,e_1)} = \min\limits_{\substack{v \in H_0^1(\Omega) \\ v \neq 0}} \dfrac{a(v,v)}{(v,v)}$

<u>Consequence</u> characterization of the Poincaré constant for $H_0^1(\Omega)$: The Poincaré constant $C(\Omega)$ is the <u>smallest</u> constant such that

$$\int_\Omega u^2 \leq C(\Omega) \int_\Omega |\nabla u|^2, \quad \forall u \in H_0^1(\Omega).$$

Therefore, $\boxed{C(\Omega) = \dfrac{1}{\lambda_1}}$

where λ_1 is the principal eigenvalue of the Laplacian with Dirichlet boundary conditions:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}.$$

Ritz - Galerkin framework to approximate solution

Consider the variational problem in the Hilbert space H:

Find $u \in H$: $a(u,v) = (f,v), \ \forall v \in H$

where $f \in H$ and $a: H \times H \to \mathbb{R}$ satisfies Lax-Milgram, and $a(u,v) = a(v,u)$, $\forall u, v \in H$ (symmetric).

Consider a finite dimensional subspace $H_K \subset H$,

$H_K = \operatorname{span}\{\phi_1, \dots, \phi_K\}$

Formulate the variational problem in H_K as

(*) Find $u_K \in H_K$ such that $a(u_K, v) = (f, v)$, $\forall v \in H_K$.

Notice that (*) is equivalent to

(**) Find $u_K \in H_K$: $a(u_K, \phi_j) = (f, \phi_j)$, $j = 1:K$

Since $u_K \in H_K$, $u_K = \sum_{i=1}^{K} \xi_i \phi_i$, where $\xi_i \in \mathbb{R}$ are the coefficients (coordinates) in basis $\{\phi_1, \dots, \phi_K\}$.

Then (**) is equivalent to

(***) $$\sum_{i=1}^{K} \xi_i \, a(\phi_i, \phi_j) = (f, \phi_j), \quad j = 1:K$$

Denote
$$a_{ji} = a(\phi_i, \phi_j) = a_{ij} \quad \left(\text{since } a(\cdot,\cdot) \text{ symmetric}\right)$$

$A \in \mathbb{R}^{K \times K}$, $A = (a_{ij})_{i,j=1:K}$ $\to$ stiffness matrix

$b_j = (f, \phi_j)$,

$b \in \mathbb{R}^K$, $b = (b_j)_{j=1:K}$

Then we obtain a linear system for the vector of coefficients $\xi \in \mathbb{R}^K$:

$$A\xi = b.$$

<u>Remark</u>: The matrix A is symmetric and positive definite since $a(\cdot,\cdot)$ is assumed to be symmetric and elliptic.

[if $\beta \in \mathbb{R}^K$ arbitrary, let $v_K = \sum_{i=1}^{K} \beta_i \phi_i \in H_K$.
then $$a(v_K, v_K) = \xi^T A \xi \geq \alpha \|v\|^2 > 0 \text{ if } v \neq 0$$]

therefore, there is a unique solution to the linear system $A\xi = b$

$\Rightarrow$ there is a unique solution $u_K \in H_K$ to $(*)$.

Error estimate (Céa's Lemma)

Let u solution to $a(u,v) = (f,v), \ \forall v \in H$

u_K solution to $a(u_K, v) = (f,v), \ \forall v \in H_K$

Then $\boxed{\|u - u_K\| \le \sqrt{\frac{M}{\alpha}}\, \|u - v\|, \ \forall v \in H_K}$

Proof $\forall v \in H_K$: $a(u - u_K, v) = 0$ $\left(\begin{array}{l} u_K = P_{H_K} v \\ \text{w.r.t. } (\cdot,\cdot)_a \end{array} \right)$

such that

$$\alpha \|u - u_K\|^2 \le a(u - u_K, u - u_K) =$$

$$= \inf_{v \in H_K} a(u - v, u - v)$$

$$\le M \|u - v\|^2, \quad \forall v \in H_K$$

$$\Rightarrow \|u - u_K\| \le \sqrt{\frac{M}{\alpha}}\, \|u - v\|, \quad \forall v \in H_K.$$

Consequence if we consider an approximating sequence ~~to~~ $u_K \in H_K$ such that $\overline{\bigcup_{K=1}^{\infty} H_K} = H$

then $u_K \to u$ (convergence to the solution of the variational problem)

Remark Spectral methods if $H_K = \{e_1, \ldots, e_K\}$ where $\{e_i\}_{i \geq 1}$ is a Hilbert basis to H that consists of the eigenvectors to

$$a(e_i, v) = \lambda_i (e_i, v), \quad \forall v \in H.$$

then

$$a(e_i, e_j) = 0, \quad i \neq j$$

$$a(e_i, e_i) = \lambda_i$$

such that $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_K)$

and $\xi = A^{-1} b$, $\xi_j = \frac{b_j}{\lambda_j}$, $j = 1:K$.

The approximate solution is expressed as

$$u_K = \sum_{i=1}^{K} \xi_i e_i = \sum_{i=1}^{K} \frac{1}{\lambda_i} (f, e_i) e_i =$$

$$\left[= \sum_{i=1}^{K} a\left(u, \frac{1}{\sqrt{\lambda_i}} e_i\right) \frac{1}{\sqrt{\lambda_i}} e_i \right]$$

the truncated eigenfunction series, projection on H_K w.r.t. $(\cdot, \cdot)_a$ inner product.