Spectral theory of elliptic operators (continued)

Recall that if $T: H \to H$ is a linear and continuous operator then $\lambda \in \mathbb{R}$ is eigenvalue to T if $N(T-\lambda I) \neq 0$, That is $\exists u \neq 0 : Tu = \lambda u$

The eigenspace associated with λ is

$N(T-\lambda I) = \{u \in H : Tu = \lambda u\}$, $\sigma_p(T)$ denotes the set of eigenvalues

Theorem (Fredholm Alternative) if $T: H \to H$ is a compact operator and $\lambda \neq 0$ then

i) $N(T-\lambda I)$ is finite dimensional

ii) $R(T-\lambda I)$ is closed subspace of H and

$$R(T-\lambda I) = \{N(T^* - \lambda I)\}^{\perp}$$

iii) $N(T-\lambda I) = \{0\} \iff R(T-\lambda I) = H$

Remark: As a consequence to i) the eigenspace associated with each eigenvalue of the elliptic operator $\mathcal{L}$ is finite dimensional (we assume $a(\cdot,\cdot)$ elliptic so, $\lambda > 0$).

Remark : Consider $a(u,v) = \lambda(u,v)$

Let $\{u_n\}_{n\geq 1}$ such that $\{u_n\}$ linearly independent,

$a(u_n, v) = \lambda(u_n, v)$

we may assume that $\{u_n\}$ orthonormal in L^2

then $a(u_n, u_n) = \lambda(u_n, u_n) = \lambda$

So, $\{u_n\}$ bounded in H^1- norm.

$\Rightarrow$ there is a subsequence weakly convergent in H^1 and convergent in L^2.

Contradiction with $\{u_n\}$ orthonormal in L^2.

Remark Enough to show the results for $\lambda = 1$

Let $H_1 = N(T - I) = \{u \in H : Tu = u\}$.

Then $H_1 \subset H$ closed thus Hilbert and

$\overline{B}_{H_1}(0,1) = \{u \in H : Tu = u, \ \|u\| \leq 1\} \subset T(\overline{B}_H(0,1))$

is compact. Thus H_1 must be finite dimensional.

Theorem Let H an infinite dimensional Hilbert space and $T : H \to H$ a compact operator. Then

a) $0 \in \sigma(T)$ (T is not bijective)

b) $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$
(the non-zero elements of the spectrum are the non-zero eigenvalues)

c) One of the following holds:

i) There is a finite number of eigenvalues

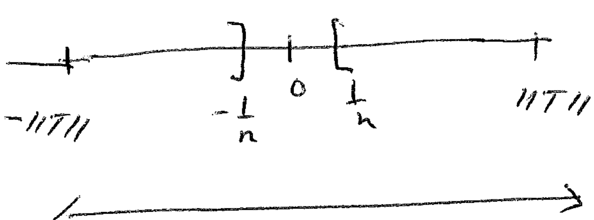
ii) There is an infinite number of eigenvalues, $\{\lambda_n\}_{n \geq 1}$. Then $\lambda_n \xrightarrow[n]{} 0$
(the sequence of eigenvalues converges to zero).

Proof a) By contradiction. if $0 \notin \sigma(T)$ then T is bijective. Then T^{-1} exists and is continuous, thus $I = T \circ T^{-1}$ is compact thus $\overline{B}(0,1)$ is compact, contradiction with $\dim(H) = \infty$.

b) Let $\lambda \in \sigma(T) \setminus \{0\}$. We show that $\lambda \in \sigma_p(T)$.
Assume that $\lambda \notin \sigma_p(T)$ then $N(T-\lambda I) = \{0\}$ such that by Fredholm Alternative $R(T-\lambda I) = H$, thus $T-\lambda I$ is bijection so $\lambda \in \rho(T)$. Contradiction.

c) Notice that any eigenvalue of T is such that $|\lambda| \leq \|T\|$, that is $\boxed{\sigma(T) \subset [-\|T\|, \|T\|]}$
indeed, $Tu = \lambda u$ implies $|\lambda| \|u\| = \|Tu\| \leq \|T\| \|u\|$.
such that $|\lambda| \leq \|T\|$

Next we show that for any $n \geq 1$ the set
$\sigma(T) \cap \{\lambda \in \mathbb{R} : |\lambda| \geq \frac{1}{n}\}$ is either empty or has a <u>finite</u> number of elements.



includes $\sigma(T)$

or $[-\|T\|, -\frac{1}{n}]$

By contradiction, assume that there is $n \geq 1$ such that the interval $[\frac{1}{n}, \|T\|]$ has an infinite number of elements of $\sigma(T)$.

Then there must be a sequence of eigenvalues $\{\lambda_k\}_{k\geq 1}$ such that $\lambda_k \neq 0$ and $\lambda_k \to \lambda \neq 0$ (since all λ_k are with $|\lambda_k| \geq \frac{1}{n}$).

This is a contradiction with the following:

<u>Lemma</u> Let $\{\lambda_n\}_{n\geq 1}$ a sequence of distinct eigenvalues such that $\lambda_n \neq 0$, $\forall n$ and $\lambda_n \to \lambda$. Then $\underline{\lambda = 0}$.

(all non-zero eigenvalues are "isolated").

<u>Proof</u>: Let $\lambda_n \neq 0$ eigenvalue, let e_n eigenvector, denote $E_n = \text{span}\{e_1, \dots, e_n\}$.

Notice that $e_1, \dots, e_n$ are <u>linearly independent</u> since $\lambda_i \neq \lambda_j$, $i \neq j$

proof

$$\left[\begin{array}{l} \text{if } e_{n+1} = \sum_{i=1}^{n} \alpha_i e_i \text{ then } Te_{n+1} = \sum_{i=1}^{n} \alpha_i Te_i \\ \text{thus } \lambda_{n+1} e_{n+1} = \lambda_{n+1} \sum_{i=1}^{n} \alpha_i e_i = \sum_{i=1}^{n} \alpha_i \lambda_i e_i \\ \qquad \Rightarrow \alpha_i (\lambda_i - \lambda_{n+1}) = 0 \text{ so } \alpha_i = 0 \end{array}\right]$$

Therefore, $E_n \subsetneq E_{n+1}$ such that there is

$u_{n+1} \in E_{n+1} \setminus E_n$: $\|u_{n+1}\| = 1$, $\text{dist}(u_{n+1}, E_n) \geq 1$, $\forall n \geq 1$

Notice that the sequence $\{u_n\}_{n \geq 2}$ is such that

$u_n \in E_n$ and $Tu_n - \lambda_n u_n \in E_{n-1}$, $n \geq 2$

for any $2 \leq m < n$ we have

$$E_{m-1} \subset E_m \subset E_{n-1} \subset E_n$$

and

$$\left\| \frac{Tu_n}{\lambda_n} - \frac{Tu_m}{\lambda_m} \right\| = \left\| \frac{Tu_n - \lambda_n u_n}{\lambda_n} - \frac{Tu_m - \lambda_m u_m}{\lambda_m} + u_n - u_m \right\|$$

$$= \Big\| u_n - \underbrace{\left[u_m + \frac{Tu_m - \lambda_m u_m}{\lambda_m} - \frac{Tu_n - \lambda_n u_n}{\lambda_n} \right]}_{\in E_{n-1}} \Big\| \geq 1$$

Since $\|u_n\| = 1$, $\forall n \geq 2$ and T is compact,

$\{Tu_n\}$ has a convergent subsequence.

if $\lambda_n \longrightarrow \lambda \neq 0$ then

$\left\{ \frac{Tu_n}{\lambda_n} \right\}$ has a convergent subsequence

Contradiction. *** end Lemma's proof.

Spectrum of a self-adjoint operator Property:

Let $T: H \to H$ linear, continuous, self-adjoint.

Denote $$m = \inf_{\substack{u \in H \\ \|u\|=1}} (Tu, u), \quad M = \sup_{\substack{u \in H \\ \|u\|=1}} (Tu, u)$$

Then $\sigma(T) \subset [m, M]$ and $m \in \sigma(T)$, $M \in \sigma(T)$.

Consequence if $T: H \to H$ is linear, continuous and self-adjoint then

$$\boxed{\sigma(T) = \{0\} \text{ implies } T \equiv 0.}$$

Remark Notice that if $\lambda > M$ then

$a(u, v) = (\lambda u - Tu, v)$ is elliptic

$$a(u, u) = \lambda \|u\|^2 - (Tu, u) \geq \underbrace{(\lambda - M)}_{>0} \|u\|^2$$

such that $\lambda I - T$ is bijective, so $\lambda \in \rho(T)$

if $\sigma(T) = \{0\}$ then $(Tu, u) = 0$, $\forall u \in H$.

then $$2(Tu, v) = (T(u+v), u+v) - (Tu, u) - (Tv, v) = 0$$

$$\Rightarrow (Tu, v) = 0, \ \forall u, v \in H \Rightarrow Tu = 0, \ \forall u \in H$$

<u>Hilbert-Schmidt theorem</u> Let H a separable Hilbert space and $T: H \to H$ a self-adjoint and compact operator. Then there is a Hilbert basis to H that consists of the eigenvectors of T.

<u>Proof</u> Let $\{\lambda_n\}_{n\geq 1}$ denote the distinct eigenvalues of T, except 0 : $\lambda_i \neq \lambda_j$, $i \neq j$

$\lambda_i \neq 0$

there is $e_n \neq 0$: $Te_n = \lambda_n e_n$

Denote $\lambda_0 = 0$, $E_0 = N(T)$, $E_n = N(T - \lambda_n I)$

We have that $0 \leq \dim E_0 \leq \infty$ and $0 < \dim E_n < \infty$

Each of E_n, $n \geq 0$ is a closed subspace of H and $E_n \perp E_m$ if $n \neq m$: if $u \in E_m$, $v \in E_n$

$Tu = \lambda_m u \qquad Tv = \lambda_n v$

$\Rightarrow (Tu, v) = \lambda_m (u,v) = (Tv, u) = \lambda_n (v, u) \Rightarrow \underline{(u,v) = 0}$

We choose in each E_n, $n \geq 0$ a Hilbert basis (each element being an eigenvector of T) and show that the vector space generated by $(E_n)_{n\geq 0}$ is <u>dense</u> in H.

Clearly, $T(E) \subset E$ such that $T(E^\perp) \subset E^\perp$

[if $u \in E^\perp$ and $v \in E$: $(\underbrace{Tu}_{E^\perp}, v) = (u, \underbrace{Tv}_{E}) = 0$.]

Let $T_0 = T|_{E^\perp}$ then T_0 is self-adjoint and compact operator on $E^\perp$: $T_0 : E^\perp \to E^\perp$ and such that $\sigma(T_0) = \{0\}$.

[if $\lambda \in \sigma(T_0) \setminus \{0\}$ then $\lambda \in \sigma_p(T_0)$ and there is $u \in E^\perp$, $u \neq 0$: $T_0 u = \lambda u$ $\to$ then λ is eigenvalue of T, thus $u \in E$ so $u \in E \cap E^\perp = \{0\}$]

Therefore, $T_0 \equiv 0$ such that $T(E^\perp) = 0$. Thus

$$E^\perp \subset N(T) = E_0 \subset E \quad \text{implies} \quad E^\perp = \{0\}$$

such that E is dense in H.

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Theorem (the minimization principle)
Let $\lambda_1 > 0$ denote the principal (smallest) eigenvalue of the elliptic operator L.
Then
$$\lambda_1 = \inf_{\substack{v \in H_0^1 \\ v \neq 0}} \frac{a(v,v)}{(v,v)}$$

Proof Let $v \in H_0^1(\Omega)$, $v \neq 0$.
As an element of $L^2(\Omega)$,
$$v = \sum_{i=1}^{\infty} (v, e_i) e_i , \qquad (v,v) = \sum_{i=1}^{\infty} (v, e_i)^2$$
Since $\left\{ \frac{1}{\sqrt{\lambda_i}} e_i \right\}$ is orthonormal basis to H_0^1 w.r.t. $(\cdot,\cdot)_a$ - inner product, as an element of H_0^1,
$$v = \sum_{i=1}^{\infty} \frac{1}{\lambda_i} a(v, e_i) e_i \quad \text{and}$$
$$\|v\|_a^2 = a(v,v) = \sum_{i=1}^{\infty} \frac{1}{\lambda_i} a(v, e_i)^2 = \sum_{i=1}^{\infty} \lambda_i (v, e_i)^2$$
$$\geq \lambda_1 \sum_{i=1}^{\infty} (v, e_i)^2$$
Thus, $\forall v \in H_0^1$: $$\frac{a(v,v)}{(v,v)} \geq \lambda_1 = \frac{a(e_1, e_1)}{(e_1, e_1)}$$