Maximum principles for elliptic problems

introductive example: Let $f \in C(\bar{I})$ and $u \in C^2(I)$ solution to the BVP

$$\begin{cases} -u'' + u = f, & \text{in } I = (0,1) \\ u(0) = \alpha, \quad u(1) = \beta \end{cases}$$

Denote $\quad m = \min_{x \in \bar{I}} f(x), \quad M = \max_{x \in \bar{I}} f(x)$

Then

$$\min\{\alpha, \beta, m\} \leq u(x) \leq \max\{\alpha, \beta, M\}, \quad \forall x \in \bar{I}$$

Proof Let x_0 min point to $u(x)$ and assume that x_0 is in the interior of $\bar{I}$, i.e. $0 < x_0 < 1$.

Then $u''(x_0) \geq 0$ such that

$$u(x_0) = u''(x_0) + f(x_0) \geq f(x_0) \geq m$$

Therefore, $\quad u(x) \geq u(x_0) \geq \min\{\alpha, \beta, m\}$

Let $\bar{x}_0$ max point to $u(x)$ and assume $0 < \bar{x}_0 < 1$ (interior to I). Then $u''(\bar{x}_0) \leq 0$ such that

$$u(\bar{x}_0) = f(\bar{x}_0) + u''(\bar{x}_0) \leq f(\bar{x}_0) \leq M$$

Therefore, $\quad u(x) \leq u(\bar{x}_0) \leq M \leq \max\{\alpha, \beta, M\}$

<u>Lemma 1</u> Let $\Omega \subset \mathbb{R}^n$ bounded domain and $u \in C^2(\Omega) \cap C(\bar{\Omega})$

i) if $\begin{cases} \Delta u \leq 0 & \text{in } \Omega \\ u \geq 0 & \text{on } \partial\Omega \end{cases}$ then $u \geq 0$ in Ω

ii) if $\begin{cases} \Delta u \geq 0 & \text{in } \Omega \\ u \leq 0 & \text{on } \partial\Omega \end{cases}$ then $u \leq 0$ in Ω

<u>Proof by contradiction</u>

i) Assume that there is $x \in \Omega$ such that $u(x) < 0$.
Since $u \in C(\bar{\Omega})$, there is $\bar{x} \in \Omega = \bar{\Omega} \setminus \partial\Omega$ such that

$$u(\bar{x}) = \min_{x \in \bar{\Omega}} u(x) < 0$$

($\bar{x}$ can't be on the boundary $\partial\Omega$ since $u(x) \geq 0$ on $\partial\Omega$. Thus $\bar{x}$ is in the interior)

Let $\varepsilon > 0$ arbitrary. Define

$$v : \bar{\Omega} \to \mathbb{R}, \qquad v(x) = u(x) - \varepsilon |x - \bar{x}|^2$$

Then, $\forall x \in \partial\Omega$, since $u_{|\partial\Omega} \geq 0$ we have

$$v(x) \geq -\varepsilon |x - \bar{x}|^2 \geq -\varepsilon d^2$$

where
$$d = \max_{(x,y)\in\bar{\Omega}} |x-y| < \infty$$ denotes the diameter of Ω

(finite number since Ω is bounded).

Choose $\varepsilon > 0$ such that $-\varepsilon d^2 > u(\bar{x})$.

Then
$$\left.\begin{array}{l} v\big|_{\partial\Omega} > u(\bar{x}) \\ v(\bar{x}) = u(\bar{x}) \end{array}\right\} \Rightarrow$$ $v(x)$ reaches its minimum value at a point $x^* \in \Omega$, interior to Ω (not on $\partial\Omega$).

Then $\boxed{\Delta v(x^*) \geq 0}$ (at a min point the Hessian matrix is positive semi-definite thus the Laplacian is non-negative).

But,
$$\Delta v(x) = \Delta u(x) - 2n\varepsilon < 0, \quad \forall x \in \Omega$$
(since $\Delta(|x-\bar{x}|^2) = 2n$)

and thus $\boxed{\Delta v(x^*) < 0}$

Contradiction.

ii) Apply i) to the function $-u(x)$.

Theorem (weak maximum – minimum principle)

Let $\Omega \subset \mathbb{R}^n$ open and bounded domain and $u \in C^2(\Omega) \cap C(\bar{\Omega})$. Denote

$$M = \max_{x \in \partial\Omega} u(x) , \qquad m = \min_{x \in \partial\Omega} u(x) .$$

Then

i) if $\Delta u(x) \geq 0$ in Ω then

$$u(x) \leq M, \quad \forall x \in \Omega$$

ii) if $\Delta u(x) \leq 0$ in Ω then

$$u(x) \geq m, \quad \forall x \in \Omega$$

iii) if $\Delta u = 0$ in Ω then

$$m \leq u(x) \leq M, \quad \forall x \in \Omega .$$

Proof i) Define $v(x) = M - u(x)$

Then

$$\left.\begin{aligned} \Delta v = -\Delta u &\leq 0 \text{ in } \Omega \\ v|_{\partial\Omega} &\geq 0 \end{aligned}\right\} \Rightarrow v(x) \geq 0 \text{ in } \bar{\Omega}$$

$\Rightarrow u(x) \leq M, \quad x \in \bar{\Omega}$

ii) Use $v(x) = u(x) - m$ | iii) Consequence of i), ii)

<u>Remarks</u> As a consequence to the max-min principle,

i) There is at most one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ to
$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases}$$

ii) Let $f_1, f_2 : \Omega \to \mathbb{R}$, $g_1, g_2 : \partial\Omega \to \mathbb{R}$ such that
$$f_1(x) \geq f_2(x), \quad x \in \Omega$$
$$g_1(x) \geq g_2(x), \quad x \in \partial\Omega$$

Let u_1, u_2 solve $\begin{cases} -\Delta u_1 = f_1 & \text{in } \Omega \\ u_1 = g_1 & \text{on } \partial\Omega \end{cases}$ $\begin{cases} -\Delta u_2 = f_2 & \text{in } \Omega \\ u_2 = g_2 & \text{on } \partial\Omega \end{cases}$

Then $\underline{u_1(x) \geq u_2(x), \quad \forall x \in \bar{\Omega}}$.

<u>Proof</u>: Let $u = u_1 - u_2$. Then
$$\left.\begin{aligned} \Delta u &= \Delta u_1 - \Delta u_2 = f_2 - f_1 \leq 0, \quad \text{in } \Omega \\ u(x) &= g_1(x) - g_2(x) \geq 0 \quad \text{on } \partial\Omega \end{aligned}\right\} \Rightarrow$$

$\Rightarrow u(x) \geq 0, \quad x \in \bar{\Omega}$, thus $u_1(x) \geq u_2(x)$ $\forall x \in \bar{\Omega}$

<u>Theorem</u> Assume that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ solves

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \quad \text{where } f \in C(\bar{\Omega}),\ g \in C(\partial\Omega)$$

Denote $F = \sup_{x\in\Omega} |f(x)|$, $\quad G = \max_{x \in \partial\Omega} |g(x)|$

Then $\quad |u(x)| \leq cF + G \quad , \quad \forall x \in \bar{\Omega}$

where $c > 0$ is a constant that depends on Ω only (on diameter of Ω).

<u>Remark</u>: If $v : \Omega \to \mathbb{R}$ is such that

$$\begin{cases} -\Delta v(x) \geq F & \text{in } \Omega \\ v(x) \geq \underbrace{|u(x)|}_{|g(x)|} & \text{on } \partial\Omega \end{cases} \quad \text{then } |u(x)| \leq v(x) \text{ on } \bar{\Omega}$$

since

$$-\Delta(v \pm u) = -\Delta v \pm (-\Delta u) \geq F \pm f(x) \geq 0, \text{ in } \Omega$$

and $\quad (v \pm u)\big|_{\partial\Omega} \geq v(x) - |u(x)| \geq 0$

thus $\quad v(x) \pm u(x) \geq 0, \quad x \in \bar{\Omega} \quad$ thus

$$|u(x)| \leq v(x), \quad x \in \bar{\Omega}$$

is an "a priori" estimate (bound) for u.

<u>Proof of the theorem</u> Since Ω is bounded, assume for simplicity, that

$$0 < x_1 < \beta \quad \text{for } x = (x_1, x_2, \dots, x_n) \in \Omega .$$

Define
$$v(x) = F(e^{\beta} - e^{x_1}) + G, \quad x \in \bar{\Omega} .$$

Then
$$-\Delta v = F e^{x_1} > F \quad (\text{since } x_1 > 0)$$

and for $x \in \partial\Omega$: $v(x) \geq G$ since $x_1 < \beta$

From the previous <u>Remark</u> it follows that

$$|u(x)| \leq v(x) < F e^{\beta} + G$$

thus with $c = e^{\beta}$, $|u(x)| \leq cF + G$.

<u>Consequence</u> <u>Continuous dependence on data</u>

Assume $u_1, u_2 \in C^2(\Omega) \cap C(\bar{\Omega})$ are such that

$$\begin{cases} -\Delta u_1 = f_1 & \text{in } \Omega \\ u_1 = g_1 & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} -\Delta u_2 = f_2 & \text{in } \Omega \\ u_2 = g_2 & \text{on } \partial\Omega \end{cases}$$

Then $\|u_1 - u_2\|_\infty = \max_{x \in \bar{\Omega}} |u_1(x) - u_2(x)| \leq (1 + c)\varepsilon$

where $c > 0$ is a constant (depends on Ω only) and $\varepsilon = \max\{\|f_1 - f_2\|_\infty, \|g_1 - g_2\|_\infty\}$