MTH 621: Homework # 3, due 11/24, in class

To receive full credit, present complete answers that show all work.

P1 (15 points). Let ϵ a constant such that $0 < \epsilon < 1$ and consider the two-dimensional domain Ω defined as the open ball with center at origin and radius ϵ in $\mathbb{R}^2$:

$$\Omega = \{(x,y) | r < \epsilon, \text{ where } r = \sqrt{x^2+y^2}\}$$

Let $0 < k < 1/2$ a constant coefficient. Decide whether the given function belongs to $H^1(\Omega)$. Justify your answer.

(5 points) $u(x,y) = r^{-k}, \; r \neq 0$

(10 points) $u(x,y) = |\ln(r)|^k, \; r \neq 0$

P2 (10 points). Consider $\Omega \subset \mathbb{R}^n$ a bounded domain and the Sobolev space $W_0^{1,p}(\Omega)$ defined as the closure of $C_c^\infty(\Omega)$ in the $W^{1,p}$-norm, $1 \leq p < \infty$,

$$W_0^{1,p}(\Omega) = \overline{\{C_c^\infty(\Omega), \|\cdot\|_{1,p}\}}$$

Show that the Poincaré inequality holds on $W_0^{1,p}(\Omega)$: there is a constant $C(\Omega,p) > 0$ such that

$$\int_\Omega |u|^p \, dx \leq C(\Omega,p) \int_\Omega \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^p dx, \quad \forall u \in W_0^{1,p}(\Omega)$$

P3 (15 points). Consider the following problem in a bounded domain $\Omega \subset \mathbb{R}^2$:

$$\begin{aligned} -\Delta u + b_1 \frac{\partial u}{\partial x_1} + b_2 \frac{\partial u}{\partial x_2} &= f, \; x \in \Omega \\ u &= 0, \; x \in \partial\Omega \end{aligned}$$

where b_1 and b_2 are arbitrary constants.

(5 points) Give a variational formulation to this problem.

(10 points) Show that for every $f \in L^2(\Omega)$ and any values of the constants b_1 and b_2 there is a unique solution to the variational problem.

P4 (15 points). Let $\Omega = (0,1) \times (0,1)$.

(5 points) Show that

$$\int_\Omega v^2 \, dx\, dy \leq \int_\Omega |\nabla v|^2 \, dx\, dy, \quad \forall v \in H_0^1(\Omega)$$

(10 points) Show that for every $f \in L^2(\Omega)$ and any constant $c > -1$ there is a unique weak solution to the problem

$$\begin{aligned} -\Delta u + cu &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$