Harmonic functions. Mean value property. Strong maximum principle.

Definition Let $\Omega \subset \mathbb{R}^n$ bounded domain.
$u : \Omega \to \mathbb{R}$ is harmonic function if $\Delta u(x) = 0$, $\forall x \in \Omega$.

Definition A function $u : \Omega \to \mathbb{R}$ is said to have the mean value property in Ω if for any $x_0 \in \Omega$ and any $r > 0$ such that $\bar{B}(x_0, r) \subset \Omega$ we have

$$u(x_0) = \frac{1}{|S(x_0,r)|} \int_{S(x_0,r)} u(x)\, ds$$

where $S(x_0, r)$ denotes the $n-1$ dimensional sphere in $\mathbb{R}^n$ of center x_0 and radius r and $|S(x_0,r)|$ denotes its area

$$\left(n=2 : \ |S(x_0,r)| = 2\pi r \ ; \quad n=3 \ \ |S(x_0,r)| = 4\pi r^2 \right)$$

$n=2$ Poisson's formula:

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} g(\bar\theta)\, \frac{a^2 - r^2}{r^2 + a^2 - 2ar\cos(\theta - \bar\theta)}\, d\bar\theta$$

for $\begin{cases} \Delta u = 0 \\ u(x,y) = g(\theta), \text{ at } r = a. \end{cases}$ Laplace eq. on $B(a,0)$.

<u>Mean value theorem</u> Let u be a harmonic function in a bounded domain $\Omega \subset \mathbb{R}^n$. Then u has the mean value property in Ω.

<u>Proof</u> (for $n=2$, similar idea for $n \geq 3$).

Let $(x_0, y_0) \in \Omega$, arbitrary fixed. For $r > 0$ define

$$g(r) = \frac{1}{2\pi r} \int_{\mathcal{C}(x_0, y_0; r)} u(x,y)\, ds$$

We show that $g'(r) = 0$, for $0 < r < \varepsilon$

where $\varepsilon > 0$ is such that $B(x_0, y_0; \varepsilon) \subset \Omega$.

$$g(r) = \frac{1}{2\pi r} \int_0^{2\pi} u(x_0 + r\cos\theta, y_0 + r\sin\theta) \cdot r\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos\theta, y_0 + r\sin\theta)\, d\theta$$

$$g'(r) = \frac{1}{2\pi} \int_0^{2\pi} u_x \cdot \cos\theta + u_y \cdot \sin\theta\, d\theta$$

$$= \frac{1}{2\pi r} \int_{\mathcal{C}(x_0, y_0; r)} \nabla u(x,y) \cdot \vec{\nu}\, ds = 0$$

since $\displaystyle\int_{\mathcal{C}(x_0, y_0; r)} \frac{\partial u}{\partial \vec{\nu}} = \int_{B(x_0, y_0; r)} \Delta u = 0$.

in addition, we have that $g(0) = u(x_0, y_0)$.

Thus, $u(x_0, y_0) = \frac{1}{2\pi r} \int_{C(x_0, y_0; r)} u(x,y)\, ds$ for any $r > 0$ such that $\bar{B}(x_0, y_0; r) \subset \Omega$.

(since $g(r)$ is constant).

Property if u satisfies the mean value property in $\Omega \subset \mathbb{R}^n$ bounded domain and $x^* \in \Omega$ is a max point for $u(x)$ in Ω, then

$u(x) \equiv u(x^*)$ for any ball $B(x^*, r) \subset \Omega$

(u is constant in any ball that contains x^* and is included in Ω).

Proof Let $x^* \in \Omega$ max point: $u(x^*) \geq u(x)$, $\forall x \in \Omega$

Let $B(x^*, r) \subset \Omega$ and assume that there is

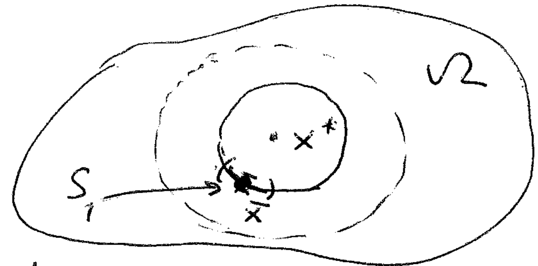
$\bar{x} \in B(x^*, r)$: $u(\bar{x}) < u(x^*)$

Let $\delta = |\bar{x} - x^*| > 0$ and consider $S(x^*, \delta)$

Since $\bar{x} \in S(x^*, \delta)$ and $u(\bar{x}) < u(x^*)$

there is a subset $S_1 \subset S$, $|S_1| > 0$ such that

$u(x) < u(x^*)$, $\forall x \in S_1$



Then

$$u(x^*) = \frac{1}{|S(x^*,\delta)|}\left[\int_{S \setminus S_1} u(x)ds + \int_{S_1} u(x)ds\right] <$$

$$< u(x^*) \quad \rightarrow \text{contradiction.}$$

<u>Consequence</u>

<u>Strong maximum principle</u> Assume that $\Omega \subset \mathbb{R}^n$ is an open, bounded domain and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $\Delta u = 0$ in Ω (satisfies the mean value property in Ω)

Then u reaches its max/min values on $\partial\Omega$. in addition, if u reaches its max or min value at a point in Ω (interior domain) then <u>u is constant in</u> $\bar{\Omega}$.

<u>Theorem</u> Let $\Omega \subset \mathbb{R}^n$ bounded domain. if $u : \Omega \to \mathbb{R}$ is continuous and satisfies the mean value property in Ω then u is harmonic in Ω.

<u>Proof</u> Let $x_0 \in \Omega$, $\bar{B}(x_0, r) \subset \Omega$.

Consider the problem

$$\begin{cases} -\Delta v = 0 & \text{on } B(x_0, r) \\ v = u & \text{on } S(x_0, r) = \partial B(x_0, r) \end{cases}$$

Since v is harmonic, it has the mean value property and by construction $v = u$ on $S(x_0, r)$

Then $v - u$ has the mean value property on $B(x_0, r)$ and $v - u = 0$ on $S(x_0, r)$

By the strong maximum principle, $v - u$ takes max/min values in $\bar{B}(x_0, r)$ on the boundary $S(x_0, r)$ $\Rightarrow$ $u(x) = v(x)$. $\forall x \in \bar{B}(x_0, r)$

Since $x_0 \in \Omega$ is arbitrary, $\Delta u(x) = 0$, $\forall x \in \Omega$.

Maximum principles for parabolic problems

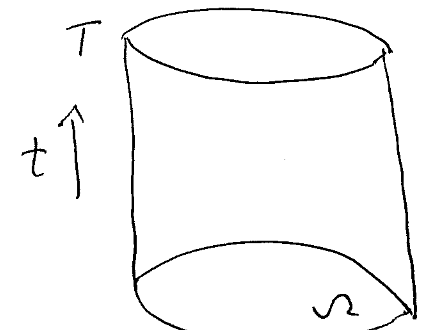
Let $\Omega \subset \mathbb{R}^n$ open, bounded domain, denote $\Omega_T = \Omega \times (0, T]$

if $u \in C^{2,1}(\Omega_T) \cap C(\bar{\Omega}_T)$ satisfies

(a) $u_t - \Delta u \leq 0 \quad , \quad (x,t) \in \Omega_T$

(b) $u|_{\partial\Omega} \leq 0 \quad , \quad 0 \leq t \leq T$

(c) $u(x,0) \leq 0 \quad , \quad x \in \Omega$



then

$$u(x,t) \leq 0, \quad (x,t) \in \bar{\Omega}_T$$

Proof by contradiction first we show that the result holds if (a) is replaced by a strict inequality: (d): $u_t - \Delta u < 0 \ , \ (x,t) \in \Omega_T$

Assume that there is $(\bar{x}, \bar{t}) \in \bar{\Omega}_T$ such that max point $u(\bar{x}, \bar{t}) > 0$. Then one of the following holds:

i) $(\bar{x}, \bar{t})$ is in the interior of $\bar{\Omega}_T$, that is $(\bar{x}, \bar{t}) \in \Omega \times (0, T)$. Then $\left.\begin{array}{l} u_t(\bar{x}, \bar{t}) = 0 \\ \Delta u(\bar{x}, \bar{t}) \leq 0 \end{array}\right\} \Rightarrow$

$\Rightarrow u_t(\bar{x}, \bar{t}) - \Delta u(\bar{x}, \bar{t}) \geq 0$ contradiction.

$\boxed{\text{OR}}$

ii) $\bar{t} = T$, $\bar{x} \in \Omega$. Then $u_t(\bar{x}, \bar{t}) \geq 0$

since $\bar{t}$ is max point to $t \to u(\bar{x}, t)$.

and $\Delta u(\bar{x}, \bar{t}) \leq 0$ since $\bar{x}$ is max point

to $x \longrightarrow u(x, \bar{t})$.

thus $u_t(\bar{x}, \bar{t}) - \Delta u(\bar{x}, \bar{t}) \geq 0$ contradiction.

Next we assume that (a) holds, thus

$$u_t - \Delta u \leq 0 \quad \text{in } \Omega_T .$$

Let $\varepsilon > 0$ arbitrary and define

$$u_\varepsilon(x,t) = u(x,t) - \varepsilon t .$$

Then $u_\varepsilon(x,t) \leq u(x,t)$ in $\bar{\Omega}_T$ thus

(b) and (c) are satisfied for u_ε

and

$$u_{\varepsilon,t} - \Delta u_\varepsilon = u_t - \Delta u - \varepsilon < 0$$

Thus u_ε satisfies (d) and therefore

$u_\varepsilon(x,t) \leq 0$ in $\bar{\Omega}_T$.

then $u(x,t) \leq \varepsilon t$, in $\bar{\Omega}_T$ for any $\varepsilon > 0$.

thus $u(x,t) \leq 0$ in $\bar{\Omega}_T$.

Maximum principle for heat equation

Let $u : \bar{\Omega} \times [0,T] \to \mathbb{R}$ strong solution to

$$u_t - \Delta u = 0 \quad \text{in} \quad \Omega_T = \Omega \times (0,T]$$

Let $M = \max\left\{ \max_{x\in\bar{\Omega}} u(x,0), \max_{\substack{x\in\partial\Omega \\ 0\le t\le T}} u(x,t) \right\} = \max_{\partial\Omega_T} u(x,t)$

$m = \min_{\partial\Omega_T} u(x,t)$

<u>Notice</u>: here $\partial\Omega_T$ does not include $\Omega \times \{t=T\}$

Then $m \le u(x,t) \le M$

[in other words: u reaches its max/min values on the <u>parabolic boundary</u> of Ω_T)

<u>Proof</u>: Let $v = u - M$

Then $v_t = u_t$, $\Delta v = \Delta u$ thus $v_t - \Delta v = 0$

and $v\big|_{\partial\Omega} \le 0$

$v(x,0) \le 0$

$\Rightarrow v \le 0$ in $\bar{\Omega}_T$

Let $w = u - m$. Then $w \ge 0$ in $\bar{\Omega}_T$.

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__Corollary__: Let $u : \bar{\Omega} \times [0,T] \to \mathbb{R}$ strong solution

to
$$\begin{cases} u_t - \Delta u = f, & (x,t) \in \Omega \times (0,T] \\ u|_{\partial\Omega} = g, & 0 \le t \le T \\ u(x,0) = u_0(x) \end{cases}$$

Then
$$\|u\|_\infty = \max_{(x,t)\in\bar{\Omega}_T} |u(x,t)| \le T\underbrace{\|f\|_\infty}_{\text{denote } F} + \underbrace{\max\{\|g\|_\infty, \|u_0\|_\infty\}}_{\text{denote } M}$$

__Proof__: Let $v = u - (tF + M)$

$$\left.\begin{array}{l} v_t = u_t - F \\ \Delta v = \Delta u \end{array}\right\} \Rightarrow \left.\begin{array}{l} v_t - \Delta v = f - F \le 0 \\ v|_{\partial\Omega_T} \le 0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \quad v \le 0 \quad \Rightarrow \quad u \le tF + M \le TF + M.$$

Let $v = u + (tF + M)$

then
$$\left.\begin{array}{l} v_t - \Delta v = f + F \ge 0 \\ v|_{\partial\Omega_T} \ge 0 \end{array}\right\} \Rightarrow v \ge 0$$

$$\Rightarrow \quad u \ge -(tF + M) \ge -(TF + M).$$

Thus, $\quad |u(x,t)| \le TF + M$ in $\bar{\Omega}_T$